

# Contents

<b>1</b>	<b>Tensor Products</b> .....	1
1.1	Multilinear Maps .....	1
1.1.1	Multilinear Maps Between Free Modules .....	1
1.1.2	Universal Multilinear Map .....	3
1.2	Tensor Product of Modules .....	4
1.2.1	Existence of Tensor Product .....	5
1.2.2	Linear Maps as Tensors .....	7
1.2.3	Tensor Products of Abelian Groups .....	9
1.3	Commutativity, Associativity, and Distributivity Isomorphisms ...	10
1.4	Tensor Product of Linear Maps .....	13
1.5	Tensor Product of Modules Presented by Generators and Relations .....	15
	Problems for Independent Solution to Chapter 1 .....	17
<b>2</b>	<b>Tensor Algebras</b> .....	21
2.1	Free Associative Algebra of a Vector Space .....	21
2.2	Contractions .....	22
2.2.1	Complete Contraction .....	22
2.2.2	Partial Contractions .....	23
2.2.3	Linear Support and Rank of a Tensor .....	25
2.3	Quotient Algebras of a Tensor Algebra .....	26
2.3.1	Symmetric Algebra of a Vector Space .....	26
2.3.2	Symmetric Multilinear Maps .....	27
2.3.3	The Exterior Algebra of a Vector Space .....	29
2.3.4	Alternating Multilinear Maps .....	30
2.4	Symmetric and Alternating Tensors .....	31
2.4.1	Symmetrization and Alternation .....	32
2.4.2	Standard Bases .....	33
2.5	Polarization of Polynomials .....	35
2.5.1	Evaluation of Polynomials on Vectors .....	36
2.5.2	Combinatorial Formula for Complete Polarization .....	37

2.5.3	Duality .....	38
2.5.4	Derivative of a Polynomial Along a Vector .....	38
2.5.5	Polars and Tangents of Projective Hypersurfaces .....	40
2.5.6	Linear Support of a Homogeneous Polynomial .....	43
2.6	Polarization of Grassmannian Polynomials .....	45
2.6.1	Duality .....	45
2.6.2	Partial Derivatives in an Exterior Algebra .....	46
2.6.3	Linear Support of a Homogeneous Grassmannian Polynomial .....	47
2.6.4	Grassmannian Varieties and the Plücker Embedding .....	49
2.6.5	The Grassmannian as an Orbit Space .....	49
	Problems for Independent Solution to Chapter 2 .....	51
<b>3</b>	<b>Symmetric Functions</b> .....	<b>57</b>
3.1	Symmetric and Sign Alternating Polynomials .....	57
3.2	Elementary Symmetric Polynomials .....	60
3.3	Complete Symmetric Polynomials .....	61
3.4	Newton's Sums of Powers .....	62
3.4.1	Generating Function for the $p_k$ .....	62
3.4.2	Transition from $e_k$ and $h_k$ to $p_k$ .....	63
3.5	Giambelli's Formula .....	65
3.6	Pieri's Formula .....	67
3.7	The Ring of Symmetric Functions .....	69
	Problems for Independent Solution to Chapter 3 .....	71
<b>4</b>	<b>Calculus of Arrays, Tableaux, and Diagrams</b> .....	<b>75</b>
4.1	Arrays .....	75
4.1.1	Notation and Terminology .....	75
4.1.2	Vertical Operations .....	76
4.1.3	Commutation Lemma .....	77
4.2	Condensing .....	79
4.2.1	Condensed Arrays .....	79
4.2.2	Bidense Arrays and Young Diagrams .....	80
4.2.3	Young Tableaux .....	81
4.2.4	Yamanouchi Words .....	82
4.2.5	Fiber Product Theorem .....	83
4.3	Action of the Symmetric Group on DU-Sets .....	86
4.3.1	DU-Sets and DU-Orbits .....	86
4.3.2	Action of $S_m = \text{Aut}(J)$ .....	86
4.4	Combinatorial Schur Polynomials .....	88
4.5	The Littlewood–Richardson Rule .....	91
4.5.1	The Jacobi–Trudi Identity .....	93
4.5.2	Transition from $e_\lambda$ and $h_\lambda$ to $s_\lambda$ .....	93
4.6	The Inner Product on $\Lambda$ .....	95
	Problems for Independent Solution to Chapter 4 .....	96

- 5 Basic Notions of Representation Theory** ..... 99
  - 5.1 Representations of a Set of Operators ..... 99
    - 5.1.1 Associative Envelope ..... 99
    - 5.1.2 Decomposability and (Semi)/Simplicity ..... 100
    - 5.1.3 Homomorphisms of Representations ..... 103
  - 5.2 Representations of Associative Algebras ..... 104
    - 5.2.1 Double Centralizer Theorem ..... 104
    - 5.2.2 Digression: Modules Over Noncommutative Rings ..... 106
  - 5.3 Isotypic Components ..... 107
  - 5.4 Representations of Groups ..... 109
    - 5.4.1 Direct Sums and Tensor Constructions ..... 109
    - 5.4.2 Representations of Finite Abelian Groups ..... 111
    - 5.4.3 Reynolds Operator ..... 113
  - 5.5 Group Algebras ..... 114
    - 5.5.1 Center of a Group Algebra ..... 115
    - 5.5.2 Isotypic Decomposition of a Finite Group Algebra ..... 115
  - 5.6 Schur Representations of General Linear Groups ..... 121
    - 5.6.1 Action of  $GL(V) \times S_n$  on  $V^{\otimes n}$  ..... 122
    - 5.6.2 The Schur–Weyl Correspondence ..... 124
- Problems for Independent Solution to Chapter 5 ..... 124
- 6 Representations of Finite Groups in Greater Detail** ..... 131
  - 6.1 Orthogonal Decomposition of a Group Algebra ..... 131
    - 6.1.1 Invariant Scalar Product and Plancherel’s Formula ..... 131
    - 6.1.2 Irreducible Idempotents ..... 133
  - 6.2 Characters ..... 134
    - 6.2.1 Definition, Properties, and Examples of Computation ... 134
    - 6.2.2 The Fourier Transform ..... 137
    - 6.2.3 Ring of Representations ..... 140
  - 6.3 Induced and Coinduced Representations ..... 141
    - 6.3.1 Restricted and Induced Modules Over  
Associative Algebras ..... 141
    - 6.3.2 Induced Representations of Groups ..... 142
    - 6.3.3 The Structure of Induced Representations ..... 143
    - 6.3.4 Coinduced Representations ..... 146
- Problems for Independent Solution to Chapter 6 ..... 148
- 7 Representations of Symmetric Groups** ..... 151
  - 7.1 Action of  $S_n$  on Filled Young Diagrams ..... 151
    - 7.1.1 Row and Column Subgroups Associated with  
a Filling ..... 151
    - 7.1.2 Young Symmetrizers  $s_T = r_T \cdot c_T$  ..... 153
    - 7.1.3 Young Symmetrizers  $s'_T = c_T \cdot r_T$  ..... 155
  - 7.2 Modules of Tabloids ..... 157

7.3	Specht Modules .....	159
7.3.1	Description and Irreducibility .....	159
7.3.2	Standard Basis Numbered by Young Tableaux .....	160
7.4	Representation Ring of Symmetric Groups .....	161
7.4.1	Littlewood–Richardson Product .....	162
7.4.2	Scalar Product on $\mathfrak{R}$ .....	163
7.4.3	The Isometric Isomorphism $\mathfrak{R} \cong \Lambda$ .....	164
7.4.4	Dimensions of Irreducible Representations .....	168
	Problems for Independent Solution to Chapter 7 .....	170
<b>8</b>	<b><math>\mathfrak{sl}_2</math>-Modules</b> .....	173
8.1	Lie Algebras .....	173
8.1.1	Universal Enveloping Algebra .....	173
8.1.2	Representations of Lie Algebras .....	174
8.2	Finite-Dimensional Simple $\mathfrak{sl}_2$ -Modules .....	176
8.3	Semisimplicity of Finite-Dimensional $\mathfrak{sl}_2$ -Modules .....	179
	Problems for Independent Solution to Chapter 8 .....	183
<b>9</b>	<b>Categories and Functors</b> .....	187
9.1	Categories .....	187
9.1.1	Objects and Morphisms .....	187
9.1.2	Mono-, Epi-, and Isomorphisms .....	189
9.1.3	Reversing of Arrows .....	190
9.2	Functors .....	191
9.2.1	Covariant Functors .....	191
9.2.2	Presheaves .....	192
9.2.3	The Functors <i>Hom</i> .....	195
9.3	Natural Transformations .....	197
9.3.1	Equivalence of Categories .....	198
9.4	Representable Functors .....	200
9.4.1	Definitions via Universal Properties .....	203
9.5	Adjoint Functors .....	205
9.5.1	Tensor Products Versus Hom Functors .....	206
9.6	Limits of Diagrams .....	213
9.6.1	(Co) completeness .....	217
9.6.2	Filtered Diagrams .....	218
9.6.3	Functorial Properties of (Co) limits .....	219
	Problems for Independent Solution to Chapter 9 .....	222
<b>10</b>	<b>Extensions of Commutative Rings</b> .....	227
10.1	Integral Elements .....	227
10.1.1	Definition and Properties of Integral Elements .....	227
10.1.2	Algebraic Integers .....	230
10.1.3	Normal Rings .....	231
10.2	Applications to Representation Theory .....	232
10.3	Algebraic Elements in Algebras .....	234

- 10.4 Transcendence Generators ..... 236
- Problems for Independent Solution to Chapter 10 ..... 239
- 11 Affine Algebraic Geometry** ..... 241
- 11.1 Systems of Polynomial Equations ..... 241
- 11.2 Affine Algebraic–Geometric Dictionary ..... 243
  - 11.2.1 Coordinate Algebra ..... 243
  - 11.2.2 Maximal Spectrum ..... 244
  - 11.2.3 Pullback Homomorphisms ..... 246
- 11.3 Zariski Topology ..... 250
  - 11.3.1 Irreducible Components ..... 251
- 11.4 Rational Functions ..... 253
  - 11.4.1 The Structure Sheaf ..... 254
  - 11.4.2 Principal Open Sets as Affine Algebraic Varieties ..... 255
- 11.5 Geometric Properties of Algebra Homomorphisms ..... 256
  - 11.5.1 Closed Immersions ..... 257
  - 11.5.2 Dominant Morphisms ..... 257
  - 11.5.3 Finite Morphisms ..... 258
  - 11.5.4 Normal Varieties ..... 259
- Problems for Independent Solution to Chapter 11 ..... 261
- 12 Algebraic Manifolds** ..... 265
- 12.1 Definitions and Examples ..... 265
  - 12.1.1 Structure Sheaf and Regular Morphisms ..... 268
  - 12.1.2 Closed Submanifolds ..... 268
  - 12.1.3 Families of Manifolds ..... 269
  - 12.1.4 Separated Manifolds ..... 269
  - 12.1.5 Rational Maps ..... 271
- 12.2 Projective Varieties ..... 272
- 12.3 Resultant Systems ..... 274
  - 12.3.1 Resultant of Two Binary Forms ..... 276
- 12.4 Closeness of Projective Morphisms ..... 278
  - 12.4.1 Finite Projections ..... 279
- 12.5 Dimension of an Algebraic Manifold ..... 281
  - 12.5.1 Dimensions of Subvarieties ..... 283
  - 12.5.2 Dimensions of Fibers of Regular Maps ..... 285
- 12.6 Dimensions of Projective Varieties ..... 286
- Problems for Independent Solution to Chapter 12 ..... 290
- 13 Algebraic Field Extensions** ..... 295
- 13.1 Finite Extensions ..... 295
  - 13.1.1 Primitive Extensions ..... 296
  - 13.1.2 Separability ..... 297
- 13.2 Extensions of Homomorphisms ..... 300
- 13.3 Splitting Fields and Algebraic Closures ..... 302
- 13.4 Normal Extensions ..... 304

13.5 Compositum ..... 306

13.6 Automorphisms of Fields and the Galois Correspondence ..... 307

Problems for Independent Solution to Chapter 13 ..... 311

**14 Examples of Galois Groups** ..... 315

14.1 Straightedge and Compass Constructions ..... 315

    14.1.1 Effect of Accessory Irrationalities ..... 318

14.2 Galois Groups of Polynomials ..... 319

    14.2.1 Galois Resolution ..... 321

    14.2.2 Reduction of Coefficients ..... 322

14.3 Galois Groups of Cyclotomic Fields ..... 323

    14.3.1 Frobenius Elements ..... 324

14.4 Cyclic Extensions ..... 326

14.5 Solvable Extensions ..... 328

    14.5.1 Generic Polynomial of Degree  $n$  ..... 331

    14.5.2 Solvability of Particular Polynomials ..... 332

Problems for Independent Solution to Chapter 14 ..... 333

**Hints to Some Exercises** ..... 335

**References** ..... 355

**Index** ..... 357



<http://www.springer.com/978-3-319-50852-8>

Algebra II

Textbook for Students of Mathematics

Gorodentsev, A.L.

2017, XV, 370 p. 155 illus., 2 illus. in color., Hardcover

ISBN: 978-3-319-50852-8