Chapter 2
One-Dimensional Singular Perturbed Problems

Abstract In this chapter, we introduce a singular perturbed problem and a numerical difficulty associate with its discretization. We first consider the one-dimensional advective dominated advection-diffusion problem, both in terms of numerical solutions and its asymptotic expansion. We then consider a more general asymptotic expansion, including a reaction term in the equation and considering the situation when the coefficients might depend on x as well.

2.1 Introduction

One of the most important techniques in asymptotic analysis is the *matching asymptotic expansion* [62, 133, 145, 186, 195], important to understand how certain ODEs or PDEs depends on one or more parameters. Sometimes it is possible to express the solution as a *regular (or outer) expansion* that satisfies the equation, but maybe not some or all boundary conditions. In the simplest case, the expansion is a formal infinite power series in terms of the equation parameters. It is formal in the sense that we do not require the series to converge. As such expansion is being obtained, other terms are added to it to force the series to satisfy the boundary conditions. Those are the *boundary correctors* and constitute the *inner expansion*.

In what follows we work the details of these ideas considering simple examples in one-dimension, to avoid many technicalities. The differential equations have a second order (diffusion) term, and lower order terms: first order (advection), and possibly zeroth order (reaction). In the examples, the diffusion term is multiplied by a small positive scalar ε.

A main application of these problems is to understand fluids with low viscosity, in particular the Navier-Stokes system of equations (we do not consider this here). Numerically, it is not easy to design computational methods that perform well for all $\varepsilon > 0$, and equations like we consider below (specially in higher dimensions) are a perfect playground for those who want to develop and analyze robust numerical schemes [42–44, 58, 98, 100–105, 110, 121, 124, 139, 143, 146, 186].

In the next section, we consider a simple advection-diffusion equation and its finite element approximation. We then develop an asymptotic expansion of the exact solution. In Sect. 2.3 we consider a more general equation that includes non-constant advection and reaction terms, before concluding the chapter with some comments.

A.L. Madureira, *Numerical Methods and Analysis of Multiscale Problems*, SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-50866-5_2

2.2 Advection-Diffusion with Constant Coefficients

We consider a very simple case of singular perturbed problem as well as its finite element discretization in Sect. 2.2.1. Then, in Sect. 2.2.2 we analyze what goes wrong with the classical discretization. Finally, in Sect. 2.2.3 we develop the asymptotic expansion of the solution.

2.2.1 The Problem and Its Finite Element Discretization

Consider the following boundary value problem:

$$\begin{aligned} -\varepsilon \frac{d^2 u^\epsilon}{dx^2} + \frac{du^\epsilon}{dx} &= 0, \\ u^\epsilon(0) = 1, \quad u^\epsilon(1) &= 0, \end{aligned} \tag{2.1}$$

where ε is a positive real number. It is convenient to assume that $\varepsilon \leq 1$. The exact solution is simply

$$u^\epsilon(x) = 1 - \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}. \tag{2.2}$$

The function plots for $\varepsilon = 1$, $\varepsilon = 0.1$, and $\varepsilon = 0.01$ follow in Fig. 2.1. It is clear that when ε approaches zero, there is the onset of a boundary layer close to $x = 1$. This is also highlighted by the following fact:

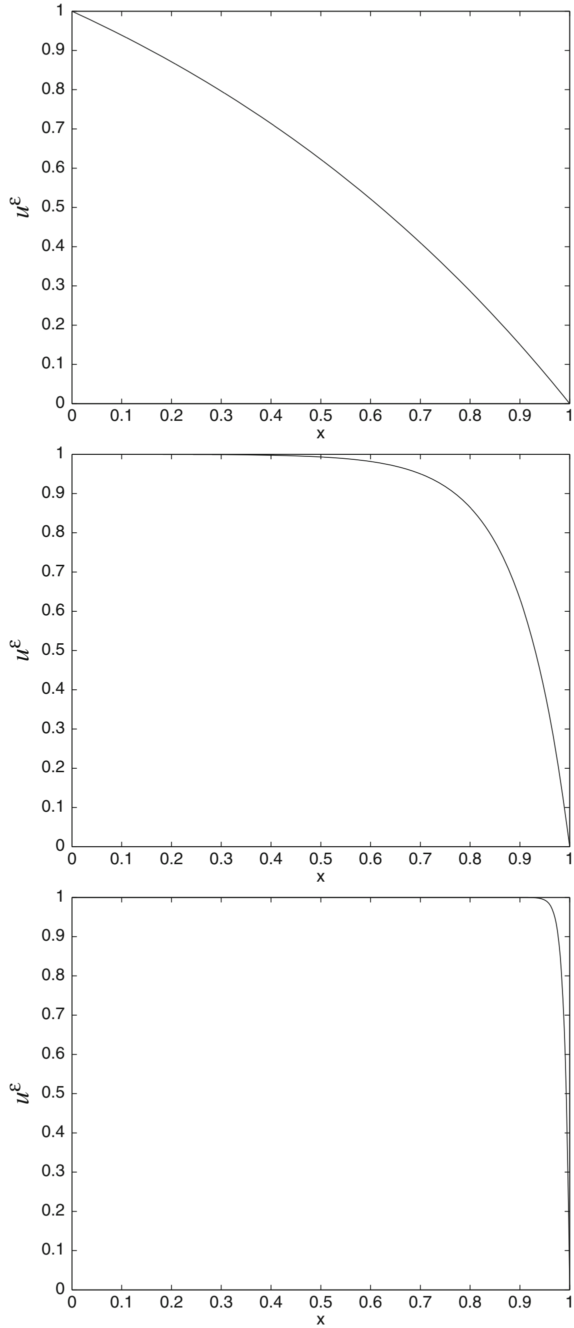
$$1 = \lim_{\varepsilon \to 0} \lim_{\substack{x \to 1 \\ x < 1}} u^\epsilon(x) \neq \lim_{\substack{x \to 1 \\ x < 1}} \lim_{\varepsilon \to 0} u^\epsilon(x) = 0.$$

The classification *singular perturbed* just means that we cannot impose $\varepsilon = 0$ in (2.1) and hope that $\lim_{\varepsilon \to 0} u^\epsilon$ would solve the resulting equation. Indeed, if $\varepsilon = 0$, then it follows from (2.1) a first order equation with two boundary conditions, and such problem does not have solutions in general. The limiting equation imposes that u^ϵ is a constant, but from the boundary conditions such constant has to be equal to one at $x = 0$, and to zero at $x = 1$, a clear impossibility. Note that the limit of $u^\epsilon(x)$ is the discontinuous function

$$\lim_{\varepsilon \to 0} u^\epsilon(x) = \begin{cases} 1 & \text{in } [0, 1), \\ 0 & \text{at } x = 1. \end{cases}$$

So, the pointwise limit of the exact solution satisfies the exact equation for $\varepsilon = 0$, and one of the boundary conditions. But not both.

Fig. 2.1 Exact solutions (2.2) of the singular perturbed problem (2.1) for $\varepsilon = 1, 0.1, 0.01$. Note the onset of a boundary layer close to $x = 1$ as the value of ε decreases



Remark 2.1 System (2.1) yields the simplest equations that capture essential features of fluid flow problems, where ε represents the fluid viscosity, the second derivative term models the diffusion, and the first derivative term models the advection. A reaction, “zeroth order term”, could be added as well, see Sect. 2.3 and Chap. 4. In general, at the limit case, the equation change type, from elliptic to hyperbolic. The inflow boundary conditions are preserved, but the outflow conditions are lost.

Let us proceed with a straightforward Galerkin discretization of (2.1) using finite element method. We first rewrite (2.1) in a weak form, i.e, the exact solution

$$u^\epsilon \in V = \{v \in H^1(0,1) \,:\, v(0) = 1 \text{ and } v(1) = 0\},$$

satisfies

$$a(u^\epsilon, v) = \varepsilon \int_0^1 \frac{du^\epsilon}{dx}\frac{dv}{dx}\,dx + \int_0^1 \frac{du^\epsilon}{dx} v\,dx = 0 \quad \text{for all } v \in H_0^1(0,1). \tag{2.3}$$

Consider a discretization as described in (1.11), and let

$$V_h = \{v_h \in V \,:\, v_h \text{ is linear in } (x_{j-1}, x_j) \text{ for } j = 1, \dots, N+1\},$$
$$V_0^h = \{v_h \in H_0^1(0,1) \,:\, v_h \text{ is piecewise linear}\}.$$

The finite element approximation to u^ϵ is $u_h \in V_h$ such that

$$a(u_h, v) = 0 \quad \text{for all } v \in V_0^h. \tag{2.4}$$

Remark 2.2 Note that u_h depends on ε, although this is not explicitly indicated in the notation.

As depicted in Fig. 2.2, for a uniform mesh with $h = 1/16$ and $\varepsilon = 1$ the exact (2.2) and finite element (2.4) solutions seem quite close. However, for $\varepsilon = 0.01$ the finite element approximation presents spurious oscillations that are more prominent close to the boundary layer. Reducing the mesh to $h = 1/32$ improves the approximation, but the oscillations are still present. Refining the mesh even further would eventually result in an accurate approximation.

2.2.2 So, What Goes Wrong?

To better understand, or, at least, have a feeling of what goes wrong, we develop an error analysis for this problem.

We first investigate the continuity of the bilinear form $a(\cdot,\cdot)$. In fact, it follows from its definition and Cauchy–Schwartz inequality that

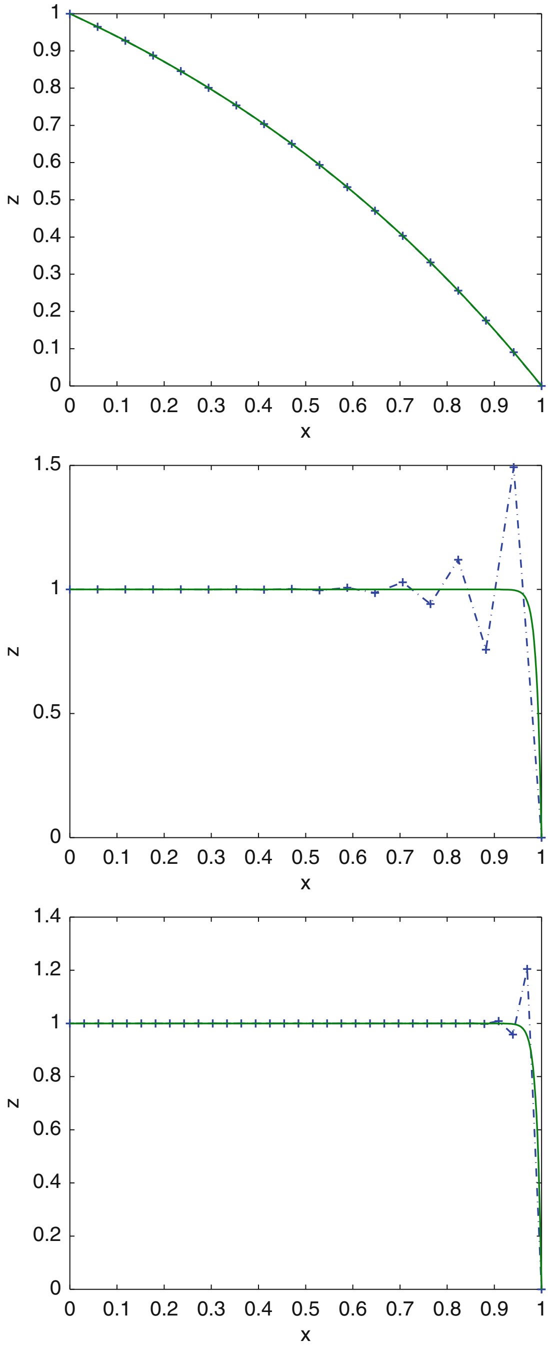


Fig. 2.2 For a uniform mesh with $h = 1/16$ we plot exact (*solid line*) and approximate (*dash-dot line*) solutions of (2.1) for $\varepsilon = 1$ (*top*) and $\varepsilon = 0.01$ (*center*). Note that the numerical solution is highly oscillatory close to the boundary layer at $x = 1$. For the same value of ε $(= 0.01)$ a more refined mesh $(h = 1/32)$ yields a numerical solution that is less oscillatory (*bottom*), but still unsatisfactory

$$a(u, v) \le c\|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)} \quad \text{for all } u, v \in H_0^1(0, 1). \tag{2.5}$$

The difficulty starts when trying to derive the coercivity estimate:

$$\begin{aligned} a(v, v) &= \varepsilon \int_0^1 \left(\frac{dv}{dx}\right)^2 dx + \int_0^1 \frac{dv}{dx} v \, dx \\ &= \varepsilon \int_0^1 \left(\frac{dv}{dx}\right)^2 dx \ge C\varepsilon\|v\|^2_{H^1(0,1)} \quad \text{for all } v \in H_0^1(0, 1), \end{aligned} \tag{2.6}$$

since integration by parts yields $\int_0^1 (dv/dx)v \, dx = 0$, for $v \in H_0^1(0, 1)$. We also used Poincaré's inequality (Lemma 1.1) at the last step. So, estimate (1.16) yields

$$\|u^\epsilon - u_h\|_{H^1(0,1)} \le C\varepsilon^{-1} h |u^\epsilon|_{H^2(0,1)}. \tag{2.7}$$

We stop now to try interpret the error estimate we just obtained. First of all, *there is convergence in h*. Indeed, for a fixed ε, the error goes to zero as the mesh size goes to zero. The problem is that the convergence in h is not uniform in ε. Hence, for ε small, unless the mesh size is also very small, the H^1 norm error estimate becomes large. The estimate is even worse than one can think at first glance, since $|u^\epsilon|_{H^2(0,1)} = O(\varepsilon^{-3/2})$. This makes (2.7) and the traditional Galerkin method almost useless.

Another way to look at this problem is by first noticing that we would like to have

$$\lim_{\varepsilon \to 0} u_h = \lim_{\varepsilon \to 0} u^\epsilon = 1.$$

After all, it would be just perfect to have a method that converges (with ε) to the correct solution *for a fixed mesh*. This is not happening. Indeed, looking at the matrix problem coming from (2.4), it is matter of computation to show that [146]

$$-\frac{\varepsilon}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{1}{2h}(u_{j+1} - u_{j-1}) = 0, \quad u_0 = 1, \; u_{N+1} = 0, \tag{2.8}$$

where $u_j = u_h(x_j)$. Assume N even. At the $\varepsilon = 0$ limit, $u_{j+1} = u_{j-1}$. This and the boundary conditions originate the oscillatory behavior of the approximate solution. See Fig. 2.3.

Remark 2.3 Note that although we used a finite element scheme to derive (2.8), this scheme is also a finite difference scheme which uses a central difference approximation for the convective term du/dx. The more naive finite difference approximation

$$-\frac{\varepsilon}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{1}{h}(u_j - u_{j-1}) = 0, \quad u_0 = 1, \; u_{N+1} = 0, \tag{2.9}$$

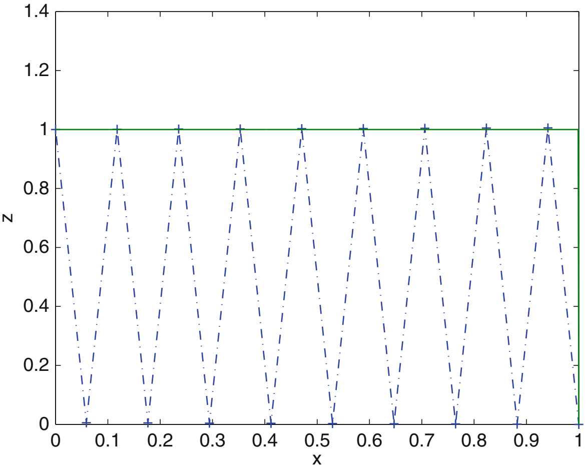


Fig. 2.3 Exact (*solid line*) and numerical (*dash-dot line*) solutions defined by (2.2) and (2.3) for $\varepsilon = 10^{-5}$ and $N = 16$. Note that the approximate solution has nodal values u_j close to zero for j odd and close to one for j even

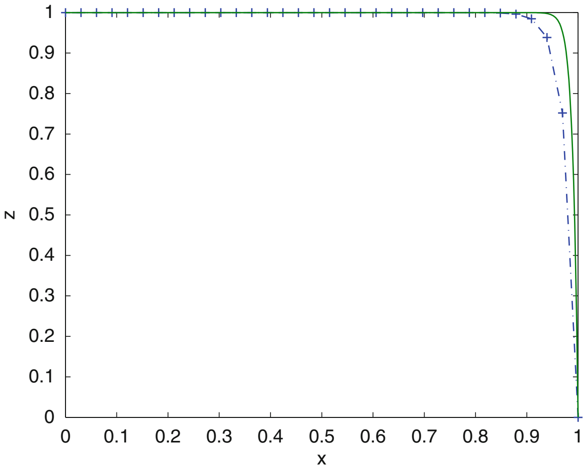


Fig. 2.4 Finite difference approximation (*dash-dot line*) with first order discretization given by (2.9) for the convection term, for $\varepsilon = 0.01$ and $h = 1/32$. Note the absence of oscillations

yields, however, a better result. See Fig. 2.4. In fact, for this scheme, $u_j = u_{j-1}$, as ε goes to zero. Since $u_0 = 1$, it holds that $u_j = 1$ in the $\varepsilon \to 0$ limit:

$$\lim_{\varepsilon \to 0} u_h(x_j) = \lim_{\varepsilon \to 0} u^\epsilon(x_j) = 1, \quad \text{for } j = 1, \ldots, N.$$

2.2.3 *Matching Asymptotic Expansions*

The behavior we described above is typical in singular perturbed PDEs, where the onset of boundary layers is a common phenomenon. But this is not all that can happen. For instance, in plate models, in particular for the Reissner–Mindlin equation, as the plate thickness goes to zero (that is, the small parameter in this case), numerical "locking" occurs, i.e., if a careless method is used, the computed solution goes to zero (a wrong limit) [93].

Several numerical methods try to somehow overcome these and other difficulties related to asymptotic limits. Some methods perform well for a certain asymptotic range, for instance by assuming $\varepsilon \ll 1$. Some other methods try to be performing for a broader range of parameters. See, for instance, [44, 101, 139, 186].

Looking at these difficulties (and their corresponding solutions!) it becomes more clear that it is important to have a full understanding of the solution's behavior. This is useful not only to help designing new numerical methods, but also to analyze and estimate old ones. A valuable analysis tool is the method of *matching asymptotics*, where the exact solution for a given singular perturbed PDE is expressed in terms of a *formal* power series with respect to a small parameter. We call it formal since we are not concerned with convergence at this point, and so we assume that all manipulations are valid from the mathematical point of view. We shall explain how an asymptotic expansion can be developed by looking at a simple example.

Consider problem (2.1), and the formal asymptotic series

$$u^\epsilon \sim u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots , \tag{2.10}$$

and formally substitute it in (2.1). Then

$$\frac{du^0}{dx} + \varepsilon\left(-\frac{d^2u^0}{dx^2} + \frac{du^1}{dx}\right) + \cdots + \varepsilon^i\left(-\frac{d^2u^{i-1}}{dx^2} + \frac{du^i}{dx}\right) + \cdots = 0.$$

By comparing the different powers of ε, it is natural to require that

$$\frac{du^0}{dx} = 0, \qquad \frac{du^1}{dx} = \frac{d^2u^0}{dx^2}, \qquad \ldots, \qquad \frac{du^i}{dx} = \frac{d^2u^{i-1}}{dx^2}, \qquad \ldots. \tag{2.11}$$

Then all functions u^i are constants.

From the boundary conditions in (2.1), it would be natural to impose $u^0(0) = 1$, $u^i(0) = 0$ for $i \geq 1$, and $u^i(1) = 0$ for all i. However, the equations in (2.11) are of first order, and only one boundary condition is to be imposed. We *choose* the inflow boundary condition at $x = 0$, and set then

$$u^0(0) = 1, \qquad u^i(0) = 0 \quad \text{for } i > 0.$$

Since u^i are constants, we conclude that $u^0 = 1$ and $u^i = 0$ for $i \geq 0$. Thus the formal series (2.10) simplifies to

$$u^\epsilon \sim 1. \tag{2.12}$$

Of course such expansion does not satisfy the boundary condition at $x = 1$, although it seems to deliver a good approximation at the interior of the domain. We correct the boundary discrepancy by introducing the *boundary corrector* U. We would like to have

$$-\varepsilon\frac{d^2U}{dx^2} + \frac{dU}{dx} = 0, \quad U(0) = 0, \quad U(1) = -1.$$

Note that if we make the change of coordinates $\hat{\rho} = (1-x)/\varepsilon$, and set $\hat{U}(\hat{\rho}) = U(1-\varepsilon\hat{\rho})$, then

$$-\frac{d^2\hat{U}}{d\hat{\rho}^2}(\hat{\rho}) - \frac{d\hat{U}}{d\hat{\rho}}(\hat{\rho}) = 0, \qquad \hat{U}(0) = 1.$$

Noting that we need another boundary condition for $\hat{U}$, we try to ensure a "local behavior" by imposing

$$\lim_{\hat{\rho}\to\infty} \hat{U}(\hat{\rho}) = 0.$$

Hence, $\hat{U}(\hat{\rho}) = -e^{-\hat{\rho}}$ and

$$U(x) = -e^{(x-1)/\varepsilon} \tag{2.13}$$

The asymptotic expansion then becomes

$$u^{\epsilon}(x) \sim u^0 + \hat{U}(\hat{\rho}) = 1 - e^{(x-1)/\varepsilon}. \tag{2.14}$$

Although this is a very simple problem, some characteristics of asymptotic expansion are present here. First, the *outer expansion* (2.12) satisfies the operator, but fails to satisfy the boundary conditions. Then an *inner expansion* (2.13) has to be added. The terms of the inner expansion depend on ε through a change of coordinates, and become exponentially small in the interior of the domain. Finally, note that the expansion does not give the exact solution—indeed it fails to satisfy the boundary condition at $x = 0$ (an exponentially small miss). The idea is that, as $\varepsilon \to 0$, the expansion approximates the solution.

Remark 2.4 Here we could have multiplied U by a smooth cut-off function that values one in the vicinity of $x = 1$ (say, in $(1/2, 1)$), and zero at $x = 0$. Then the asymptotic expansion would satisfy all boundary conditions, at the expense of no longer satisfying the operator exactly. That would only introduce an exponential small error with respect to $1/\varepsilon$. For higher dimensional problems, the introduction of the cut-off function is "mandatory," as the boundary correctors are defined only in a neighborhood of the boundary, and not in the whole domain as here. See Chap. 4.

For the sake of comparison, the exact solution is

$$\begin{aligned} u^\epsilon(x) &= 1 - \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1} = 1 - \frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \\ &= 1 - (e^{(x-1)/\varepsilon} - e^{-1/\varepsilon})(1 + e^{-1/\varepsilon} + e^{-2/\varepsilon} + e^{-3/\varepsilon} + \cdots) \\ &= 1 - (e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}) - s(e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}) = 1 - e^{(x-1)/\varepsilon} + r, \end{aligned}$$

where

$$s = \frac{e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}, \qquad r = e^{-1/\varepsilon}\left(1 - \frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right).$$

Thus, the difference between the exact solution and the asymptotic expansion (2.14) is r, and, for all $x \in (0,1)$, $|r| \le e^{-1/\varepsilon}$ is exponentially small with respect to $1/\varepsilon$.

2.3 A More General Singular Perturbed Second Order ODE

The asymptotic expansion of Sect. 2.2 is remarkably simple, but that does not represent the general case. Consider the differential operator

$$\mathscr{L}^\epsilon u^\epsilon = -\varepsilon \frac{d^2 u^\epsilon}{dx^2} + \beta(x)\frac{du^\epsilon}{dx} + \sigma(x) u^\epsilon,$$

and the problem

$$\mathscr{L}^\epsilon u^\epsilon = f \quad \text{in } (0,1), \tag{2.15}$$

$$u^\epsilon(0) = u^\epsilon(1) = 0. \tag{2.16}$$

We assume that $\varepsilon > 0$, that β, σ, and f are smooth functions, and that both $\sigma + \beta/2 > 0$ and $\beta > 0$ are positive.

In Sect. 2.3.1 we shall develop asymptotic expansion for u^ϵ, and in Sect. 2.3.2 we show error estimates.

2.3.1 Asymptotic Expansion

Consider the asymptotic series

$$u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$$

and formally substitute it in (2.15). Then

$$\beta(x)\frac{du^0}{dx}+\sigma(x)u^0+\varepsilon\left(-\frac{d^2u^0}{dx^2}+\beta(x)\frac{du^1}{dx}+\sigma(x)u^1\right)+\cdots$$
$$+\varepsilon^i\left(-\frac{d^2u^{i-1}}{dx^2}+\beta(x)\frac{du^i}{dx}+\sigma(x)u^i\right)+\cdots=f.$$

Grouping the different powers of ε, we set

$$\mathscr{L}^0\, u^0=f,\qquad \mathscr{L}^0\, u^1=\frac{d^2u^0}{dx^2},\qquad \ldots,\qquad \mathscr{L}^0\, u^i=\frac{d^2u^{i-1}}{dx^2},\qquad \ldots, \tag{2.17}$$

where $\mathscr{L}^0\, v=\beta(x)dv/dx+\sigma(x)v$.

Since the equations in (2.17) are of first order, it is not possible to impose $u^i(0)=u^i(1)=0$, conforming (2.16). We then set

$$u^i(0)=0.$$

We correct the discrepancy at $x=1$ by introducing the boundary corrector U. We would like to have

$$\mathscr{L}^\epsilon\, U=0,\quad U(0)=0,\quad U(1)=u^0(1)+\varepsilon u^1(1)+\varepsilon^2u^2(1)+\cdots$$

From the change of coordinates $\hat\rho=\varepsilon^{-1}(1-x)$, and making $\hat U(\hat\rho)=U(1-\varepsilon\hat\rho)$, it follows that

$$-\frac{d^2\hat U}{d\hat\rho^2}(\hat\rho)-\beta(1-\varepsilon\hat\rho)\frac{d\hat U}{d\hat\rho}(\hat\rho)+\varepsilon\sigma(1-\varepsilon\hat\rho)\hat U(\hat\rho)=0,$$
$$\hat U(0)=u^0(1)+\varepsilon u^1(1)+\varepsilon^2u^2(1)+\cdots$$

Going one step further, we develop the Taylor expansions

$$\beta(1-\varepsilon\hat\rho)=\beta(1)-\varepsilon\hat\rho\frac{db}{dx}(1)+\frac{\varepsilon^2\hat\rho^2}{2}\frac{d^2b}{dx^2}(1)-\cdots,$$
$$\sigma(1-\varepsilon\hat\rho)=\sigma(1)-\varepsilon\hat\rho\frac{dc}{dx}(1)+\frac{\varepsilon^2\hat\rho^2}{2}\frac{d^2c}{dx^2}(1)-\cdots.$$

Finally, assuming the asymptotic expansion

$$\hat U\sim\hat U^0+\varepsilon\hat U^1+\varepsilon^2\hat U^2+\cdots,$$

we gather that

$$-\frac{d^2\hat U^0}{d\hat\rho^2}-\beta(1)\frac{d\hat U^0}{d\hat\rho}=0,$$
$$U^0(0)=u^0(1).$$

Noting that we need another boundary condition for $\hat{U}^0$, we try to ensure a "local behavior" by imposing

$$\lim_{\hat{\rho}\to\infty} \hat{U}^0(\hat{\rho}) = 0.$$

Hence, $\hat{U}^0(\hat{\rho}) = u(1)\exp\big(-\beta(1)\hat{\rho}\big)$.

Similarly,

$$-\frac{d^2\hat{U}^1}{d\hat{\rho}^2} - \beta(1)\frac{d\hat{U}^1}{d\hat{\rho}} = -\hat{\rho}\frac{d\beta}{dx}(1)\frac{d\hat{U}^0}{d\hat{\rho}} - \sigma(1)\hat{U}^0,$$
$$\hat{U}^1(0) = u^1(1), \qquad \lim_{\hat{\rho}\to\infty} \hat{U}^1(\hat{\rho}) = 0,$$

etc. It is possible to show that for every positive integer i there exist ε-independent positive constants c and α such that

$$\frac{d\hat{U}^i}{d\hat{\rho}}(\hat{\rho}) + \hat{U}^i(\hat{\rho}) \leq c\exp(-\alpha\hat{\rho}). \tag{2.18}$$

So, putting everything together, we have that

$$\begin{aligned} u^\epsilon(x) &\sim u^0(x) + \varepsilon u^1(x) + \varepsilon^2 u^2(x) + \cdots \\ &\quad - \hat{U}^0(\varepsilon^{-1}(1-x)) - \varepsilon\hat{U}^1(\varepsilon^{-1}(1-x)) - \varepsilon^2\hat{U}^2(\varepsilon^{-1}(1-x)) - \cdots . \end{aligned} \tag{2.19}$$

By construction, the above infinite power series *formally* solves the ODE (2.15). We did not make any comment regarding convergence of the above expansion. Actually, what we will prove is that a *truncated expansion* approximates well the exact solution.

Remark 2.5 Note that each term u^i in the series (2.19) is independent of ε, and each boundary corrector terms U^i depends on ε but only up to a change of coordinates. Also, U^i does not satisfy the boundary condition at $x = 0$, but this error is exponentially small. As pointed out in Remark 2.4, it is possible to correct this by introducing a smooth cut-off function; see also Remark 2.9.

2.3.2 Truncation Error Analysis

We start by developing here an analysis quite similar to that of Sect. 2.2.2. To simplify the computations, we assume here that the functions β and $\sigma \geq 0$ are actually positive constants—otherwise it would be necessary to take into account the effect of replacing them by their truncated Taylor expansions.

We first obtain stability estimates.

Lemma 2.1 *If* $\mathscr{L}^\epsilon\, v = g$ *weakly, and* $v \in H_0^1(0,1)$*, then*

$$\|v\|_{H^1(0,1)} \le c\varepsilon^{-1}\|g\|_{H^{-1}(0,1)}.$$

Proof From (2.6), we conclude that

$$\|v\|^2_{H^1(0,1)} \le c\varepsilon^{-1}a(v,v) = c\varepsilon^{-1}(g,v) \le c\varepsilon^{-1}\|g\|_{H^{-1}(0,1)}\|v\|_{H^1(0,1)}.$$

□

Corollary 2.1 *If* $w \in H^1(0,1)$ *is the weak solution of*

$$\mathscr{L}^\epsilon\, w = g, \qquad w(0) = w_0, \quad w(1) = w_1,$$

then

$$\|w\|_{H^1(0,1)} \le c\varepsilon^{-1}\big(\|g\|_{H^{-1}(0,1)} + |w_0| + |w_1|\big).$$

Proof Take any function $w_{bc} \in H^1(0,1)$ be such that $w_{bc}(0) = w_0$, and $w_{bc} = w_1$, with $\|w_{bc}\|_{H^1(0,1)} \le c\big(|w_0| + |w_1|\big)$. Then Lemma 2.1 with $v = w - w_{bc}$ yields the result since $\|w\|_{H^1(0,1)} \le \|w - w_{bc}\|_{H^1(0,1)} + \|w_{bc}\|_{H^1(0,1)}$. □

Lemma 2.2 *Let* $\mathscr{L}^0\, v = g$ *in* $(0,1)$*. Then there exists a constant* c *depending on* β*,* σ*, and* g *such that*

$$\|\frac{d^2v}{dx^2}\|_{L^\infty(0,1)} + \|\frac{dv}{dx}\|_{L^\infty(0,1)} + \|v\|_{L^\infty(0,1)} \le c.$$

Proof Note that since β is always positive, v solves $dv/dx + pv = \bar{g}$ with $p = \sigma/\beta$ and $\bar{g} = g/\beta$. Then [33]

$$u(x) = \frac{\int_0^x \mu(s)\bar{g}(s)\,ds}{\mu(x)}, \qquad \mu(x) = \exp\int_0^x p(s)\,ds.$$

Thus, $\|\mu\|_{L^\infty(0,1)} \le \exp\|p\|_{L^\infty(0,1)}$ and $|\mu(x)| \ge \exp(-\|p\|_{L^\infty(0,1)})$ for all $x \in (0,1)$. So, $\|v\|_{L^\infty(0,1)} \le \exp(2\|p\|_{L^\infty(0,1)})\|\bar{g}\|_{L^\infty(0,1)}$. The bound for dv/dx follows from $\|dv/dx\|_{L^\infty(0,1)} = \|\bar{g} - pv\|_{L^\infty(0,1)}$. The bound for $\|d^2v/dx^2\|_{L^\infty(0,1)}$ follows from the previous arguments, since $w = dv/dx$ solves $dw/dx + pw = \bar{g} - (dp/dx)v$. □

From (2.17) and the above lemma, we find that for all $i \in \mathbb{N}$, there exists a constant c (depending on i, f, β, and σ) such that

$$\left\|\frac{d^2u_i}{dx^2}\right\|_{L^\infty(0,1)} + \left\|\frac{du_i}{dx}\right\|_{L^\infty(0,1)} + \|u_i\|_{L^\infty(0,1)} \le c. \tag{2.20}$$

With the aid of Corollary 2.1, we are ready to estimate how well the asymptotic expansion approximates the exact solution of (2.15), (2.16). Let

$$e_N(x) = u^\epsilon(x) - \sum_{i=0}^{N} \varepsilon^i u^i(x) + \sum_{i=0}^{N} \varepsilon^i \hat{U}^i(\varepsilon^{-1}(1-x)), \tag{2.21}$$

From its construction, $e_N \in H^1(0,1)$, and

$$\mathscr{L}^\epsilon e_N = \varepsilon^{N+1} \frac{du^i}{dx^2}, \qquad e_N(0) = \sum_{i=0}^{N} \varepsilon^i \hat{U}^i(\varepsilon^{-1}), \quad e_N(1) = 0. \tag{2.22}$$

Using now Corollary 2.1, Eqs. (2.20), (2.22), and (2.18), we gather that there exists a constant c such that

$$\|e_N\|_{H^1(0,1)} \le c\varepsilon^N.$$

This estimate is not sharp. We improve it by adding and subtracting the $(N+1)$th term of the expansion:

$$\begin{aligned} \|e_N\|_{H^1(0,1)} &\le \|e_{N+1}\|_{H^1(0,1)} + \|e_N - e_{N+1}\|_{H^1(0,1)} \\ &\le c\big[\varepsilon^{N+1} + \varepsilon^{N+1}\|u^{N+1}\|_{H^1(0,1)} + \varepsilon^{N+1}\|\hat{U}^{N+1}\big(\varepsilon^{-1}(1-\cdot)\big)\|_{H^1(0,1)}\big] \\ &\le c\varepsilon^{N+1/2}. \end{aligned} \tag{2.23}$$

The last estimate was possible since

$$\begin{aligned} &\|\hat{U}^{N+1}\big(\varepsilon^{-1}(1-\cdot)\big)\|^2_{H^1(0,1)} \\ &\quad = \int_0^1 \left|\frac{d\hat{U}^{N+1}}{dx}\big(\varepsilon^{-1}(1-x)\big)\right|^2 + \left|\hat{U}^{N+1}\big(\varepsilon^{-1}(1-x)\big)\right|^2 dx \\ &\quad \le \varepsilon \int_0^\infty \varepsilon^{-2}\left|\frac{d\hat{U}^{N+1}}{dx}(\hat{\rho})\right|^2 + \left|\hat{U}^{N+1}(\hat{\rho})\right|^2 d\hat{\rho} \\ &\quad \le c\varepsilon \int_0^\infty \varepsilon^{-2} \exp(-2\alpha\hat{\rho})\, d\hat{\rho} \le c\varepsilon^{-1}. \end{aligned}$$

Estimates in other norms can be obtained in a similar fashion:

$$\|e_N\|_{L^2(0,1)} \le \|e_{N+1}\|_{H^1(0,1)} + \|e_N - e_{N+1}\|_{L^2(0,1)} \le c\varepsilon^{N+1}.$$

Remark 2.6 It is also possible to derive interior error estimates, by considering the interval $I_\delta = (0, 1-\delta)$, for $\delta \in (0, 1/2)$. Then compute $\|e_N\|_{H^1(I_\delta)}$ and $\|e_N\|_{L^2(I_\delta)}$, proceeding as in (2.23). We leave the details to the reader.

Interior estimates are important because global estimates in H^1 norms are dominated by boundary layer terms (high derivatives). Such terms, however, are exponentially small in the interior of the domain, allowing for better convergence rates away from the boundary. No improvements are obtained in L^2 norms.

We obtained then the following important result.

Theorem 2.1 *Let u^ϵ be the solution of the ODE (2.15), and let e_N be as in (2.21). Then, for every non-negative integer N, there exists a constant c such that*

$$\|e_N\|_{H^1(0,1)} \leq c\varepsilon^{N+1/2}. \qquad \|e_N\|_{L^2(0,1)} \leq c\varepsilon^{N+1}, \qquad \|e_N\|_{H^1(I_\delta)} \leq c_\delta \varepsilon^{N+1},$$

where I_δ is defined in Remark 2.6. The constant c might depend on N, f, β, and σ, but not on ε. The constant c_δ might also depend on δ.

Remark 2.7 Theorem 2.1 does not imply convergence of the power series as N goes to infinity, since the constants that appear in the right-hand side of the estimates depend on N. What the theorem provides is a convergence in ε, i.e., if ε is quite small, then the asymptotic expansion truncation error gets small as well.

Remark 2.8 The asymptotic *rule of the thumb* works: the error estimates present in Theorem 2.1 are of the same order as the terms left out of the truncated asymptotic expansion.

Remark 2.9 It is possible to derive an error estimate under the presence of a cut-off function χ using the following result.

Lemma 2.3 *For every non-negative integer i, there exist ε-independent positive constants C and α such that*

$$\sup_{x\in(0,1)} \mathscr{L}^\epsilon \left(\chi(x)U^i(\varepsilon^{-1}(1-x))\right) \leq C\exp(\alpha\varepsilon^{-1})$$

Proof It follows from the definition of U^i that

$$\begin{aligned}\mathscr{L}^\epsilon\left(\chi U^i\right) = \mathscr{L}^\epsilon\left(\chi U^i\right) - \chi\,\mathscr{L}^\epsilon\left(U^i\right) &= -\varepsilon\big(\chi U^i\big)'' + \beta(x)\big(\chi' U^i\big) + \varepsilon\chi\big(U^i\big)'' \\ &= -\varepsilon\chi'' U^i - 2\varepsilon\chi' U^{i\prime} + \beta(x)\big(\chi' U^i\big).\end{aligned}$$

The result follows from the definition of χ and estimate (2.18). $\square$

2.4 Conclusions

In general, it is not easy to design numerical methods schemes for singular perturbed PDEs. The layers that characterize such problems cause numerical instabilities that are hard to tame. Understanding such layers is then important, and there is no easier way to do so than considering one-dimensional cases.

Asymptotic expansions, even when they are formal, allow a clear description of the solution. It is also important to note that the derivation of the expansion might be formal, but its outcome can be justified rigorously, by estimating the difference between the exact solution and truncated expansions. Such estimates do not guarantee that the series converge. Instead, it tells us that the first few terms of the expansion yield a remarkable approximation to the exact solution. And that is enough for most applications.

Numerical Methods and Analysis of Multiscale Problems
Madureira, A.L.
2017, X, 123 p. 31 illus., 9 illus. in color., Softcover
ISBN: 978-3-319-50864-1