Chapter 2
The Geometrical Character of Physics Theories

One of the powerful features of the variational approach is its generality. It is in fact based on a simple principle, stating that physical laws share the common characteristics of minimizing the action of a system, and as we have seen this can be applied to mechanics (i.e., the dynamics of particles) as well as to the dynamics of fields. This obviously cannot be regarded as the silver bullet of the physics problems, inasmuch as no one can anticipate which is the right expression of the Lagrangian for a given topic, and therefore that of the action to be minimized. The solution of such a task, however, is far from being purely arbitrary, and often many of the characteristics of a correct action can be deduced by some basic principles characterizing the physics theory of reference. Moreover, it is somewhat surprising to realize, as we do in the remainder of the book, that this technique can be applied to many different theories: Newtonian or relativistic, classical or quantum.[1] It is therefore extremely useful to understand which are these principles and how the differences among them can lead to completely different theories.

This is the goal of the next chapter but, even before that, it is worth anticipating that these principles are closely related to the first reason we advanced to justify the introduction of the variational approach, namely the possibility of deducing the equations of motion in any coordinate system. They in fact are the expression of a "geometrical connection" that stands at the very basis of our way of formulating physics theories.

[1]Given the scope of this book, the exposition is limited to classical physics. However, as long as the mechanism of the so-called *quantization* is understood, the same techniques can be applied to quantum models.

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2.1 A Scientific Theory as a Model

It might seem starting from afar, but inasmuch as this book is dealing with physics theories it is reasonable to ask ourselves if we actually realize what a physics theory is. As in the previous chapter, we choose to refrain from giving an abstract definition and instead start by describing what can probably be considered the first historical example of a scientific theory, when not even of a real theory of mathematical physics: Euclidean geometry.

Euclidean geometry as a "physics" theory

In today's view it is difficult to see Euclidean geometry in its physical "nuance," but in the past, at least until Newton, the feeling was completely different. To this aim it is sufficient to remember that Galileo considered the world "written in geometrical characters," or that many demonstrations of Newton's *Principia* are based on some theorems of Euclidean geometry which, albeit now considered exotic, at that time were part of the common knowledge of scientists as it presently for calculus. But this is even more meaningful if we consider Newton's position precisely with regard to calculus. It is well known that he was one of the inventors of this powerful mathematical technique, which a modern physicist would consider essential to explain the theories of the English scientist. It thus can be surprising to realize that Newton himself never used calculus in the *Principia*, that rather his exposition is entirely based on Euclidean geometry, and that even the structure of the work closely reproduces the axiomatic and deductive one of Euclid's *Elements*.

If we go even farther in the past, to the times of Hellenistic science, we can see more examples of such a connections because the physics and astronomy works of the time used Euclidean geometry to such an extent that it cannot be clearly stated, in terms of our mindset, whether they are mathematical or physical works (Russo 2004). The way this theory is now presented favors its misconception as a highly abstract mathematical work, with no connection to "reality", but this vision might change if we realize that in its original version Euclid's *Elements* contained not only definitions and theorems, but also what we could call *problems,* showing how some geometrical figures could be drawn, and in a certain sense could be considered as applications of the theory to practical tasks. This is not so strange if we think that geometrical methods are needed for many practical purposes and were used by other ancient civilizations way before Euclidean systematization. Does this mean that these cultures were using a scientific theory? The answer is no, because there are fundamental differences between their knowledge and how we specify a scientific theory today, and these are the same differences they show with respect to Euclid's exposition.

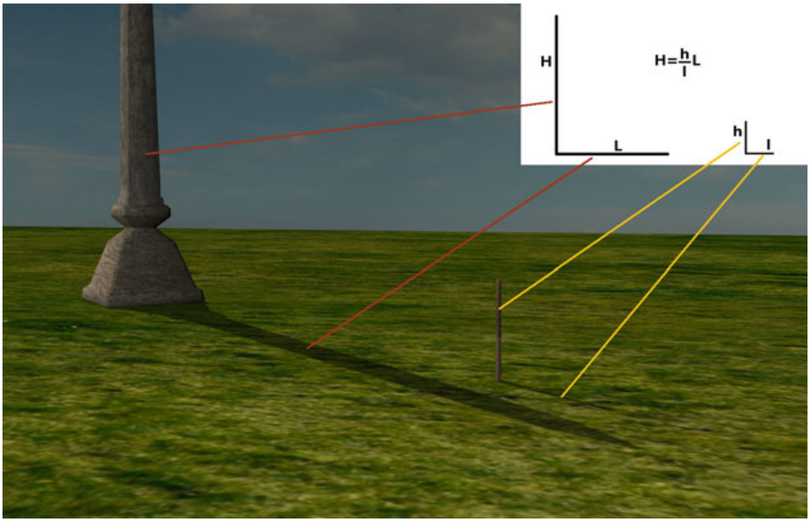

Fig. 2.1 The "rules of correspondence" between Euclidean geometry and reality in action.

Model versus reality in scientific theories

The latter starts with some definitions, concerning the concepts of point, line, surface, and so on, and with five assumptions or axioms,[2] that is, some statements which cannot be demonstrated by other, more elementary facts and thus taken as true *by hypothesis*.

Whereas the role of the latter is clear as the "game rules" of a theory, those of the definitions are usually less emphasized. Nonetheless they too play a fundamental role in a scientific theory, precisely that of establishing the "rules of correspondence" between the *real world* and *its representation given in the scientific model*.[3] For example, when we appeal to the triangles' similarity theorems to deduce the height of an obelisk from that of a smaller stick and the lengths of the two shadows (Fig. 2.1) we are implicitly doing the following reasoning: because the only property of interest in this case is a length, in any essential sense we can represent the two real objects and their shadows as segments of a triangle in our *model of reality* as far as the

[2] And also five common notions, which can be regarded as other axioms.

[3] The term "rules of correspondence" was first used in this sense by the philosopher Rudolf Carnap (1891–1970) in his concept of a scientific theory as an axiomatic formal system. In the case under discussion, strictly speaking, the definitions tell us which are the "characters of the game." Their correspondence with entities of the real world, however, is implicit in its use. For Euclid's *Elements* these regard the constructions that could be done with rulers and compasses, whereas in other works using these geometric theories as its fundamental tool this "mapping" could concern different objects.

solution of this problem is concerned. We are then allowed to say that the results of the theorems, which "live" in our model, correspond to a correct result in the real world.

We can also sketch a more elaborate picture. Let us imagine that initially a scientist starts with these "rules" (i.e., the "mapping" between the reality and our model, and the axioms) and wants to understand how accurately this model works. Then one could start off by deducing the consequences from the hypotheses, i.e., the theorems, and apply them back into reality using the rules of correspondence backwards. Sooner or later our imaginary scientist will run across the problem of the obelisk, which will represent a twofold step:

- If by direct measurement the *predictions* of our model are shown true, this constitutes an *experimental verification* that our model is a correct representation of that part of the reality covered by the rules of correspondence.
- On the other hand, the theorem will have given us access to a *new application/technology* that was previously unknown.

It is worth stressing that the last finding was made possible by the simplifications imposed in our model by the rules of correspondence. It would probably have been much harder, when not infeasible, to get the same results if one had to take into account, e.g., the color of the obelisk, its material, the day it was built, or also simply its actual shape. In other words, *simplification* and *schematization* play important roles in the definition of a scientific theory by filtering out of the model all the characteristics deemed in essential to its goals.

Although oversimplified, this schema well represents what in practice happens every day in science. It can have a more evident connection with mathematics in works such as those of Archimedes or of Hipparchus, where the rules of correspondence of the geometric entities refer to physical or astronomical objects, but the same principles apply, for example, to "softer" sciences such as biology, although more loosely.

2.2 Geometry and Physics: Tools for Modeling the Reality

Just the fact that Euclidean geometry can be regarded as the first historical example of a scientific theory, and that it is at the base of many ancient examples of physics theories, would be enough to show the argued connection between geometry and physics. But actually that is stronger than this.

Reference system and physical space

The concept of *reference system* is ubiquitous in physics, as well as that of geometrical objects including vectors, tensors, and the like that are directly connected with the former and constitute the building blocks of the equations of physics. In the framework depicted in the previous section, our rules of correspondence are set in such a way that the reference system represents the *model of physical space*, and therefore the geometrical objects "living" in it have to be matched with appropriate physical objects.

Choosing the right geometry

However, as we have stressed above, one geometry is defined by its postulates, examples of which are represented by the five axioms of Euclidean geometry, and in principle we have no limitations in choosing them. The choice, indeed, is sometimes believed to be governed by their alleged "self-evident truth," however:

1. The mere fact that we require them to correspond to some kind of truth implies that we are making a comparison with our perception of reality in the physical world, which means that geometry takes its origin from the need of modeling it.
2. Such so-called "self-evidence" can be just apparent, as in the famous case of Euclid's fifth postulate.

The history of this axiom represents an enlightening example of the difficulty of identifying the "right" set of postulates. First of all, a fundamental characteristic of an axiom is that it must be independent of the others, namely that it cannot be derived from other assumptions. It is well known that for at least 10 centuries many brilliant scientists unsuccessfully tried to demonstrate that the fifth postulate was not independent of the previous four, and these attempts continued until in the nineteenth century Beltrami (1868) proved that it was indeed the case.

Such proof came together with the discovery that other equally self-consistent geometries could be generated by taking another version of this postulate, which raises another strictly related problem: if these geometries are equivalent to each other from the point of view of their internal consistency, why should we prefer one or another as the correct model of the physical world? One might be tempted to say that Euclidean geometry is self-evident because every day it shows its adherence to reality. This, however, would just show once again the tight connection between geometry and physics, and moreover that the way we built it up was experience-driven.

We return to this concept in the next chapter, but now we want to remark that adherence to reality is what we ask of a physical theory, not geometry as an abstract mathematical construction. Thus physics and geometry are mutually interrelated because the goal of physics is that of finding the "governing principles" of the physical world, whereas geometry provides a way to translate them in a mathematical language through its rules of correspondence.[4]

[4] It has to be stressed that this connection must not be interpreted as a one-to-one identification between physics and geometry. For example, a particle in classical physics can correspond to a geometric point, but in a gravity theory this model includes a property called mass, and in electrodynamics we have an electric charge.

It can thus happen that the selection of specific physics principles implies the selection of a specific geometry which therefore represents the "best fit" for the mathematical description of that theory.

This process is based on a requirement called the "principle of covariance," which we begin to explore in detail in the next chapter.

2.3 When Should a Scientific Theory Be Changed?

The possibility of having different physics theories even at the geometric level further enhances the need to understand the criteria that can help the selection of the best model. We can summarize them in the following list.

1. Comparison with experimental results
2. Compatibility issues between different theories
3. Unsolved "philosophical" or self-consistency issues

Actually one should not be deceived by this rigid classification. All these events can appear intermingled, which indeed is what commonly happens in the real world.

The first point of the list should be the most obvious for us inasmuch as it has already been mentioned in Sect. 2.1. A scientific theory can make predictions that can be seen as: (a) deductions of a chain of consequences starting from its postulates (theorems); (b) the translation of such theorems in the physical world by means of its rules of correspondence. These predictions are in principle subject to direct verification that can support or disprove the theory.

A typical example of the second point is the general relativity versus quantum physics issue, and it is also well known.

Finally, an easy example for the third case is Newtonian gravity theory, with its implication of an "action at distance" (see Sect. 4.1) which was "philosophically" difficult to accept even for Newton himself and moreover had a further problem of compatibility with classical electrodynamics, which instead required an interaction propagating at finite speed. Other less known examples are special-relativistic theories of gravity and their predictions for particles traveling at the speed of light.[5] Notably, it is possible to have theories accepting the existence of particles moving at the speed of light and getting deflected by a mass, but only if they are massive particles, whereas special relativity requires that only massless particles can travel at the speed of light, but they are not deflected by special-relativistic gravity.

In this book we find and explore in more detail examples of each of these cases.

[5] There are many such theories, and their predictions for these particles can vary considerably.

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