# Chapter 2
# Tensor Algebras

## 2.1 Free Associative Algebra of a Vector Space

Let $V$ be a vector space over an arbitrary field $\Bbbk$. We write $V^{\otimes n} \stackrel{\text{def}}{=} V \otimes V \otimes \cdots \otimes V$ for the tensor product of $n$ copies of $V$ and call it the *nth tensor power* of $V$. We also put $V^{\otimes 0} \stackrel{\text{def}}{=} \Bbbk$ and $V^{\otimes 1} \stackrel{\text{def}}{=} V$. The infinite direct sum

$$\mathsf{T}V \stackrel{\text{def}}{=} \bigoplus_{n \geqslant 0} V^{\otimes n}$$

is called the *tensor algebra* of $V$. The multiplication in $\mathsf{T}V$ is provided by the tensor multiplication of vectors $V^{\otimes k} \times V^{\otimes m} \to V^{\otimes (k+m)}$, $(t_k, t_m) \mapsto t_k \otimes t_m$. For every basis $E$ of $V$ over $\Bbbk$, all the tensor monomials $e_1 \otimes e_2 \otimes \cdots \otimes e_d$ with $e_i \in E$ form a basis of $V^{\otimes d}$. These monomials are multiplied just by writing them sequentially with the sign $\otimes$ between them:

$$\begin{aligned}(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) \cdot \left(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_m}\right) \\ = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_m}.\end{aligned}$$

Thus, $\mathsf{T}V$ is an associative but not commutative $\Bbbk$-algebra. It can be thought of as the algebra of polynomials in noncommuting variables $e \in E$ with coefficients in $\Bbbk$. From this point of view, the subspace $V^{\otimes d} \subset \mathsf{T}V$ consists of all homogeneous polynomials of degree $d$.

Another name for $\mathsf{T}V$ is the *free associative $\Bbbk$-algebra with unit* spanned by the vector space $V$. This name emphasizes the following *universal property* of the $\Bbbk$-linear map $\iota : V \hookrightarrow \mathsf{T}V$ embedding $V$ into $\mathsf{T}V$ as the subspace $V^{\otimes 1}$ of linear homogeneous polynomials.

**Proposition 2.1 (Universal Property of Free Associative Algebras)** *For every associative $\Bbbk$-algebra $A$ with unit and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : \mathsf{T}V \to A$ such that $f = \widetilde{f} \circ \iota$. Thus, for every*



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*$\Bbbk$-algebra $A$, the homomorphisms of $\Bbbk$-algebras $\mathsf{T}V \to A$ are in bijection with the linear maps $V \to A$.*

**Exercise 2.1** Let $\iota' : V \to T'$, where $T'$ is an associative $\Bbbk$-algebra with unit, be another linear map satisfying the universal property from Proposition 2.1. Show that there exists a unique isomorphism of $\Bbbk$-algebras $\psi : \mathsf{T}V \xrightarrow{\sim} T'$ such that $\psi\iota = \iota'$.

*Proof (of Proposition 2.1)* A homomorphism of $\Bbbk$-algebras $\widetilde{f} : \mathsf{T}V \to A$ such that $f = \widetilde{f} \circ \iota$ maps every decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ to the product $f(v_1) \cdot f(v_2) \cdots f(v_n)$ in $A$, and therefore $\widetilde{f}$ is unique, because the decomposable tensors span $\mathsf{T}V$. Since the product $f(v_1) \cdot f(v_2) \cdots f(v_n)$ is multilinear in $v_i$, for each $n \in \mathbb{N}$ there exists the linear map

$$f_n : V \otimes V \otimes \cdots \otimes V \to A, \quad v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto f(v_1) \cdot f(v_2) \cdots f(v_n).$$

We put $f_0 : \Bbbk \to A$, $1 \mapsto 1$, and define $\widetilde{f} : \mathsf{T}V \to A$ to be the sum of all the $f_n$:

$$\widetilde{f} : \bigoplus_{n\geqslant 0} V^{\otimes n} \to A, \quad \sum_{n\geqslant 0} t_n \mapsto \sum_{n\geqslant 0} \varphi_n(t_n) \in A.$$

Since every tensor polynomial $t = \sum t_n \in \mathsf{T}V$ has a finite number of nonzero homogeneous components $t_n \in V^{\otimes n}$, the map $\widetilde{f}$ is a well-defined algebra homomorphism. □

## 2.2 Contractions

### *2.2.1 Complete Contraction*

For dual vector spaces $V$, $V^*$ and two decomposable tensors of equal degree $t = v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$, $\vartheta = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$, the product

$$\langle t, \vartheta \rangle \overset{\text{def}}{=} \prod_{i=1}^{n} \xi_i(v_i) = \prod_{i=1}^{n} \langle v_i, \xi_i \rangle \in \Bbbk \tag{2.1}$$

is called the *complete contraction* of $t$ with $\xi$. For a fixed

$$\vartheta = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n},$$

the constant $\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n, \vartheta \rangle \in \Bbbk$ depends multilinearly on the vectors $v_1, v_2, \ldots, v_n \in V$. Hence, there exists a unique linear form

$$c_\vartheta : V^{\otimes n} \to \Bbbk, \quad v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n, \vartheta \rangle .$$

Since the covector $c_\vartheta \in V^{\otimes n*}$ depends multilinearly on $\xi_1, \xi_2, \dots, \xi_n$, there exists a unique linear map

$$V^{*\otimes n} \to V^{\otimes n*}, \quad \vartheta \mapsto c_\vartheta . \tag{2.2}$$

In other words, the complete contraction assigns a well-defined pairing[1] between the vector spaces $V^{\otimes n}$ and $V^{*\otimes n}$,

$$V^{\otimes n} \times V^{*\otimes n} \to \Bbbk, \quad (t, \vartheta) \mapsto \langle\, t,\, \vartheta\, \rangle . \tag{2.3}$$

**Proposition 2.2** *For a finite-dimensional vector space V, the pairing* (2.3) *is perfect, i.e., the linear map* (2.2) *is an isomorphism.*

*Proof* Choose dual bases $e_1, e_2, \dots, e_n \in V$ and $x_1, x_2, \dots, x_n \in V^*$. Then the tensor monomials $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ and $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_s}$ form bases in $V^{\otimes n*}$ and $V^{*\otimes n}$ dual to each other with respect to the full contraction pairing (2.1). □

**Corollary 2.1** *For every finite-dimensional vector space V, there is a canonical isomorphism*

$$\left(V^*\right)^{\otimes n} \simeq \operatorname{Hom}(V, \dots, V; \Bbbk) \tag{2.4}$$

*mapping the decomposable tensor* $\vartheta = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$ *to the n-linear form*

$$V \times V \times \cdots \times V \to \Bbbk, \quad (v_1, v_2, \dots, v_n) \mapsto \prod_{i=1}^{n} \xi_i(v_i).$$

*Proof* The universal property of tensor product $V^{\otimes n}$ asserts that the dual space $\left(V^{\otimes n}\right)^*$, that is, the space of linear maps $V^{\otimes n} \to \Bbbk$, is isomorphic to the space of $n$-linear forms $V \times V \times \cdots \times V \to \Bbbk$. It remains to compose this isomorphism with the isomorphism (2.2). □

### *2.2.2 Partial Contractions*

Given a pair of injective but not necessarily order-preserving maps

$$\{1, 2, \dots, p\} \overset{I}{\hookleftarrow} \{1, 2, \dots, m\} \overset{J}{\hookrightarrow} \{1, 2, \dots, q\},$$

[1] See Sect. 7.2 of Algebra I.

we write $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_m)$ for the sequences of their values $i_\nu = I(\nu), j_\nu = J(\nu)$. The *partial contraction* in the indices $I, J$ is the linear map

$$c^I_J : V^{*\otimes p} \otimes V^{\otimes q} \to V^{*\otimes(p-m)} \otimes V^{\otimes(q-m)} \tag{2.5}$$

sending a decomposable tensor $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_q$ to the product

$$\prod_{\nu=1}^{m} \langle v_{j_\nu}, \xi_{i_\nu} \rangle \cdot \Big(\bigotimes_{i \notin I} \xi_i\Big) \otimes \Big(\bigotimes_{j \notin J} v_j\Big), \tag{2.6}$$

obtained by contracting the $i_\nu$th tensor factor of $V^{*\otimes p}$ with the $j_\nu$th tensor factor of $V^{\otimes q}$ for $\nu = 1, 2, \ldots, m$ and leaving all the other tensor factors in their initial order. Note that the different choices of injective maps $I, J$ lead to different partial contraction maps (2.5) even if the maps have equal images and differ only in the order of sequences $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_m$.

**Exercise 2.2** Verify that the linear map (2.5) is well defined by its values (2.6) on the decomposable tensors.

*Example 2.1 (Inner Product of Vector and Multilinear Form)* Consider an $n$-linear form $\varphi : V \times V \times \cdots \times V \to \Bbbk$ as a tensor from $V^{*\otimes n}$ by means of the isomorphism from Corollary 2.1, and contract this tensor with a vector $v \in V$ at the first tensor factor. The result of such a contraction is called the *inner product* of the $n$-linear form $\varphi$ with the vector $v$, and is denoted by $v \llcorner \varphi \in V^{*\otimes(n-1)}$. This tensor can be viewed as the $(n-1)$-linear form on $V$ obtained from the form $\varphi$ by setting the first argument equal to $v$. In other words,

$$v \llcorner \varphi(u_1, u_2, \ldots, u_{n-1}) = \varphi\,(v,\, u_1, u_2, \ldots, u_{n-1})$$

for all $u_1, u_2, \ldots, u_{n-1} \in V$. Indeed, since both sides of the equality are linear in $\varphi$, it is enough to verify it only for the $n$-linear forms $\varphi$ coming from the decomposable tensors

$$\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n},$$

because the latter span $V^{*\otimes n}$. For such $\varphi$, we have

$$\begin{aligned}
\varphi\,(v,\, u_1, u_2, \ldots, u_{n-1}) &= \langle v \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_{n-1},\, \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \rangle \\
&= \langle v,\, \xi_1 \rangle \cdot \langle u_1,\, \xi_2 \rangle \cdot \langle u_2,\, \xi_3 \rangle \cdots \langle u_{n-1},\, \xi_n \rangle \\
&= \langle u_1 \otimes u_2 \otimes \cdots \otimes u_{n-1},\, \langle v,\, \xi_1 \rangle \cdot \xi_2 \otimes \cdots \otimes \xi_n \rangle \\
&= \big\langle u_1 \otimes u_2 \otimes \cdots \otimes u_{n-1},\, c^1_1\,(v \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) \big\rangle \\
&= v \llcorner \varphi\,(u_1, u_2, \ldots, u_{n-1})\,.
\end{aligned}$$

**Exercise 2.3** Verify that for every pair of vector subspaces $U, W \subset V$, one has $U^{\otimes n} \cap W^{\otimes n} = (U \cap W)^{\otimes n}$ in $V^{\otimes n}$.

### 2.2.3 Linear Support and Rank of a Tensor

It follows from Exercise 2.3 that for every tensor $t \in V^{\otimes n}$, the intersection of all vector subspaces $U \subset V$ such that $t \in U^{\otimes n}$ is the minimal subspace of $V$ with respect to inclusions whose $n$th tensor power contains $t$. It is called the *linear support* of $t$ and denoted by $\operatorname{Supp}(t) \subset V$. Its dimension is denoted by $\operatorname{rk} t \stackrel{\text{def}}{=} \dim \operatorname{Supp}(t)$ and called the *rank* of the tensor $t$. Tensors $t$ with $\operatorname{rk} t < \dim V$ are called *degenerate*. If we think of tensors as polynomials in noncommutative variables, then the degeneracy of a tensor $t$ means that $t$ depends on fewer than $\dim V$ variables for an appropriate choice of basis in $V$. For example, every tensor $t \in V^{\otimes n}$ of rank 1 can be written as $\lambda \cdot e^{\otimes n} = \lambda \cdot e \otimes e \otimes \cdots \otimes e$ for some nonzero vector $e \in \operatorname{Supp}(t)$ and $\lambda \in \Bbbk$. For a practical choice of such special coordinates and the computation of $\operatorname{rk} t$, we need a more effective description of $\operatorname{Supp}(t)$.

Let $t \in V^{\otimes n}$ be an arbitrary tensor. For every sequence $J = j_1 j_2 \dots j_{n-1}$ of $n-1$ distinct but not necessarily increasing indices $1 \leqslant j_\nu \leqslant n$, write

$$c_t^J : V^{*\otimes(n-1)} \to V, \quad \xi \mapsto c^{1,2,\dots,(n-1)}_{j_1, j_2, \dots, j_{n-1}}(\xi \otimes t) \tag{2.7}$$

for the contraction map that pairs all $(n-1)$ factors of $V^{*\otimes(n-1)}$ with the $(n-1)$ factors of $t$ chosen in the order determined by $J$, that is, the $\nu$th factor of $V^{*\otimes(n-1)}$ is contracted with the $j_\nu$th factor of $t$ for each $\nu = 1, 2, \dots, n-1$. The result of such a contraction is a linear combination of vectors that appear in monomials of $t$ at the position not represented in $J$. This linear combination certainly belongs to $\operatorname{Supp}(t)$.

**Theorem 2.1** *For every $t \in V^{\otimes n}$, the subspace* $\operatorname{Supp}(t) \subset V$ *is spanned by the images of the $n!$ contraction maps* (2.7) *corresponding to all possible choices of $J$.*

*Proof* Let $\operatorname{Supp}(t) = W \subset V$. We have to show that every linear form $\xi \in V^*$ annihilating all the subspaces $\operatorname{im}\left(c_t^J\right) \subset W$ has to annihilate all of $W$ as well. Assume the contrary. Let $\xi \in V^*$ be a linear form having nonzero restriction on the subspace $W$ and annihilating all the subspaces $c_t^J\left(V^{*\otimes(n-1)}\right)$. Write $\xi_1, \xi_2, \dots, \xi_d$ for a basis in $V^*$ such that $\xi_1 = \xi$ and the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ to $W$ form a basis in $W^*$. Let $w_1, w_2, \dots, w_k$ be the dual basis of $W$. Expand $t$ as a linear combination of tensor monomials built out of the $w_i$. Then

$$\xi\left(c_t^J(\xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}})\right)$$

is equal to the complete contraction of $t$ with the monomial $\xi_{\mu_1} \otimes \xi_{\mu_2} \otimes \cdots \otimes \xi_{\mu_n}$ whose indices $\mu_1, \mu_2, \dots, \mu_n$ form the permutation of the indices $1, \nu_1, \nu_2, \dots, \nu_{n-1}$ uniquely determined by $J$. The result of this contraction equals the coefficient of the monomial $w_{\mu_1} \otimes w_{\mu_2} \otimes \cdots \otimes w_{\mu_n}$ in the expansion of $t$. Varying $J$ and $\nu_1, \nu_2, \ldots, \nu_{n-1}$ allows us to obtain every monomial $w_{\mu_1} \otimes w_{\mu_2} \otimes \cdots \otimes w_{\mu_n}$ containing $w_1$. Our assumption on $\xi = \xi_1$ forces the coefficients of all these monomials in $t$ to vanish. Therefore, $w_1 \notin \mathrm{Supp}(t)$. Contradiction. □

## 2.3 Quotient Algebras of a Tensor Algebra

There are three kinds of ideals in a noncommutative ring $R$. A subring $I \subset R$ is called a *left ideal* if $xa \in I$ for all $a \in I$, $x \in R$. Symmetrically, $I$ is called a *right ideal* if $ax \in I$ for all $a \in I$, $x \in R$. If $I \subset R$ is both a left and right ideal, then $I$ is called a *two-sided ideal* or simply an *ideal* of $R$. The two-sided ideals are exactly the kernels of ring homomorphisms, because for a homomorphism of rings $\varphi : R \to S$ and $a \in R$ such that $\varphi(a) = 0$, the equality $\varphi(xay) = \varphi(x)\varphi(a)\varphi(y) = 0$ holds for all $x, y \in R$. Conversely, if an additive abelian subgroup $I \subset R$ is a two-sided ideal, then the quotient group[2] $R/I$ inherits the well-defined multiplication by the usual rule $[a][b] \stackrel{\text{def}}{=} [ab]$.

**Exercise 2.4** Check this.

Therefore, the quotient map $R \twoheadrightarrow R/I$ is a homomorphism of rings with kernel $I$. It follows from the factorization theorem for a homomorphism of abelian groups[3] that an arbitrary homomorphism of rings $\varphi : R \to S$ is factorized into a composition of the surjective quotient map $R \twoheadrightarrow R/\ker\varphi \simeq \operatorname{im}\varphi$ followed by the monomorphism $R/\ker\varphi \simeq \operatorname{im}\varphi \hookrightarrow S$.

The algebra of polynomials on a vector space $V$ introduced in Sect. 11.2.1 of Algebra I and the algebra of Grassmannian polynomials from Sect. 9.4 of Algebra I can be described uniformly as the quotient algebras of the free associative algebra by appropriate two-sided ideals spanned by the *commutativity* and *skew-commutativity* relations. The details follow in the next four sections.

### *2.3.1 Symmetric Algebra of a Vector Space*

Let $V$ be a vector space over an arbitrary field $\Bbbk$. Write $\mathcal{I}_{\mathrm{sym}} \subset \mathsf{T}V$ for the two-sided ideal generated by the $\Bbbk$-linear span of all the differences

$$u \otimes w - w \otimes u \in V \otimes V. \tag{2.8}$$

The ideal $\mathcal{I}_{\mathrm{sym}}$ consists of finite linear combinations of the tensors obtained from the differences (2.8) by taking left and right products with arbitrary elements of $\mathsf{T}V$.

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[2]See Sect. 6.6.1 of Algebra I.

[3]See Proposition 2.1 of Algebra I.

Therefore, the intersection $\mathcal{I}_{\text{sym}} \cap V^{\otimes n}$ is linearly spanned by the differences

$$(\cdots \otimes v \otimes w \otimes \cdots) - (\cdots \otimes w \otimes v \otimes \cdots), \tag{2.9}$$

where the right dotted fragments in both decomposable tensors are the same, as are the left dotted fragments as well. The whole ideal $\mathcal{I}_{\text{sym}}$ is the direct sum of these homogeneous components:

$$\mathcal{I}_{\text{sym}} = \bigoplus_{n\geqslant 0} \left(\mathcal{I}_{\text{sym}} \cap V^{\otimes n}\right).$$

The quotient algebra $SV \stackrel{\text{def}}{=} \mathsf{T}V/\mathcal{I}_{\text{sym}}$ is called the *symmetric algebra* of the vector space $V$. The multiplication in $SV$ is induced by the tensor multiplication in $\mathsf{T}V$ and denoted by the dot sign $\cdot$, which is, however, usually omitted. The relations (2.8) force all vectors $u, w \in V$ to commute in $SV$. As a vector space, the symmetric algebra splits into the direct sum of homogeneous components

$$SV = \bigoplus_{n\geqslant 0} S^n V, \text{ where } S^n V \stackrel{\text{def}}{=} V^{\otimes n}/(\mathcal{I}_{\text{sym}} \cap V^{\otimes n}).$$

The space $S^n V$ is called the *nth symmetric power* of $V$. Note that $S^0 V = \Bbbk$ and $S^1 V = V$. The inclusion $\iota : V \hookrightarrow SV$, which maps $V$ to $S^1 V$, has the following universal property.

**Exercise 2.5 (Universal Property of Free Commutative Algebras)** Show that for every commutative $\Bbbk$-algebra $A$ and linear map $f : V \to A$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : SV \to A$ such that $f = \widetilde{\varphi} \circ \iota$. Also show that for every linear map $\iota' : V \to S'$ to a commutative algebra $S'$ that possesses the same universal property, there exists a unique isomorphism of algebras $\psi : S' \xrightarrow{\sim} SV$ such that $\psi\iota' = \iota$.

For this reason, the symmetric algebra $SV$ is also called the *free commutative $\Bbbk$-algebra with unit* spanned by $V$. For every basis $e_1, e_2, \dots, e_d$ of $V$, the commutative monomials $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ of total degree $\sum_i m_i = n$ form a basis in $S^n V$, as we have seen in Proposition 11.2 of Algebra I. Thus, the choice of basis in $V$ assigns the isomorphism of $\Bbbk$-algebras $SV \simeq \Bbbk[e_1, e_2, \dots, e_d]$.

**Exercise 2.6** Calculate $\dim S^n V$ for $\dim V = d$.

### *2.3.2 Symmetric Multilinear Maps*

An $n$-linear map $\varphi : V \times V \times \cdots \times V \to U$ is called *symmetric* if $\varphi(v_{g_1}, v_{g_2}, \dots, v_{g_n}) = \varphi(v_1, v_2, \dots, v_n)$ for all permutations $g \in S_n$. The symmetric multilinear maps form a subspace of the vector space $\operatorname{Hom}(V, \dots, V; U)$ of all $n$-linear maps. We denote this subspace by $\operatorname{Sym}^n(V, U) \subset \operatorname{Hom}(V, \dots, V; U)$.

Given a symmetric $n$-linear map $\varphi : V \times V \times \cdots \times V \to U$, then for every vector space $W$, the right composition of linear maps $F : U \to W$ with $\varphi$ assigns the linear map

$$\varrho_\varphi : \mathrm{Hom}(U, W) \to \mathrm{Sym}^n(V, W), \quad F \mapsto F \circ \varphi.$$

A symmetric multilinear map $\varphi$ is called *universal* if $\varrho_\varphi$ is an isomorphism for all $W$. The universal symmetric $n$-linear map is also called the $n$-ary *commutative multiplication* of vectors.

**Exercise 2.7** Verify that the target spaces of any two universal symmetric $n$-linear maps are isomorphic by means of the unique linear map commuting with the commutative multiplication.

**Proposition 2.3** *The universal symmetric n-linear map*

$$\sigma_n : V \times V \times \cdots \times V \to U$$

*is provided by tensor multiplication followed by factorization through the commutativity relations, i.e.,*

$$\sigma_n : V \times V \times \cdots \times V \overset{\tau}{\dashrightarrow} V^{\otimes n} \overset{\pi}{\twoheadrightarrow} S^n(V).$$

*Proof* By the universal property of tensor multiplication $\tau : V \times V \times \cdots \times V \to V^{\otimes n}$, every $n$-linear map $\varphi : V \times V \times \cdots \times V \to W$ is uniquely factorized as $\varphi = \widetilde{F} \circ \tau$ for some linear map $\widetilde{F} : V^{\otimes n} \to W$. If the multilinear map $\varphi$ is symmetric, then the linear map $\widetilde{F}$ annihilates the commutativity relations (2.8):

$$\begin{aligned}
&\widetilde{F}\big((\cdots \otimes v \otimes w \otimes \cdots) - (\cdots \otimes w \otimes v \otimes \cdots)\big) \\
&\quad = \widetilde{F}(\cdots \otimes v \otimes w \otimes \cdots) - \widetilde{F}(\cdots \otimes w \otimes v \otimes \cdots) \\
&\quad = \varphi(\ldots, v, w, \ldots) - \varphi(\ldots, w, v, \ldots) = 0.
\end{aligned}$$

Hence, there exists a linear map $F : S^n V \to W$ such that

$$F(v_1 v_2 \ldots v_n) = \varphi(v_1, v_2, \ldots, v_n)$$

and $\widetilde{F} = F\pi$, where $\pi : V^{\otimes n} \twoheadrightarrow S^n V$ is the factorization by the symmetry relation. Therefore, $\varphi = \widetilde{F} \circ \tau = F\pi\tau = F\sigma$. Given another linear map $F' : S^n V \to W$ such that $\varphi = F'\sigma = F'\pi\tau$, the universal property of $\tau$ forces $F'\pi = F\pi$. Since $\pi$ is surjective, this leads to $F' = F$. □

**Corollary 2.2** *For an arbitrary (not necessarily finite-dimensional) vector space V, the nth symmetric power* $S^n V$ *and the space* $\mathrm{Sym}^n(V, \Bbbk)$ *of symmetric n-linear forms* $V \times V \times \cdots \times V \to \Bbbk$ *are canonically dual to each other.*

*Proof* Right composition with the commutative multiplication

$$\sigma_n : V \times V \times \cdots \times V \to S^n V,$$

which takes a covector $\xi : S^n V \to \Bbbk$ to the symmetric $n$-linear form

$$\xi \circ \sigma_n : V \times V \times \cdots \times V \to \Bbbk,$$

establishes an isomorphism $(S^n V)^* \xrightarrow{\sim} \mathrm{Sym}^n(V, \Bbbk)$ by the universal property of $\sigma_n$. □

### 2.3.3 The Exterior Algebra of a Vector Space

Write $\mathcal{I}_{\mathrm{skew}} \subset \mathsf{T}V$ for the two-sided ideal generated by the $\Bbbk$-linear span of all proper squares $v \otimes v \in V \otimes V$, $v \in V$.

**Exercise 2.8** Convince yourself that the $\Bbbk$-linear span of all proper squares $v \otimes v \in V \otimes V$ contains all the sums $u \otimes w + w \otimes u$ with $u, w \in V$ and is linearly generated by these sums if $\mathrm{char}\, \Bbbk \neq 2$.

As in the commutative case, the ideal $\mathcal{I}_{\mathrm{skew}}$ splits into the direct sum of homogeneous components

$$\mathcal{I}_{\mathrm{skew}} = \bigoplus_{n \geqslant 0} \left( \mathcal{I}_{\mathrm{skew}} \cap V^{\otimes n} \right),$$

where the degree-$n$ component $\mathcal{I}_{\mathrm{skew}} \cap V^{\otimes n}$ is linearly generated over $\Bbbk$ by the decomposable tensors $\cdots \otimes v \otimes v \otimes \cdots$, containing a pair of equal sequential factors. By Exercise 2.8, all the sums

$$(\cdots \otimes v \otimes w \otimes \cdots) + (\cdots \otimes w \otimes v \otimes \cdots). \tag{2.10}$$

also belong to $\mathcal{I}_{\mathrm{skew}} \cap V^{\otimes n}$. The quotient algebra $\Lambda V \stackrel{\mathrm{def}}{=} \mathsf{T}V / \mathcal{I}_{\mathrm{skew}}$ is called the *exterior* or *Grassmannian algebra* of the vector space $V$. The multiplication in $\Lambda V$ is induced by the tensor multiplication in $\mathsf{T}V$. It is called the *exterior* or *Grassmannian* multiplication and is denoted by the wedge sign $\wedge$. The skew-symmetry relations imply that all the vectors from $V$ anticommute and have zero squares in $\Lambda V$, i.e., $u \wedge w = -w \wedge u$ and $u \wedge u = 0$ for all $u, w \in V$. A permutation of factors in any monomial multiplies the monomial by the sign of the permutation,

$$v_{g_1} \wedge v_{g_2} \wedge \cdots \wedge v_{g_k} = \mathrm{sgn}(g) \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_k \quad \forall\, g \in S_k.$$

The algebras possessing this property are commonly called *skew commutative* in mathematics and *supercommutative* in physics. We will shorten both names to *s-commutativity*.

As a vector space over $\Bbbk$, the Grassmannian algebra splits into the direct sum of homogeneous components

$$\Lambda V = \bigoplus_{n \geqslant 0} \Lambda^n V, \text{ where } \Lambda^n V = V^{\otimes n} / (\mathcal{I}_{\text{skew}} \cap V^{\otimes n}).$$

The vector space $\Lambda^n V$ is called the *nth exterior power* of $V$. Note that $\Lambda^0 V = \Bbbk$ and $\Lambda^1 V = V$. As in the symmetric case, the inclusion $\iota : V \hookrightarrow \Lambda V$, mapping $V$ to $\Lambda^1 V$, has a universal property.

**Exercise 2.9 (Universal Property of Free s-Commutative Algebras)** Show that for every s-commutative $\Bbbk$-algebra $L$ and linear map $f : V \to L$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : \Lambda V \to L$ such that $f = \widetilde{f} \circ \iota$. Also show that for every linear map $\iota' : V \to \Lambda'$ to an s-commutative algebra $\Lambda'$ possessing the same universal property, there exists a unique isomorphism of algebras $\psi : \Lambda' \xrightarrow{\sim} \Lambda V$ such that $\psi \iota' = \iota$.

For this reason, the algebra $\Lambda V$ is also called the *free s-commutative* $\Bbbk$*-algebra* spanned by $V$.

### 2.3.4 Alternating Multilinear Maps

An $n$-linear map $\varphi : V \times V \times \cdots \times V \to U$ is called *alternating* if

$$\varphi(v_{g_1}, v_{g_2}, \ldots, v_{g_n}) = \operatorname{sgn}(g) \cdot \varphi(v_1, v_2, \ldots, v_n)$$

for all permutations $g \in S_n$. We write $\operatorname{Alt}^n(V, U) \subset \operatorname{Hom}(V, \ldots, V; U)$ for the subspace of alternating $n$-linear maps.

Associated with every alternating $n$-linear map $\varphi : V \times V \times \cdots \times V \to U$ and vector space $W$ is the linear map

$$\operatorname{Hom}(U, W) \to \operatorname{Alt}^n(V, W), \quad F \mapsto F \circ \varphi. \tag{2.11}$$

The map $\varphi$ is called the *universal alternating n-linear map* or the *n-ary s-commutative multiplication* of vectors if the linear map (2.11) is an isomorphism for all vector spaces $W$.

**Exercise 2.10** Prove that the universal alternating $n$-linear map

$$\alpha_n : V \times V \times \cdots \times V \to U$$

is provided by tensor multiplication followed by factorization by the skew-commutativity relations, i.e., $\alpha_n : V \times \ldots \times V \overset{\tau}{\dashrightarrow} V^{\otimes n} \overset{\pi}{\twoheadrightarrow} \Lambda^n(V)$, and verify that the target spaces of every two universal symmetric $n$-linear maps are isomorphic by means of the unique linear map commuting with the s-commutative multiplication.

**Corollary 2.3** *For an arbitrary (not necessarily finite-dimensional) vector space $V$, the nth exterior power $\Lambda^n V$ and the space $\mathrm{Alt}^n(V, \Bbbk)$ of alternating n-linear forms $V \times V \times \cdots \times V \to \Bbbk$ are canonically dual to each other.*

*Proof* The same as for Corollary 2.2 on p. 28. $\square$

**Proposition 2.4** *For every basis $e_1, e_2, \ldots, e_d$ of $V$, a basis in $\Lambda^d V$ is formed by the Grassmannian monomials*

$$e_I \overset{\text{def}}{=} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \tag{2.12}$$

*numbered by all $I = (i_1, i_2, \ldots, i_n)$ with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$. In particular,* $\dim \Lambda^n V = \binom{d}{n}$ *and* $\dim \Lambda V = 2^d$.

*Proof* Write $U$ for the vector space of dimension $\binom{d}{n}$ with the basis $\{u_I\}$ numbered by the same multi-indices $I$ as the Grassmannian monomials (2.12). We know from Sect. 1.1.1 on p. 1 that every $n$-linear map $\alpha : V \times V \times \cdots \times V \to U$ is uniquely determined by its values on all the collections of basis vectors $\alpha(e_{j_1}, e_{j_2}, \ldots, e_{j_n})$, and these values may be arbitrary. Let us put $\alpha(e_{j_1}, e_{j_2}, \ldots, e_{j_n}) = 0$ if some arguments coincide, and $\alpha(e_{j_1}, e_{j_2}, \ldots, e_{j_n}) = \mathrm{sgn}(g) \cdot u_I$, where $I = (j_{g_1}, j_{g_2}, \ldots, j_{g_n})$ is the strictly increasing permutation of the indices $j_1, j_2, \ldots, j_n$ if all the indices are distinct. The resulting $n$-linear map $\alpha : V \times V \times \cdots \times V \to U$ is alternating and universal, because for every $n$-linear alternating map $\varphi : V \times V \times \cdots \times V \to W$, there exists a unique linear operator $F : U \to W$ such that $\varphi = F \circ \alpha$, namely, the operator acting on the basis of $U$ as $F(u_I) = \varphi(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$. By Exercise 2.10, there exists a linear isomorphism $U \xrightarrow{\sim} \Lambda^n V$ sending the basis vectors $u_I$ to the s-symmetric products $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. This forces the latter to form a basis in $\Lambda^n V$. $\square$

**Corollary 2.4** *For every basis $e_1, e_2, \ldots, e_d$ of $V$, the exterior algebra $\Lambda V$ is isomorphic to the Grassmannian polynomial algebra $\Bbbk\langle e_1, e_2, \ldots, e_d \rangle$ defined in Sect. 9.4 of Algebra I.* $\square$

## 2.4 Symmetric and Alternating Tensors

Starting from this point, we will always assume by default that $\mathrm{char}\,\Bbbk = 0$. For every $n \in \mathbb{N}$, the symmetric group $S_n$ acts on $V^{\otimes n}$ by permutations of factors in the decomposable tensors:

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{g_1} \otimes v_{g_2} \otimes \cdots \otimes v_{g_n} \quad \forall\, g \in S_n. \tag{2.13}$$

Since $v_{g_1} \otimes v_{g_2} \otimes \cdots \otimes v_{g_n}$ is multilinear in $v_1, v_2, \dots, v_n$, there exists a well-defined linear operator $g : V^{\otimes n} \to V^{\otimes n}$ acting on decomposable tensors by formula (2.13). The subspaces of $S_n$-invariant and sign-alternating tensors are denoted by

$$\operatorname{Sym}^n V \stackrel{\text{def}}{=} \{t \in V^{\otimes n} \mid \forall\, g \in S_n,\ g(t) = t\}, \tag{2.14}$$

$$\operatorname{Alt}^n V \stackrel{\text{def}}{=} \{t \in V^{\otimes n} \mid \forall\, g \in S_n,\ g(t) = \operatorname{sgn}(g) \cdot t\}, \tag{2.15}$$

and called, respectively, the spaces of *symmetric* and *alternating* tensors of degree $n$ on $V$.

### *2.4.1 Symmetrization and Alternation*

If $\operatorname{char} \Bbbk = 0$, then for all $n \geqslant 2$, the tensor power $V^{\otimes n}$ is projected onto the subspaces of symmetric and alternating tensors, respectively, by means of the *symmetrization* and *alternation* maps

$$\operatorname{sym}_n : V^{\otimes n} \to \operatorname{Sym}^n V, \quad t \mapsto \frac{1}{n!} \sum_{g \in S_n} g(t), \tag{2.16}$$

$$\operatorname{alt}_n : V^{\otimes n} \to \operatorname{Alt}^n V, \quad t \mapsto \frac{1}{n!} \sum_{g \in S_n} \operatorname{sgn}(g) \cdot g(t). \tag{2.17}$$

**Exercise 2.11** For all $t \in V^{\otimes n}$, $s \in \operatorname{Sym}^n V$, $a \in \operatorname{Alt}^n V$, and $n \geqslant 2$, prove that **(a)** $\operatorname{sym}_n(s) = s$, **(b)** $\operatorname{alt}_n(a) = a$, **(c)** $\operatorname{sym}_n(a) = \operatorname{alt}_n(s) = 0$, **(d)** $\operatorname{sym}_n(t) \in \operatorname{Sym}^n V$, **(e)** $\operatorname{alt}_n(t) \in \operatorname{Alt}^n V$.

Therefore, the symmetrization and alternation maps satisfy the relations

$$\operatorname{sym}_n^2 = \operatorname{sym}_n, \quad \operatorname{alt}_n^2 = \operatorname{alt}_n, \quad \operatorname{sym}_n \circ \operatorname{alt}_n = \operatorname{alt}_n \circ \operatorname{sym}_n = 0. \tag{2.18}$$

*Example 2.2 (Tensor Square Decomposition)* For $n = 2$, the symmetrization and alternation maps form a pair of complementary projectors,[4] that is,

$$\operatorname{sym}_2 + \operatorname{alt}_2 = (\mathrm{Id} + s_{12})/2 + (\mathrm{Id} - s_{12})/2 = \mathrm{Id},$$

where $s_{12} \in S_2$ is a transposition. Therefore, there exists the direct sum decomposition

$$V^{\otimes 2} = \operatorname{Sym}^2 V \oplus \operatorname{Alt}^2 V. \tag{2.19}$$

[4] See Example 15.3 in Algebra I.

If we interpret $V^{\otimes 2}$ as the space of bilinear forms on $V^*$, then the decomposition (2.19) turns out to be the decomposition of the space of bilinear forms into the direct sum of subspaces of symmetric and alternating forms considered in Sect. 16.1.6 of Algebra I.

*Example 2.3 (Tensor Cube Decomposition)* For $n = 3$, the direct sum $\mathrm{Sym}^3 V \oplus \mathrm{Alt}^3 V$ does not exhaust all of $V^{\otimes 3}$.

**Exercise 2.12** Find $\operatorname{codim}\left(\mathrm{Sym}^3 V \oplus \mathrm{Alt}^3 V\right)$ in $V^{\otimes 3}$.

To find the complement to $\mathrm{Sym}^3 V \oplus \mathrm{Alt}^3 V$ in $V^{\otimes 3}$, write $T = |123\rangle \in S_3$ for the cyclic permutation and consider the difference

$$p = \mathrm{Id} - \mathrm{sym}_3 - \mathrm{alt}_3 = \mathrm{Id} - \left(\mathrm{Id} + T + T^2\right)/3. \tag{2.20}$$

**Exercise 2.13** Verify that $p^2 = p$ and $p \circ \mathrm{alt}_3 = \mathrm{alt}_3 \circ p = p \circ \mathrm{sym}_3 = \mathrm{sym}_3 \circ p = 0$.

Since $\mathrm{sym}_3 + \mathrm{alt}_3 + p = \mathrm{Id}_{V^{\otimes 3}}$, there exists the direct sum decomposition

$$V^{\otimes 3} = \mathrm{Sym}^3 V \oplus \mathrm{Alt}^3 V \oplus \mathrm{im}(p),$$

where $\mathrm{im}(p) = \{t \in V^{\otimes 3} \mid t + Tt + T^2 t = 0\}$ consists of all cubic tensors annihilated by averaging over the action of a 3-cycle. An example of such a tensor is provided by $[u, [v, w]]$, where $[a, b] \stackrel{\mathrm{def}}{=} a \otimes b - b \otimes a$ means the commutator in the tensor algebra.

**Exercise 2.14 (Jacobi Identity)** Verify that $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ in $V^{\otimes 3}$ for all $u, v, w \in V$.

If we think of $V^{\otimes 3}$ as the space of 3-linear forms on $V^*$, then $\mathrm{im}(p)$ consists of all 3-linear forms $t : V^* \times V^* \times V^* \to \Bbbk$ satisfying the *Jacobi identity*:

$$t(\xi, \eta, \zeta) + t(\eta, \zeta, \xi) + t(\zeta, \xi, \eta) = 0$$

for all $\xi, \eta, \zeta \in V^*$.

For larger $n$, the decomposition of $V^{\otimes n}$ by the "symmetry types" of tensors becomes more complicated. It is the subject of the representation theory of the symmetric group, which will be discussed in Chap. 7 below.

### 2.4.2 *Standard Bases*

Let us fix a basis $e_1, e_2, \dots, e_d$ in $V$ and break the basis monomials

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \in V^{\otimes n}$$

into a disjoint union of $S_n$-orbits. Since the monomials of every $S_n$-orbit appear in the expansion of every symmetric tensor $t \in \operatorname{Sym}^n V$ with equal coefficients, a basis in $\operatorname{Sym}^n V$ is formed by the *monomial symmetric tensors*

$$e_{[m_1,m_2,\dots,m_d]} \stackrel{\text{def}}{=} \begin{pmatrix} \text{sum of all tensor monomials formed by} \\ m_1 \text{ factors } e_1,\, m_2 \text{ factors } e_2,\, \dots,\, m_d \text{ factors } e_d \end{pmatrix} \tag{2.21}$$

numbered by the sequences $(m_1, m_2, \dots, m_d)$ of nonnegative integers satisfying the condition

$$m_1 + m_2 + \cdots + m_d = n.$$

It follows from the orbit length formula[5] that the sum on the right-hand side of (2.21) consists of $n!/(m_1!m_2! \cdots m_d!)$ summands, because the stabilizer of each summand is formed by $m_1!\, m_2! \cdots m_d!$ independent permutations of equal tensor factors.

Similarly, a basis in $\operatorname{Alt}^n V$ is formed by the *monomial alternating tensors*

$$e_I = e_{\langle i_1,i_2,\dots,i_n\rangle} \stackrel{\text{def}}{=} \sum_{g \in S_n} \operatorname{sgn}(g) \cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}} \tag{2.22}$$

numbered by strictly increasing sequences of positive integers

$$I = (i_1, i_2, \dots, i_n), \quad 1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d.$$

*Remark 2.1 (Bases* (2.21) *and* (2.22) *for Infinite-Dimensional V)* We do not actually need to assume that $d = \dim V < \infty$ in both formulas (2.21), (2.22). They make sense for an arbitrary, not necessarily finite, basis $E$ in $V$ under the following agreement on notation. Let us fix some total ordering on the set $E$ and number once and for all the elements of every finite subset $X \subset E$ in increasing order by integer indices 1, 2, ... , $|X|$. Then a basis in $S^n V$ is formed by the monomial tensors (2.21), where $d, m_1, m_2, \dots, m_d \in \mathbb{N}$ are any *positive* integers such that $m_1 + m_2 + \cdots + m_d = n$, and $e_1, e_2, \dots, e_d$ run through the (numbered) subsets of cardinality $d$ in $E$. Similarly, a basis in $\operatorname{Alt}^n V$ is formed by the monomials (2.22), where $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ run through the (numbered) subsets of cardinality $n$ in $E$.

**Proposition 2.5** *If* $\operatorname{char}(\Bbbk) = 0$*, then the restriction of the quotient map*

$$V^{\otimes n} \twoheadrightarrow S^n V$$

*to the subspace* $\operatorname{Sym}^n \subset V^{\otimes n}$ *and the restriction of the quotient map*

$$V^{\otimes n} \twoheadrightarrow \Lambda^n V$$

[5]See Proposition 12.2 of Algebra I.

*to the subspace* $\mathrm{Alt}^n \subset V^{\otimes n}$ *establish the isomorphisms of vector spaces*

$$\pi_{\mathrm{sym}} : \mathrm{Sym}^n V \xrightarrow{\sim} S^n V \quad \textit{and} \quad \pi_{\mathrm{sk}} : \mathrm{Alt}^n V \xrightarrow{\sim} \Lambda^n V.$$

*These isomorphisms act on the basis monomial tensors* (2.21) *and* (2.22) *by the rules*

$$e_{[m_1,m_2,\dots,m_d]} \mapsto \frac{n!}{m_1! \cdot m_2! \cdots m_d!} \cdot e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}, \tag{2.23}$$

$$e_{\langle i_1,i_2,\dots,i_d\rangle} \mapsto n! \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}. \tag{2.24}$$

*Proof* The projection $\pi_{\mathrm{sym}}$ maps each of the $n!/(m_1!m_2!\cdots m_d!)$ summands in (2.21) to the commutative monomial $e_1^{m_1} e_2^{m_2} \dots e_d^{m_d}$. Similarly, the projection $\pi_{\mathrm{sk}}$ sends each of the $n!$ summands in (2.22) to the Grassmannian monomial $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. □

**Caution 2.1** In spite of Proposition 2.5, the subspaces $\mathrm{Sym}^n V$, $\mathrm{Alt}^n V \subset V^{\otimes n}$ should not be confused with the quotient spaces $S^n V$ and $\Lambda^n V$ of the tensor power $V^{\otimes n}$. If $\mathrm{char}\,\Bbbk = p > 0$, then many symmetric tensors and all the alternating tensors of degree larger than $p$ are *annihilated* by projections $V^{\otimes n} \twoheadrightarrow S^n V$ and $V^{\otimes n} \twoheadrightarrow \Lambda^n V$. Even if $\mathrm{char}\,\Bbbk = 0$, the isomorphisms from Proposition 2.5 *do not* identify the monomial bases of tensor spaces directly with the standard monomials in the polynomial rings. Both isomorphisms contain some combinatorial factors, which should be taken into account whenever we need either to pull back the multiplication from the polynomial (respectively exterior) algebra to the space of symmetric (respectively alternating) tensors or push forward the contractions of tensors into the polynomial algebras.

## 2.5 Polarization of Polynomials

It follows from Proposition 2.5 that for every homogeneous polynomial $f \in S^n V^*$, there exists a unique symmetric tensor $\widetilde{f} \in \mathrm{Sym}^n V^*$ mapped to $f$ under the factorization by the commutativity relations $(V^*)^{\otimes n} \twoheadrightarrow S^n V^*$ on p. 23 allows us to treat $\widetilde{f}$ as the symmetric $n$-linear form

$$\widetilde{f} : V \times V \times \dots \times V \to \Bbbk, \quad \widetilde{f}(v_1, v_2, \dots, v_n) \stackrel{\mathrm{def}}{=} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n, \widetilde{f} \right\rangle.$$

This form is called the *complete polarization* of the polynomial $f$. For $n = 2$, the polarization $\widetilde{f}$ of a quadratic form $f \in S^2 V^*$ coincides with that defined in Chap. 17 of Algebra I by the equality

$$2\widetilde{f}(u, w) = f(u + w) - f(u) - f(w).$$

For arbitrary $n$, the complete polarization of every monomial $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$ of degree $n = m_1 + m_2 + \cdots + m_d$ is given by the first formula from Proposition 2.5 and equals

$$\frac{m_1!\, m_2! \cdots m_d!}{n!} \cdot x_{[m_1, m_2, \ldots, m_d]} . \tag{2.25}$$

The complete polarization of an arbitrary polynomial can be computed using (2.25) and the linearity of the polarization map $\pi_{\mathrm{sym}}^{-1} : S^n V^* \xrightarrow{\sim} \mathrm{Sym}^n V^*, f \mapsto \widetilde{f}$. By Remark 2.1 on p. 34, this works for every (not necessarily finite) basis in $V^*$ as well.

### 2.5.1 Evaluation of Polynomials on Vectors

Associated with every polynomial $f \in S^n V^*$ is the *polynomial function*

$$f : V \to \Bbbk, \quad v \mapsto f(v) \stackrel{\text{def}}{=} \widetilde{f}(v, v, \ldots, v). \tag{2.26}$$

Note that the value of $f$ on $v$ is well defined even for infinite-dimensional vector spaces and does not depend on any extra data on $V$, such as the choice of basis. Now assume that $\dim V < \infty$, fix dual bases $e_1, e_2, \ldots, e_d \in V$, $x_1, x_2, \ldots, x_d \in V^*$, and identify the symmetric algebra $SV^*$ with the polynomial algebra $\Bbbk[x_1, x_2, \ldots, x_d]$. Then the value of a polynomial $f(x_1, x_2, \ldots, x_n)$ at a vector $v = \sum \alpha_i e_i \in V$ coincides with the result of the substitution $x_i = \alpha_i$ in $f$:

$$f(v) = f(\alpha_1, \alpha_2, \ldots, \alpha_d). \tag{2.27}$$

Indeed, for every monomial $f = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$, the complete contraction of $v^{\otimes n}$ with

$$\widetilde{f} = \frac{m_1! \cdot m_2! \cdots m_d!}{n!} x_{[m_1, m_2, \ldots, m_d]}$$

is the sum of $n!/(m_1! \cdot m_2! \cdots m_d!)$ equal products

$$\begin{aligned} &\frac{m_1! \cdot m_2! \cdots m_d!}{n!} \cdot x_1(v)^{m_1} \cdot x_2(v)^{m_2} \cdots x_d(v)^{m_d} \\ &= \frac{m_1! \cdot m_2! \cdots m_d!}{n!} \cdot \alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_d^{m_d} . \end{aligned}$$

It coincides with the result of the substitution $(x_1, x_2, \ldots, x_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in the monomial

$$\frac{n!}{m_1! m_2! \cdots m_d!} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} .$$

We conclude that the evaluation of a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_d]$ at the coordinates of a vector $v \in V$ depends only on $f \in S^n V^*$ and $v \in V$ but not on the choice of dual bases in $V$, $V^*$.

### 2.5.2 Combinatorial Formula for Complete Polarization

Since the value of a symmetric $n$-linear form does not depend on the order of arguments, let us write

$$\widetilde{f}\left(v_1^{m_1}, v_2^{m_2}, \dots, v_n^{m_k}\right)$$

for the value of $\widetilde{f}$ at $m_1$ vectors $v_1$, $m_2$ vectors $v_2, \dots, m_k$ vectors $v_k$ with $\sum_\nu m_\nu = n$.

**Exercise 2.15** Show that for every polynomial $f \in S^n V^*$ and all vectors $v_1, v_2, \dots, v_k \in V$, one has

$$\begin{aligned} f(v_1 + v_2 + \dots + v_k) &= \widetilde{f}\left((v_1 + v_2 + \dots + v_k)^n\right) \\ &= \sum_{m_1 m_2 \dots m_k} \frac{n!}{m_1! m_2! \cdots m_k!} \cdot \widetilde{f}\left(v_1^{m_1}, v_2^{m_2}, \dots, v_k^{m_k}\right), \end{aligned} \tag{2.28}$$

where the summation is over all integers $m_1, m_2, \dots, m_k$ such that

$$m_1 + m_2 + \dots + m_k = n$$

and $0 \leqslant m_\nu \leqslant n$ for all $\nu$.

**Proposition 2.6** *Let $V$ be a vector space, not necessarily finite-dimensional, over a field $\Bbbk$ of characteristic zero. Then for every homogeneous polynomial $f \in S^n V^*$,*

$$n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \sum_{I \subsetneq \{1,\dots,n\}} (-1)^{|I|} f\Big(\sum_{i \notin I} v_i\Big), \tag{2.29}$$

*where the left summation is over all subsets $I \subsetneq \{1, 2, \dots, n\}$ including $I = \varnothing$, for which $|\varnothing| = 0$. For example, for $f \in S^3 V^*$, one has*

$$6\widetilde{f}(u, v, w) = f(u+v+w) - f(u+v) - f(u+w) - f(v+w) + f(u) + f(v) + f(w).$$

*Proof* Consider the expansion (2.28) from Exercise 2.15 for $k = n = \deg f$. Its right-hand side contains the unique term depending on all the vectors $v_1, v_2, \dots, v_n$, namely $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n)$. For every proper subset $I \subsetneq \{1, 2, \dots, n\}$, the summands of (2.28) that do not contain vectors $v_i$ with $i \in I$ appear in (2.28) with the same coefficients as they do in the expansion of $f\Big(\sum_{i \notin I} v_i\Big)$, because the latter is obtained from $f(v_1+v_2+\dots+v_n)$ by setting $v_i=0$ for all $i\in I$. Therefore, all terms that do not depend on some of the $v_i$ can be removed from (2.28) by the standard combinatorial inclusion–exclusion procedure. This leads to the required formula

$$n!\cdot\widetilde{f}(v_1,v_2,\dots,v_n)$$
$$=f\Big(\sum_\nu v_\nu\Big)-\sum_{\{i\}}f\Big(\sum_{\nu\neq i}v_\nu\Big)+\sum_{\{i,j\}}f\Big(\sum_{\nu\neq i,j}v_\nu\Big)-\sum_{\{i,j,k\}}f\Big(\sum_{\nu\neq i,j,k}v_\nu\Big)+\cdots.$$

□

### 2.5.3 Duality

Assume that $\operatorname{char}\Bbbk=0$ and $\dim V<\infty$. The complete contraction between $V^{\otimes m}$ and $V^{*\otimes m}$ provides the spaces $S^mV$ and $S^mV^*$ with the perfect pairing[6] that couples polynomials $f\in S^nV$ and $g\in S^nV^*$ to a complete contraction of their complete polarizations $\widetilde{f}\in V^{\otimes m}$ and $\widetilde{g}\in V^{*\otimes m}$.

**Exercise 2.16** Verify that for every pair of dual bases

$$e_1,e_2,\dots,e_d\in V,\quad x_1,x_2,\dots,x_d\in V^*,$$

all the nonzero couplings between the basis monomials are exhausted by

$$\left\langle e_1^{m_1}e_2^{m_2}\cdots e_d^{m_d},\,x_1^{m_1}x_2^{m_2}\cdots x_d^{m_d}\right\rangle=\frac{m_1!m_2!\cdots m_d!}{n!}. \tag{2.30}$$

Note that the monomials constructed from the dual basis vectors become the dual bases of the polynomial rings only after rescaling by the same combinatorial factors as in Proposition 2.5.

### 2.5.4 Derivative of a Polynomial Along a Vector

Associated with every vector $v\in V$ is the linear map

$$i_v:V^{*\otimes n}\to V^{*\otimes(n-1)},\quad \varphi\mapsto v\llcorner\varphi, \tag{2.31}$$

provided by the inner multiplication[7] of $n$-linear forms on $V$ by $v$, which takes an $n$-linear form $\varphi\in V^{*\otimes n}$ to the $(n-1)$-linear form

$$v\llcorner\varphi(v_1,v_2,\dots,v_{n-1})=\varphi(v,\,v_1,v_2,\dots,v_{n-1}).$$

[6] See Sect. 7.1.4 of Algebra I.

[7] See Example 2.1 on p. 24.

The map (2.31) preceded by the complete polarization map

$$S^n V^* \xrightarrow{\sim} \mathrm{Sym}^n V^* \subset V^{*\otimes n}$$

and followed by the quotient map $V^{*\otimes(n-1)} \twoheadrightarrow S^{n-1}V^*$ gives the linear map

$$\mathrm{pl}_v : S^n V^* \to S^{n-1}V^*, \quad f(x) \mapsto \mathrm{pl}_v f(x) \stackrel{\text{def}}{=} \widetilde{f}(v, x, x, \dots, x), \tag{2.32}$$

which depends linearly on $v \in V$. This map fits in the commutative diagram

$$\begin{tikzcd} V^{*\otimes n} \supset \mathrm{Sym}^n V^* \arrow[r, "v \llcorner"] \arrow[d, "\pi_{\mathrm{sym}}"', "\wr"] & V^{*\otimes(n-1)} \arrow[d, "\pi_{\mathrm{sym}}"] \\ S^n V^* \arrow[r, "\mathrm{pl}_v"] & S^{n-1} V^* \end{tikzcd} \tag{2.33}$$

The polynomial $\mathrm{pl}_v f(x)\widetilde{f}(v, x, \dots x) \in S^{n-1}(V^*)$ is called the *polar* of $v$ with respect to $f$. For $n = 2$, the polar of a vector $v$ with respect to a quadratic form $f \in S^2V^*$ is the linear form $w \mapsto \widetilde{f}(v, w)$ considered[8] in Sect. 17.4.3 of Algebra I.

In terms of dual bases $e_1, e_2, \dots, e_d \in V$, $x_1, x_2, \dots, x_d \in V^*$, the contraction of the first tensor factor in $V^{*\otimes n}$ with the basis vector $e_i \in V$ maps the complete symmetric tensor $x_{[m_1, m_2, \dots, m_n]}$ either to the complete symmetric tensor containing the $(m_i - 1)$ factors $x_i$ or to zero for $m_i = 0$. By formula (2.23) from Proposition 2.5,

$$\mathrm{pl}_{e_i} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} = \frac{m_i}{n}\, x_1^{m_1} \cdots x_{i-1}^{m_{i-1}} x_i^{m_i - 1} x_{i+1}^{m_{i+1}} \cdots x_d^{m_d} = \frac{1}{n} \frac{\partial}{\partial x_i}\, x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}.$$

Since $\mathrm{pl}_v f$ is linear in both $v$ and $f$, we conclude that for every $v = \sum \alpha_i e_i$, the polar polynomial of $v$ with respect to $f$ is nothing but the *derivative* of the polynomial $f$ along the vector $v$ divided by $\deg f$, i.e.,

$$\mathrm{pl}_v f = \frac{1}{\deg(f)}\, \partial_v f = \frac{1}{\deg(f)} \sum_{i=1}^{d} \alpha_i \frac{\partial f}{\partial x_i}.$$

Note that this forces the right-hand side of the formula to be independent of the choice of dual bases in $V$ and $V^*$. It follows from the definition of polar map that the derivatives along vectors commute, $\partial_u \partial_w = \partial_w \partial_u$, and satisfy the following

[8] Recall that the zero set of this form in $\mathbb{P}(V)$ is the hyperplane intersecting the quadric $Z(f) \subset \mathbb{P}(V)$ along its apparent contour viewed from $v$.

remarkable relation:

$$m!\,\frac{\partial^m f}{\partial u^m}(w) = n!\,\widetilde{f}(\underbrace{u,u,\,\dots,u}_{m},\underbrace{w,w,\,\dots,w}_{n}) = (n-m)!\,\frac{\partial^{n-m} f}{\partial w^{n-m}}(u), \tag{2.34}$$

which holds for all $u, w \in V, f \in S^n V^*$, and $0 \leqslant m \leqslant n$.

**Exercise 2.17** Prove the *Leibniz rule* $\partial_v(f \cdot g) = \partial_v(f) \cdot g + f \cdot \partial_v(g)$.

**Exercise 2.18** Show that

$$\widetilde{f}(v_1, v_2, \dots, v_n) = \frac{1}{n!}\partial_{v_1}\partial_{v_2}\cdots\partial_{v_n} f$$

for every polynomial $f \in S^n V^*$ and all vectors $v_1, v_2, \dots, v_n \in V$.

*Example 2.4 (Taylor Expansion)* For $k = 2$, the expansion (2.28) from Exercise 2.15 turns into the identity

$$f(u+w) = \widetilde{f}(u+w,\, u+w,\, \dots,\, u+w) = \sum_{m=0}^{n}\binom{n}{m}\cdot\widetilde{f}(u^m, w^{n-m}),$$

where $n = \deg f$, which holds for every polynomial $f \in S^n V^*$ and all vectors $u, w \in V$. The relations (2.34) allow us to rewrite this identity as the *Taylor expansion* for $f$ at $u$:

$$f(u+w) = \sum_{m=0}^{\deg f}\frac{1}{m!}\,\partial_w^m f(u). \tag{2.35}$$

Note that this is an exact equality in the polynomial ring $SV^*$, and its right-hand side actually is completely symmetric in $u$, $w$, because of the same relations in (2.34).

### *2.5.5 Polars and Tangents of Projective Hypersurfaces*

Let $S = Z(F) \subset \mathbb{P}(V)$ be a projective hypersurface defined by a homogeneous polynomial equation $F(x) = 0$ of degree $n$. For every line $\ell = (pq) \subset \mathbb{P}(V)$, the intersection $\ell \cap S$ consists of all points $\lambda p + \mu q \in \ell$ such that $(\lambda : \mu)$ satisfies the homogeneous equation $f(\lambda, \mu) = 0$ obtained from the equation $F(x) = 0$ via the substitution $x \leftarrow\!\shortmid \lambda p + \mu q$. Over an algebraically closed field $\Bbbk$, the binary form $f(\lambda, \mu) \in \Bbbk[\lambda, \mu]$ either is zero or is completely factorized into a product of $n$ forms linear in $\lambda$, $\mu$:

$$f(\lambda, \mu) = \prod_i (\alpha_i''\lambda - \alpha_i'\mu)^{s_i} = \prod_i \det{}^{s_i}\begin{pmatrix}\lambda & \alpha_i' \\ \mu & \alpha_i''\end{pmatrix}, \tag{2.36}$$

where $a_i = (\alpha_i' : \alpha_i'')$ are distinct points on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ and $\sum_i s_i = n$. In the first case, the line $\ell$ lies on $S$. In the second case, the intersection $\ell \cap S$ consists of points $a_i = \alpha_i' p + \alpha_i'' q$. The exponent $s_i$ of the linear form $\alpha_i''\mu - \alpha_i'\lambda$ in the factorization (2.36) is called the *intersection multiplicity* of the hypersurface $S$ and the line $\ell$ at the point $a_i$, and is denoted by $(S,\ell)_{a_i}$. If $(S,\ell)_{a_i} = 1$, then $a_i$ is called a *simple* (or *transversal*) intersection point. Otherwise, the intersection of $\ell$ and $S$ at $a_i$ is called *multiple*. Note that the total number of intersections counted with their multiplicities equals the degree of $S$.

Let $p \in S$. Then a line $\ell = (p,q)$ is called *tangent* to the hypersurface $S = Z(F)$ at $p$ if either $\ell \subset S$ or $(S,\ell)_a \geqslant 2$. In other words, the line $\ell$ is tangent to $S$ at $p$ if the polynomial $F(p+tq) \in \Bbbk[t]$ either is the zero polynomial or has a multiple root at zero. It follows from formulas (2.35), (2.34) that the Taylor expansion of $F(p+tq)$ at $p$ starts with

$$F(p+tq) = t\binom{d}{1}\widetilde{F}(p^{n-1},q) + t^2\binom{d}{2}\widetilde{F}(p^{n-2},q^2) + \cdots .$$

Therefore, $\ell = (p,q)$ is tangent to $S$ at $p$ if and only if $\widetilde{F}(p^{n-1},q) = 0$. This is a straightforward generalization of Lemma 17.4 from Algebra I.

If $F(p^{n-1},x)$ does not vanish identically as a linear form in $x$, then the linear equation $F(p^{n-1},x) = 0$ on $x \in V$ defines a hyperplane in $\mathbb{P}(V)$ filled by the lines $(pq)$ tangent to $S$ at $p$. This hyperplane is called the *tangent space* to $S$ at $p$ and is denoted by

$$T_p = \left\{x \in \mathbb{P}(V) \mid \widetilde{F}(p^{n-1},x) = 0\right\}.$$

In this case, the point $p$ is called a *smooth point* of $S$. The hypersurface $S \subset \mathbb{P}(V)$ is called *smooth* if every point $p \in S$ is smooth.

If $F(p^{n-1},x)$ is the zero linear form in $x$, the hypersurface $S$ is called *singular* at $p$, and the point $p$ is called a *singular point* of $S$.

By formulas (2.34), the coefficients of the polynomial $F(p^{n-1},x) = \partial_x F(p)$, considered as a linear form in $x$, are equal to the partial derivatives of $F$ evaluated at the point $p$. Therefore, the singularity of a point $p \in S = Z(F)$ is expressed by the equations

$$\frac{\partial F}{\partial x_i}(p) = 0 \quad \text{for all } i,$$

in which case every line $\ell$ passing through $p$ has $(S,\ell)_p \geqslant 2$, i.e., is tangent to $S$ at $p$. Thus, the tangent lines to $S$ at $p$ fill the whole ambient space $\mathbb{P}(V)$ in this case.

If $q$ is either a smooth point on $S$ or a point outside $S$, then the polar polynomial

$$\mathrm{pl}_q F(x) = \widetilde{F}(q, x^{n-1})$$

does not vanish identically as a homogeneous polynomial of degree $n-1$ in $x$, because otherwise, all partial derivatives of $\mathrm{pl}_q F(x) = \widetilde{F}(q, x^{n-1})$ in $x$ would also vanish, and in particular,

$$\widetilde{F}(q^{n-1}, x) = \frac{\partial^{n-2}}{\partial q^{n-2}} \mathrm{pl}_q F(x) = 0$$

identically in $x$, meaning that $q$ would be a singular point of $S$, in contradiction to our choice of $q$. The zero set of the polar polynomial $\mathrm{pl}_q F \in S^{n-1}V^*$ is denoted by

$$\mathrm{pl}_q S \stackrel{\text{def}}{=} Z\left(\mathrm{pl}_q F\right) = \left\{x \in \mathbb{P}(V) \mid \widetilde{F}(q, x^{n-1}) = 0\right\} \tag{2.37}$$

and called the *polar hypersurface* of the point $q$ with respect to $S$. If $S$ is a quadric, then $\mathrm{pl}_q S$ is exactly the polar hyperplane of $q$ considered in Sect. 17.4.3 of Algebra I. As in that case, for a hypersurface $S$ of arbitrary degree, the intersection $S \cap \mathrm{pl}_q S$ coincides with the *apparent contour* of $S$ viewed from the point $q$, that is, with the locus of all points $p \in S$ such that the line $(pq)$ is tangent to $S$ at $p$.

More generally, for an arbitrary point $q \in \mathbb{P}(V)$, the locus of points

$$\mathrm{pl}_q^{n-r} S \stackrel{\text{def}}{=} \left\{x \in \mathbb{P}(V) \mid \widetilde{F}(q^{n-r}, x^r) = 0\right\}$$

is called the *rth-degree polar* of the point $q$ with respect to $S$ or the *rth-degree polar* of S at $q$ for $q \in S$. If the polynomial $\widetilde{F}(q^{n-r}, x^r)$ vanishes identically in $x$, we say that the $r$th-degree polar is *degenerate*. Otherwise, the $r$th-degree polar is a projective hypersurface of degree $r$. The linear[9] polar of $S$ at a smooth point $q \in S$ is simply the tangent hyperplane to $S$ at $q$,

$$T_q S = \mathrm{pl}_q^{n-1} S.$$

The quadratic polar $\mathrm{pl}_q^{n-2} S$ is the quadric passing through $q$ and having the same tangent hyperplane at $q$ as $S$. The cubic polar $\mathrm{pl}_q^{n-3} S$ is the cubic hypersurface passing through $q$ and having the same quadratic polar at $q$ as $S$, etc. The $r$th-degree polar $\mathrm{pl}_q^{n-2} S$ at a smooth point $q \in S$ passes through $q$ and has $\mathrm{pl}_q^{r-k} \mathrm{pl}_q^{n-r} S = \mathrm{pl}_q^{n-k} S$ for all $1 \leqslant k \leqslant r-1$, because

$$\begin{aligned} \mathrm{pl}_q^{r-k} \mathrm{pl}_q^{n-r} F(x) &= \widetilde{\mathrm{pl}_q^{n-r} F}\left(q^{r-k}, x^k\right) = \widetilde{F}\left(q^{n-r}, q^{r-k}, x^k\right) = \widetilde{F}\left(q^{n-k}, x^k\right) \\ &= \mathrm{pl}_q^{n-k} F(x). \end{aligned}$$

[9]That is, of first degree.

### 2.5.6 Linear Support of a Homogeneous Polynomial

Let $V$ be a finite-dimensional vector space and $f \in S^n V^*$ a polynomial. We write $\mathrm{Supp} f$ for the minimal[10] vector subspace $W \subset V^*$ such that $f \in S^n W$, and call this subspace the *linear support* of $f$. For $\mathrm{char}\,\Bbbk = 0$, the linear support of a polynomial $f$ coincides with the linear support of the symmetric tensor[11] $\widetilde{f} \in \mathrm{Sym}^n V^*$, the complete polarization of $f$. By Theorem 2.1, it is linearly generated by the images of the $(n-1)$-tuple contraction maps

$$c_t^J : V^{\otimes(n-1)} \to V^*, \quad t \mapsto c^{1,2,\dots,(n-1)}_{j_1,j_2,\dots,j_{n-1}}(t \otimes \widetilde{f}),$$

coupling all the $(n-1)$ factors of $V^{\otimes(n-1)}$ with some $n-1$ factors of $\widetilde{f} \in V^{*\otimes n}$ in the order indicated by the sequence $J = (j_1, j_2, \dots, j_{n-1})$. For the symmetric tensor $\widetilde{f}$, such a contraction does not depend on $J$ and maps every decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1}$ to the linear form on $V$ proportional to the $(n-1)$-tuple derivative $\partial_{v_1}\partial_{v_2}\cdots\partial_{v_{n-1}} f \in V^*$.

Therefore, $\mathrm{Supp}(f)$ is linearly generated by all $(n-1)$-tuple partial derivatives

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}}\frac{\partial^{m_2}}{\partial x_2^{m_2}}\cdots\frac{\partial^{m_d}}{\partial x_d^{m_d}} f(x), \text{ where } \sum m_\nu = n-1. \tag{2.38}$$

The coefficient of $x_i$ in the linear form (2.38) depends only on the coefficients of the monomial

$$x_1^{m_1}\cdots x_{i-1}^{m_{i-1}} x_i^{m_i+1} x_{i+1}^{m_{i+1}}\cdots x_d^{m_d}$$

in $f$. Writing the polynomial $f$ in the form

$$f = \sum_{\nu_1+\cdots+\nu_d=n} \frac{n!}{\nu_1!\nu_2!\cdots\nu_d!} a_{\nu_1\nu_2\dots\nu_d} x_1^{\nu_1}x_2^{\nu_2}\cdots x_d^{\nu_d} \tag{2.39}$$

turns the linear form (2.38) into

$$n! \cdot \sum_{i=1}^{d} a_{m_1\dots m_{i-1}(m_i+1)m_{i+1}\dots m_d} x_i. \tag{2.40}$$

Altogether, we get $\binom{n+d-2}{d-1}$ such linear forms, which are in bijection with the non-negative integer solutions $m_1, m_2, \dots, m_d$ of the equation $m_1+m_2+\cdots+m_d=n-1$.

[10] With respect to inclusions.

[11] See Sect. 2.2.3 on p. 25.

**Proposition 2.7** *Let $\Bbbk$ be a field of characteristic zero, $V$ a finite-dimensional vector space over $\Bbbk$, and $f \in S^nV^*$ a polynomial written in the form* (2.39) *in some basis of $V^*$. If $f = \varphi^n$ is the proper nth power of some linear form $\varphi \in V^*$, then the $d \times \binom{n+d-2}{d-1}$ matrix built from the coefficients of linear forms* (2.40) *has rank* 1. *In this case, there are at most n linear forms $\varphi \in V^*$ such that $\varphi^n = f$, and they differ from one another by multiplication by the nth roots of unity lying in $\Bbbk$. Over an algebraically closed field $\Bbbk$, the converse is true as well: if all the linear forms* (2.40) *are proportional, then $f = \varphi^n$ for some linear form $\varphi$, which is also proportional to the forms* (2.40).

*Proof* The equality $f = \varphi^n$ means that $\mathrm{Supp}(f) \subset V^*$ is the 1-dimensional subspace spanned by $\varphi$. In this case, all linear forms (2.40) are proportional to $\varphi$. Such a form $\psi = \lambda\varphi$ has $\psi^n = f$ if and only if $\lambda^n = 1$ in $\Bbbk$. Conversely, let all the linear forms (2.40) be proportional, and let $\psi \neq 0$ be one of them. Then $\mathrm{Supp}(f) = \Bbbk \cdot \psi$ is the 1-dimensional subspace spanned by $\psi$. Hence $f = \lambda\psi^n$ for some $\lambda \in \Bbbk$, and therefore, $f = \varphi^n$ for[12] $\varphi = \sqrt[n]{\lambda} \cdot \psi$. ☐

*Example 2.5 (Binary Forms of Rank* 1*)* We know from Example 11.6 of Algebra I that a homogeneous binary form of degree $n$,

$$f(x_0, x_1) = \sum_k a_k \cdot \binom{n}{k} \cdot x_0^{n-k} x_1^k,$$

is the proper $n$th power of some linear form $\alpha_0 x_0 + \alpha_1 x_1$ if and only if the ratio of sequential coefficients $a_i : a_{i+1} = \alpha_0 : \alpha_1$ does not depend on $i$. This is equivalent to the condition

$$\mathrm{rk}\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1,$$

which is expanded to a system of homogeneous quadratic equations $a_i a_{j+1} = a_{i+1} a_j$ in the coefficients of $f$. Proposition 2.7 leads to the same result, because the columns of the above matrix are exactly the coefficients of linear forms (2.40) divided by $n!$.

**Corollary 2.5** *The proper nth powers of linear forms $\varphi \in V^*$ form the projective algebraic variety*

$$\mathcal{V}_n \stackrel{\text{def}}{=} \{\varphi^n \mid \varphi \in V^*\} \subset \mathbb{P}(S^nV^*) \tag{2.41}$$

*in the space of all degree-n hypersurfaces*[13] *in $\mathbb{P}(V)$. This variety is described by the system of quadratic equations representing the vanishing of all* $2 \times 2$ *minors in the $d \times \binom{n+d-2}{d-1}$ matrix built from the coefficients of linear forms* (2.40). ☐

[12]Here we use that $\Bbbk$ is algebraically closed.

[13]See Sect. 11.3.3 of Algebra I.

**Definition 2.1 (Veronese Variety)** The projective algebraic variety (2.41) is called the *Veronese variety*.

**Exercise 2.19 (Veronese Embedding)** Verify that the prescription $\varphi \mapsto \varphi^n$ gives the well-defined injective map $\mathbb{P}(V^*) \hookrightarrow \mathbb{P}(S^n V^*)$ whose image coincides with the Veronese variety (2.41).

## 2.6 Polarization of Grassmannian Polynomials

It follows from Proposition 2.5 on p. 34 that for every Grassmannian polynomial $\omega \in \Lambda^n V^*$ over a field of characteristic zero, there exists a unique alternating tensor $\widetilde{\omega} \in \mathrm{Alt}^n V^* \subset V^{*\otimes n}$ mapped to $\omega$ under the factorization by the skew-commutativity relations $\pi_{\mathrm{sk}} : V^{*\otimes n} \twoheadrightarrow \Lambda^n V^*$. It can be viewed as the alternating $n$-linear form

$$\widetilde{\omega} : V \times V \times \cdots \times V \to \Bbbk, \quad \widetilde{\omega}(v_1, v_2, \dots, v_n) \overset{\text{def}}{=} \langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n,\, \widetilde{\omega} \,\rangle\,,$$

called the *complete polarization* of the Grassmannian polynomial $\omega \in \Lambda^n V^*$. If the covectors $x_i$ form a basis of $V^*$, then by formula (2.24) on p. 35, the complete polarization of the Grassmannian monomial $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n}$ equals

$$\frac{1}{n!}\, x_{\langle i_1, i_2, \dots, i_n \rangle} = \mathrm{alt}_n\left(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}\right). \tag{2.42}$$

The polarization of an arbitrary Grassmannian polynomial can be computed using this formula and the linearity of the polarization map

$$\pi_{\mathrm{sk}}^{-1} : \Lambda^n V^* \xrightarrow{\sim} \mathrm{Alt}^n V^*, \quad \omega \mapsto \widetilde{\omega}. \tag{2.43}$$

By Remark 2.1 on p. 34, this procedure is also well defined for infinite-dimensional vector spaces.

### *2.6.1 Duality*

Similarly to the symmetric case, for a finite-dimensional vector space $V$ over a field of characteristic zero, there exists a perfect pairing between the spaces $\Lambda^n V$ and $\Lambda^n V^*$ coupling polynomials $\tau \in \Lambda^n V$ and $\omega \in \Lambda^n V^*$ to the complete contraction of their complete polarizations $\widetilde{\tau} \in V^{\otimes n}$ and $\widetilde{\omega} \in V^{*\otimes n}$.

**Exercise 2.20** Convince yourself that the nonzero couplings between the basis monomials $e_I \in \Lambda^n V$ and $x_J \in \Lambda^n V^*$ are exhausted by

$$\langle\, e_I,\, x_I \,\rangle = 1/n!. \tag{2.44}$$

## 2.6.2 Partial Derivatives in an Exterior Algebra

By analogy with Sect. 2.5.4, the *derivative* of a Grassmannian polynomial $\omega \in \Lambda^n V^*$ along a vector $v \in V$ is defined by the formula

$$\partial_v \omega \stackrel{\text{def}}{=} \deg \omega \cdot \mathrm{pl}_v \omega,$$

where the *polarization map* $\mathrm{pl}_v : \Lambda^n V^* \to \Lambda^{n-1} V^*$, $\omega \mapsto \pi_{\mathrm{sk}}(v \llcorner \widetilde{\omega})$, is composed of the inner multiplication (2.31) preceded by the complete polarization (2.43) and followed by the quotient map $\pi_{\mathrm{sk}} : V^{*\otimes(n-1)} \twoheadrightarrow \Lambda^{n-1} V^*$. Thus, $\mathrm{pl}_v$ fits in the commutative diagram

$$\begin{tikzcd} V^{*\otimes n} \supset \mathrm{Alt}^n V^* \arrow[r, "v\llcorner"] \arrow[d, "\pi_{\mathrm{sk}}"'] & V^{*\otimes(n-1)} \arrow[d, "\pi_{\mathrm{sk}}"] \\ \Lambda^n V^* \arrow[r, "\mathrm{pl}_v"] & \Lambda^{n-1} V^* \end{tikzcd} \tag{2.45}$$

which is similar to the diagram from formula (2.33) on p. 39. Since $\mathrm{pl}_v \omega$ is linear in $v$, it follows that

$$\partial_v = \sum \alpha_i \, \partial_{e_i} \quad \text{for} \quad v = \sum \alpha_i e_i .$$

If $\omega$ does not depend on $x_i$, then certainly $\partial_{e_i} \omega = 0$. Therefore, a nonzero contribution to $\partial_v x_I$ is given only by the derivations $\partial_{e_i}$ with $i \in I$. Formula (2.42) implies that

$$\partial_{e_{i_1}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} = x_{i_2} \wedge x_{i_3} \wedge \cdots \wedge x_{i_n}$$

for every collection of indices $i_1, i_2, \ldots, i_n$, not necessarily increasing. Hence,

$$\begin{aligned} \partial_{e_{i_k}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} &= \partial_{e_{i_k}} (-1)^{k-1} x_{i_k} \wedge x_{i_1} \wedge \cdots \wedge x_{i_{k-1}} \wedge x_{i_{k+1}} \cdots x_{i_n} \\ &= (-1)^{k-1} \partial_{e_{i_k}} x_{i_k} \wedge x_{i_1} \wedge \cdots \wedge x_{i_{k-1}} \wedge x_{i_{k+1}} \cdots x_{i_n} \\ &= (-1)^{k-1} x_{i_1} \wedge \cdots \wedge x_{i_{k-1}} \wedge x_{i_{k+1}} \cdots x_{i_n} . \end{aligned}$$

In other words, the derivation along the basis vector that is dual to the $k$th variable from the left in the monomial behaves as $(-1)^{k-1} \frac{\partial}{\partial x_{i_k}}$, where the *Grassmannian partial derivative* $\frac{\partial}{\partial x_i}$ takes $x_i$ to 1 and annihilates all $x_j$ with $j \neq i$, exactly as in the symmetric case. However, the sign $(-1)^k$ in the previous formula forces the Grassmannian partial derivatives to satisfy the *Grassmannian Leibniz rule*, which differs from the usual one by an extra sign.

**Exercise 2.21 (Grassmannian Leibniz Rule)** Prove that for every homogeneous Grassmannian polynomial $\omega, \tau \in \Lambda V^*$ and vector $v \in V$, one has

$$\partial_v(\omega \wedge \tau) = \partial_v(\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \partial_v(\tau). \tag{2.46}$$

Since the Grassmannian polynomials are linear in each variable, it follows that $\partial_v^2 \omega = 0$ for all $v \in V$, $\omega \in \Lambda V$. The relation $\partial_v^2 = 0$ forces the Grassmannian derivatives to be skew commutative, i.e.,

$$\partial_u \partial_w = -\partial_w \partial_u \quad \forall\, u, w \in V.$$

### 2.6.3 Linear Support of a Homogeneous Grassmannian Polynomial

Let $V$ be a finite-dimensional vector space over a field $\Bbbk$ of characteristic zero. For the needs of further applications, in this section we switch between $V^*$ and $V$ and consider $\omega \in \Lambda^n V$. The *linear support* $\operatorname{Supp} \omega$ is defined to be the minimal (with respect to inclusions) vector subspace $W \subset V$ such that $\omega \in \Lambda^n W$. It coincides with the linear support of the complete polarization $\widetilde{\omega} \in \operatorname{Alt}^n V$, and is linearly generated by all $(n-1)$-tuple partial derivatives[14]

$$\partial_J \omega \stackrel{\text{def}}{=} \partial_{x_{j_1}} \partial_{x_{j_2}} \cdots \partial_{x_{j_{n-1}}} \omega = \frac{\partial}{\partial_{e_{j_1}}} \frac{\partial}{\partial_{e_{j_2}}} \cdots \frac{\partial}{\partial_{e_{j_{n-1}}}} \omega,$$

where $J = j_1 j_2 \dots j_{n-1}$ runs through all sequences of $n-1$ distinct indices from the set $\{1, 2, \dots, d\}$, $d = \dim V$. Up to a sign, the order of indices in $J$ is not essential, and we will not assume the indices to be increasing, because this simplifies the notation in what follows. Let us expand $\omega$ as a sum of basis monomials

$$\omega = \sum_I a_I e_I = \sum_{i_1 i_2 \dots i_n} \alpha_{i_1 i_2 \dots i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}, \tag{2.47}$$

where $I = i_1 i_2 \dots i_n$ also runs through the $n$-tuples of distinct but not necessarily increasing indices, and the coefficients $\alpha_{i_1 i_2 \dots i_n} \in \Bbbk$ are alternating in $i_1 i_2 \dots i_n$. Nonzero contributions to $\partial_J \omega$ are given only by the monomials $a_I e_I$ with $I \supset J$. Therefore, up to a common sign,

$$\partial_J \omega = \pm \sum_{i \notin J} \alpha_{j_1 j_2 \dots j_{n-1} i}\, e_i. \tag{2.48}$$

[14]Compare with Sect. 2.5.6 on p. 43.

**Proposition 2.8** *The following conditions on a Grassmannian polynomial* $\omega \in \Lambda^n V$ *written in the form* (2.47) *are equivalent:*

1. $\omega = u_1 \wedge u_2 \wedge \cdots \wedge u_n$ *for some* $u_1, u_2, \dots, u_n \in V$.
2. $u \wedge \omega = 0$ *for all* $u \in \mathrm{Supp}(\omega)$.
3. *for any two collections* $i_1 i_2 \dots i_{m+1}$ *and* $j_1 j_2 \dots j_{m-1}$ *consisting of* $n+1$ *and* $n-1$ *distinct indices, the following Plücker relation holds:*

$$\sum_{\nu=1}^{m+1} (-1)^{\nu-1} a_{j_1 \dots j_{m-1} i_\nu} a_{i_1 \dots \widehat{i_\nu} \dots i_{m+1}} = 0, \tag{2.49}$$

*where the hat in* $a_{i_1 \dots \widehat{i_\nu} \dots i_{m+1}}$ *means that the index* $i_\nu$ *should be omitted.*

*Proof* Condition 1 holds if and only if $\omega$ belongs to the top homogeneous component of its linear span, $\omega \in \Lambda^{\dim \mathrm{Supp}(\omega)} \mathrm{Supp}(\omega)$. Condition 2 means the same because of the following exercise.

**Exercise 2.22** Show that $\omega \in \Lambda U$ is homogeneous of degree $\dim U$ if and only if $u \wedge \omega = 0$ for $u \in U$.

The Plücker relation (2.49) asserts the vanishing of the coefficient of

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{m+1}}$$

in the product $\left(\partial_{j_1 \dots j_{m-1}} \omega\right) \wedge \omega$. In other words, (2.49) is the coordinate form of condition 2 written for the vector $u = \partial_{j_1 \dots j_{m-1}} \omega$ from the formula (2.48). Since these vectors linearly generate the subspace $\mathrm{Supp}(\omega)$, the whole set of Plücker relations is equivalent to condition 2. □

*Example 2.6 (The Plücker Quadric)* Let $n = 2$, $\dim V = 4$, and let $e_1$, $e_2$, $e_3$, $e_4$ be a basis of $V$. Then the expansion (2.47) for $\omega \in \Lambda^2 V$ looks like $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$, where the coefficients $a_{ij}$ form a skew-symmetric $4 \times 4$ matrix. The Plücker relation corresponding to $(i_1, i_2, i_3) = (2, 3, 4)$ and $j_1 = 1$ is

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0. \tag{2.50}$$

All other choices of $(i_1, i_2, i_3)$ and $j_1 \notin \{i_1, i_2, i_3\}$ lead to exactly the same relation.

**Exercise 2.23** Check this.

For $j_1 \in \{i_1, i_2, i_3\}$, we get the trivial equality $0 = 0$. Thus for $\dim V = 4$, the set of decomposable Grassmannian quadratic forms $\omega \in \Lambda^2 V$ is described by just one quadratic equation, (2.50).

**Exercise 2.24** Convince yourself that the Eq. (2.50) in $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$ is equivalent to the condition[15] $\omega \wedge \omega = 0$.

[15] Compare with Problem 17.20 of Algebra I.

### 2.6.4 Grassmannian Varieties and the Plücker Embedding

Given a vector space $V$ of dimension $d$, the set of all vector subspaces $U \subset V$ of dimension $m$ is denoted by $\mathrm{Gr}(m, V)$ and called the *Grassmannian*. When the origin of $V$ is not essential or $V = \Bbbk^d$, we write $\mathrm{Gr}(m, d)$ instead of $\mathrm{Gr}(m, V)$. Thus, $\mathrm{Gr}(1, V) = \mathbb{P}(V)$, $\mathrm{Gr}(\dim V - 1, V) = \mathbb{P}(V^*)$. The Grassmannian $\mathrm{Gr}(m, V)$ is embedded into the projective space $\mathbb{P}(\Lambda^m V)$ by means of the *Plücker map*

$$p_m : \mathrm{Gr}(m, V) \to \mathbb{P}(\Lambda^m V), \quad U \mapsto \Lambda^m U \subset \Lambda^m V, \tag{2.51}$$

sending every $m$-dimensional subspace $U \subset V$ to its highest exterior power $\Lambda^m U$, which is a 1-dimensional vector subspace in $\Lambda^m V$. If $U$ is spanned by vectors $u_1, u_2, \dots, u_m$, then $p_m(U) = u_1 \wedge u_2 \wedge \cdots \wedge u_m$ up to proportionality.

**Exercise 2.25** Check that the Plücker map is injective.

The image of the map (2.51) consists of all Grassmannian polynomials $\omega \in \Lambda^m V$ completely factorizable into a product of $m$ vectors. Such polynomials are called *decomposable*. By Proposition 2.8, they form a projective algebraic variety given by the system of quadratic Eq. (2.49) in the coefficients of the expansion (2.47).

*Example 2.7 (The Plücker Quadric, Geometric Continuation of Example 2.6)* For $\dim V = 4$, the Grassmannian $\mathrm{Gr}(2, 4) = \mathrm{Gr}(2, V)$ can be viewed as the set of lines $\ell = \mathbb{P}(U)$ in $\mathbb{P}_3 = \mathbb{P}(V)$. The Plücker embedding (2.51) maps a line $(ab) \subset \mathbb{P}_3$ to the point $a \wedge b \in \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ and establishes a bijection between the lines in $\mathbb{P}_3$ and the points of the smooth quadric

$$P = \left\{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\right\}$$

in $\mathbb{P}_5$, called the *Plücker quadric*.

### 2.6.5 The Grassmannian as an Orbit Space

The Grassmannian $\mathrm{Gr}(m, d)$ admits the following matrix description. Fix some basis $(e_1, e_2, \dots, e_d)$ in $V$. Given a vector subspace $U \subset V$ with a basis $(u_1, u_2, \dots, u_m)$, consider the $m \times d$ matrix $A_u$ whose $i$th row is formed by the coordinates of the vector $u_i$ in the chosen basis of $V$. Every other basis of $U$,

$$(w_1, w_2, \dots, w_m) = (u_1, u_2, \dots, u_m) \cdot C_{uw},$$

where $C_{wu} \in \mathrm{GL}_m(\Bbbk)$ is an invertible transition matrix, leads to the matrix $A_w$ expressed through $A_u$ by the formula

$$A_w = C_{uw}^t A_u.$$

**Exercise 2.26** Check this.

Therefore, the bases in $U$ are in bijection with the $m \times d$ matrices of rank $m$ forming one orbit of the action of $\mathrm{GL}_m(\Bbbk)$ on $\mathrm{Mat}_{m\times d}(\Bbbk)$ by left multiplication, $G : A \mapsto GA$ for $G \in \mathrm{GL}_m$, $A \in \mathrm{Mat}_{m\times d}$. Hence the Grassmannian $\mathrm{Gr}(m,d)$ can be viewed as the set of all $m \times d$ matrices of rank $m$ considered up to left multiplication by nondegenerate $m \times m$ matrices. Note that for $m = 1$, this agrees with the description of projective space $\mathbb{P}_{d-1} = \mathrm{Gr}(1,d)$ as the set of nonzero rows $(x_1, x_2, \dots, x_d) \in \Bbbk^d = \mathrm{Mat}_{1\times d}$ considered up to multiplication by nonzero constants $\lambda \in \Bbbk^* = \mathrm{GL}_1(\Bbbk)$. Thus, the matrix $A_u$ formed by the coordinate rows of some basis vectors $u_1, u_2, \dots, u_m$ in $U$ is the direct analogue of the homogeneous coordinates in projective space.

**Exercise 2.27 (Plücker Coordinates)** Verify that the coefficients $\alpha_{i_1 i_2 \dots i_n}$ in the expansion (2.47) written for $\omega = u_1 \wedge u_2 \wedge \cdots \wedge u_m$ are equal to the $m \times m$ minors of the matrix $A_u$.

These minors are called the *Plücker coordinates* of the subspace $U \subset V$ spanned by the vectors $u_i$.

*Example 2.8 (Segre Varieties Revisited, Continuation of Example 1.2)* Let $W = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ be a direct sum of finite-dimensional vector spaces $V_i$. For $k \in \mathbb{N}$ and nonnegative integers $m_1, m_2, \dots, m_n$ such that $\sum_\nu m_\nu = k$ and

$$0 \leqslant m_i \leqslant \dim V_i,$$

denote by $W_{m_1,m_2,\dots,m_n} \subset \Lambda^k W$ the linear span of all products $w_1 \wedge w_2 \wedge \cdots \wedge w_k$ formed by $m_1$ vectors from $V_1$, $m_2$ vectors from $V_2$, etc.

**Exercise 2.28** Show that the well-defined isomorphism of vector spaces

$$\Lambda^{m_1} V_1 \otimes \Lambda^{m_2} V_2 \otimes \cdots \otimes \Lambda^{m_n} V_n \xrightarrow{\sim} W_{m_1,m_2,\dots,m_n}$$

is given by the prescription $\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n \mapsto \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$, and verify that

$$\Lambda^k W = \bigoplus_{m_1,m_2,\dots,m_n} W_{m_1,m_2,\dots,m_n} \simeq \bigoplus_{m_1,m_2,\dots,m_n} \Lambda^{m_1} V_1 \otimes \Lambda^{m_2} V_2 \otimes \cdots \otimes \Lambda^{m_n} V_n.$$

Thus, the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ can be identified with the component $W_{1,1,\dots,1} \subset \Lambda^n W$. Under this identification, the decomposable tensors

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

go to the decomposable Grassmannian monomials $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. Therefore, the Segre variety from Example 1.2 on p. 6 is the intersection of the Grassmannian variety $\mathrm{Gr}(n, W) \subset \mathbb{P}(\Lambda^n W)$ with the projective subspace $\mathbb{P}(W_{1,1,\dots,1}) \subset \mathbb{P}(\Lambda^n W)$. In particular, the Segre variety is actually an algebraic variety described by the system of quadratic equations from Proposition 2.8 on p. 48 restricted to the linear subspace $W_{1,1,\dots,1} \subset \Lambda^n W$.

## Problems for Independent Solution to Chapter 2

**Problem 2.1** Let $V$ be a finite-dimensional vector space of over a field $\Bbbk$ of characteristic zero. Show that the following vector spaces are canonically isomorphic: **(a)** $\mathrm{Sym}^n(V^*)$, **(b)** $\mathrm{Sym}^n(V)^*$, **(c)** $(S^n V)^*$, **(d)** $S^n(V^*)$, **(e)** symmetric $n$-linear forms $V \times V \times \cdots \times V \to \Bbbk$, **(f)** functions $V \to \Bbbk$, $v \mapsto f(v)$, where $f$ is a homogeneous polynomial of degree $n$ in the coordinates of $v$ with respect to some basis in $V$.

**Problem 2.2** For the same $V$ and $\Bbbk$ as in the previous problem, show that the following vector spaces are canonically isomorphic: **(a)** $\mathrm{Alt}^n(V^*)$, **(b)** $\mathrm{Alt}^n(V)^*$, **(d)** $(\Lambda^n V)^*$, **(d)** $\Lambda^n(V^*)$, **(e)** alternating $n$-linear forms $V \times V \times \cdots \times V \to \Bbbk$.

**Problem 2.3** Which of the isomorphisms from the previous two problems hold

**(a)** over a field $\Bbbk$ of any positive characteristic?
**(b)** for an infinite-dimensional vector space $V$?

**Problem 2.4 (Aronhold's Principle)** Let $V$ be a finite-dimensional vector space over a field $\Bbbk$ of zero characteristic. Prove that the subspace of symmetric tensors $\mathrm{Sym}^n(V) \subset V^{\otimes n}$ is linearly generated by the proper $n$th tensor powers $v^{\otimes n} = v \otimes v \otimes \cdots \otimes v$ of vectors $v \in V$. Write the symmetric tensor

$$u \otimes w \otimes w + w \otimes u \otimes w + w \otimes w \otimes u \in \mathrm{Sym}^3(V)$$

as a linear combination of proper tensor cubes.

**Problem 2.5** Is there a linear change of coordinates that makes the polynomial

$$9x^3 - 15\,yx^2 - 6\,zx^2 + 9\,xy^2 + 18\,z^2x - 2\,y^3 + 3\,zy^2 - 15\,z^2y + 7\,z^3$$

depend on at most two variables?

**Problem 2.6** Ascertain whether the cubic Grassmannian polynomial

$$-\xi_1 \wedge \xi_2 \wedge \xi_3 + 2\,\xi_1 \wedge \xi_2 \wedge \xi_4 + 4\,\xi_1 \wedge \xi_3 \wedge \xi_4 + 3\,\xi_2 \wedge \xi_3 \wedge \xi_4$$

is decomposable. If it is, write down an explicit factorization. If not, explain why.

**Problem 2.7** Let $V$ be a vector space of dimension $n$. Fix some nonzero element $\eta \in \Lambda^n V$. Check that for all $k$, $m$ with $k + m = n$, the perfect pairing between $\Lambda^k V$ and $\Lambda^m V$ is well defined by the formula $\omega_1 \wedge \omega_2 = \langle\, \omega_1, \omega_2 \,\rangle \cdot \eta$. Given a vector $v \in V$, describe the linear operator $\Lambda^{m-1}V \to \Lambda^m V$ dual with respect to this pairing to the left multiplication by $v : \Lambda^k V \to \Lambda^{k+1}V$, $\omega \mapsto v \wedge \omega$.

**Problem 2.8** Verify that the Taylor expansion for the polynomial $\det(A)$ in the space of linear operators $A : V \to V$ has the following form:

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \lambda^p \mu^q \cdot \operatorname{tr}\left(\Lambda^p A \cdot \Lambda^q B^*\right),$$

where $\Lambda^p A : \Lambda^p V \to \Lambda^p V$, $v_1 \wedge v_2 \wedge \cdots \wedge v_p \mapsto A(v_1) \wedge A(v_2) \wedge \ldots \wedge A(v_p)$ is the $p$th exterior power of $A$ and $\Lambda^q B^* : \Lambda^p V \to \Lambda^p V$ is dual to $\Lambda^q B : \Lambda^q V \to \Lambda^q V$ with respect to the perfect pairing from Problem 2.7.

**Problem 2.9** Write $\mathbb{P}_N = \mathbb{P}(S^2 V^*)$ for the space of quadrics in $\mathbb{P}_n = \mathbb{P}(V)$, and $S \subset \mathbb{P}_N$ for the locus of all singular quadrics. Show that:

**(a)** $S$ is an algebraic hypersurface of degree $n+1$,
**(b)** a point $Q \in S$ is a smooth point of $S$ if and only if the corresponding quadric $Q \subset \mathbb{P}_n$ has just one singular point,
**(c)** the tangent hyperplane $T_Q S \subset \mathbb{P}_N$ to $S$ at such a smooth point $Q \in S$ is formed by all quadrics in $\mathbb{P}_n$ passing through the singular point of the quadric $Q \subset \mathbb{P}_n$.

**Problem 2.10** Find all singular points of the following plane projective curves[16] in $\mathbb{P}_2 = \mathbb{P}(\mathbb{C}^3)$: **(a)** $(x_0 + x_1 + x_2)^3 = 27\, x_0 x_1 x_2$, **(b)** $x^2 y + x y^2 = x^4 + y^4$, **(c)** $(x^2 - y + 1)^2 = y^2(x^2 + 1)$.

**Problem 2.11** Write an explicit rational parameterization[17] for the plane projective quartic

$$\left(x_0^2 + x_1^2\right)^2 + 3\, x_0^2 x_1 x_2 + x_1^3 x_2 = 0$$

using the projection of the curve from its singular point to some line.[18]

**Problem 2.12** For a diagonalizable linear operator $F : V \to V$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, find the eigenvalues of $F^{\otimes n}$ for all $n \in \mathbb{N}$.

**Problem 2.13** Prove that for every collection of linear operators

$$F_1, F_2, \ldots, F_m : V \to V$$

[16]Though the last two curves are given by their affine equations within the standard chart $U_0 \subset \mathbb{P}_2$, the points at infinity should also be taken into account.

[17]That is, a triple of rational functions $x_0(t), x_1(t), x_2(t) \in \Bbbk(t)$ such that $f(x_0(t), x_1(t), x_2(t)) = 0$ in $\Bbbk(t)$, where $f \in \Bbbk[x_0, x_1, x_2]$ is the equation of the curve.

[18]Compare with Example 11.7 and the proof of Proposition 17.6 in Algebra I.

and constants $\lambda_1, \lambda_2, \dots, \lambda_m \in \Bbbk$, one has $\lambda_1 F_1^{\otimes n} + \lambda_2 F_2^{\otimes n} + \cdots + \lambda_m F_m^{\otimes n} = 0$ for all $n \in \mathbb{N}$ only if $\lambda_i = 0$ for all $i$.

**Problem 2.14** Express the following quantities in terms of the coefficients of the characteristic polynomial of $F$ for an arbitrary linear operator $F : V \to V$: **(a)** $\operatorname{tr} F^{\otimes 2}$, **(b)** $\operatorname{tr} F^{\otimes 3}$, **(c)** $\det F^{\otimes 2}$, **(d)** $\det F^{\otimes 3}$, **(e)** the trace and determinant of the map $\mathrm{Ad}_F : \mathrm{End}(V) \to \mathrm{End}(V)$, $G \mapsto FGF^{-1}$, assuming that $F$ is invertible, **(f)** the trace and determinant of the map $S^2F : S^2V^* \to S^2V^*$ that sends a quadratic form $q : V \to \Bbbk$ to the composition $q \circ F : V \to \Bbbk$.

**Problem 2.15** Let $F$ be a diagonalizable linear operator on an $n$-dimensional vector space over a field $\Bbbk$ of characteristic zero. Express the eigenvalues of the operators

$$S^nF : v_1 v_2 \cdots v_n \mapsto F(v_1)F(v_2)\cdots F(v_n),$$

$$\Lambda^nF : v_1 \wedge v_2 \wedge \cdots \wedge v_n \mapsto F(v_1) \wedge F(v_2) \wedge \cdots \wedge F(v_n),$$

through the eigenvalues of $F$, and prove the following two identities in $\Bbbk[\![t]\!]$: **(a)** $\det(E - tF)^{-1} = \sum_{k\geqslant 0} \operatorname{tr}\big(S^kF\big) \cdot t^k$, **(b)** $\det(E + tF) = \sum_{k\geqslant 0} \operatorname{tr}\big(\Lambda^kF\big) \cdot t^k$.

**Problem 2.16 (Splitting Principle)** Prove that the answers you got in the previous two problems hold for nondiagonalizable linear operators $F$ as well. Use the following arguments, known as a *splitting principle*. Interpret the relation on $F$ you are going to prove as the identical vanishing of some polynomial with rational coefficients in the matrix elements $f_{ij}$ of $F$ considered as independent variables. Then prove the following claims:

**(a)** If a polynomial $f \in \mathbb{Q}[x_1, x_2, \dots, x_n]$ evaluates to zero at all points of some dense subset of $\mathbb{C}^n$, then $f$ is the zero polynomial. (Thus, it is enough to check that the relation being proved holds for some set of *complex* matrices dense in $\mathrm{Mat}_n(\mathbb{C})$.)

**(b)** The diagonalizable matrices are dense in $\mathrm{Mat}_n(\mathbb{C})$. Hint: every Jordan block[19] can be made diagonalizable by a small perturbation of the diagonal elements of the cell.

**(c)** The polynomial identity being proved is not changed under conjugation[20] $F \mapsto gFg^{-1}$ of the matrix $F = \big(f_{ij}\big)$ by any invertible matrix $g \in \mathrm{GL}_n(\mathbb{C})$. (Thus, it is enough to check the required identity only for the *diagonal* matrices.)[21]

[19] See Sect. 15.3.1 of Algebra I.

[20] This is clear if the identity in question expresses some basis-independent properties of the *linear operator* but not its matrix in some specific basis.

[21] Even for the diagonal matrices with distinct eigenvalues, because the conjugation classes of these matrices are dense in $\mathrm{Mat}_n(\mathbb{C})$ as well.

**Problem 2.17** Use the splitting principle to prove the Cayley–Hamilton identity $\chi_F(F) = 0$ by reducing the general case to the diagonal $F$.

**Problem 2.18** Prove that for every $F \in \mathrm{Mat}_{n^2}(\mathbb{C})$, one has $e^{F\otimes E+E\otimes F} = e^F \otimes e^F$ in $\mathrm{Mat}_{n^2}(\mathbb{C})$, where $E$ is the identity matrix.

**Problem 2.19*** Prove the identity $\log\det(E - A) = \operatorname{tr}\log(E - A)$ in the ring of formal power series with rational coefficients in the matrix elements $a_{ij}$ of the $n \times n$ matrix $A$. Show that for all small enough complex matrices $A \in \mathrm{Mat}_n(\mathbb{C})$, this identity becomes a true numerical identity in $\mathbb{C}$.

**Problem 2.20** Let $V$ be a vector space of dimension 4 over $\mathbb{C}$ and $g \in S^2V^*$ a nondegenerate quadratic form with the polarization $\widetilde{g} \in \mathrm{Sym}^2 V^*$. Write $G \subset \mathbb{P}_3 = \mathbb{P}(V)$ for the projective quadric defined by the equation $g(x) = 0$.

**(a)** Prove that there exists a unique symmetric bilinear form $\Lambda^2\widetilde{g}$ on the space $\Lambda^2 V$ such that

$$\Lambda^2\widetilde{g}(\, v_1 \wedge v_2,\, w_1 \wedge w_2\,) \stackrel{\text{def}}{=} \det\begin{pmatrix} \widetilde{g}(v_1, w_1) & \widetilde{g}(v_1, w_2) \\ \widetilde{g}(v_2, w_1) & \widetilde{g}(v_2, w_2) \end{pmatrix}$$

for all decomposable bivectors.

**(b)** Check that this form is symmetric and nondegenerate, and write its Gram matrix in the monomial basis $e_i \wedge e_j$ constructed from a $g$-orthonormal basis $e_1, e_2, e_3, e_4$ of $V$.

**(c)** Show that the Plücker embedding $\mathrm{Gr}(2, V) \hookrightarrow \mathbb{P}_3 = \mathbb{P}(V)$ from Example 2.7 on p. 49, which establishes a one-to-one correspondence between the lines in $\mathbb{P}_3 = \mathbb{P}(V)$ and the points of the Plücker quadric $P = \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\}$ in $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$, maps the tangent lines to $G$ bijectively to the intersection $P \cap \Lambda^2 G$, where $L^2G \subset \mathbb{P}_5$ is the quadric given by the symmetric bilinear form $\Lambda^2\widetilde{g}$.

**Problem 2.21 (Plücker–Segre–Veronese Interaction)** Let $U$ be a vector space of dimension 2 over $\mathbb{C}$. Consider the previous problem for the vector space $V = \operatorname{End} U$ and the quadratic form $g = \det$, whose value on an endomorphism $f : U \to U$ is $\det f \in \mathbb{C}$ and the zero set is the Segre quadric[22] $G \subset \mathbb{P}_3 = \mathbb{P}(V)$ consisting of endomorphisms of rank one.

**(a)** Construct canonical isomorphisms

$$S^2V \simeq \mathrm{Sym}^2 V \simeq \left(S^2U^* \otimes S^2U\right) \oplus \left(\Lambda^2 U^* \otimes \Lambda^2 U\right),$$

$$\Lambda^2 V \simeq \mathrm{Alt}^2 V \simeq \left(S^2U^* \otimes \Lambda^2 U\right) \oplus \left(\Lambda^2 U^* \otimes S^2 U\right).$$

**(b)** Show that the Plücker embedding sends two families of lines on the Segre quadric to the pair of smooth conics $P \cap \Lambda_+$, $P \cap \Lambda_-$ cut out of the Plücker

[22] See Example 1.3 on p. 8 and Example 17.6 from Algebra I.

quadric $P \subset \mathbb{P}\left(\Lambda^2 \operatorname{End}(U)\right)$ by the complementary planes

$$\Lambda_- = \mathbb{P}\left(S^2U^* \otimes \Lambda^2 U\right) \quad \text{and} \quad \Lambda_+ = \mathbb{P}\left(\Lambda^2 U^* \otimes S^2 U\right),$$

the collectivizations of components of the second decomposition in (a).

**(c)** Check that the two conics $P \cap \Lambda_-$ and $P \cap \Lambda_+$ in (b) are the images of the quadratic Veronese embeddings

$$\mathbb{P}(U^*) \hookrightarrow \mathbb{P}\left(S^2U^*\right) = \mathbb{P}\left(S^2U^* \otimes \Lambda^2 U\right), \quad \xi \mapsto \xi^2,$$
$$\mathbb{P}(U) \hookrightarrow \mathbb{P}\left(S^2U\right) = \mathbb{P}\left(\Lambda^2 U^* \otimes S^2 U\right), \quad v \mapsto v^2.$$

In other words, there is the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}_1^+ = \mathbb{P}(U) \arrow[rr, hook, "\text{Veronese}"'] & & \mathbb{P}(S^2U) = \Lambda_+ \arrow[d, hook] \\
\mathbb{P}_1^+ \times \mathbb{P}_1^- \arrow[u, "\pi_+"] \arrow[d, "\pi_-"] \arrow[r, "\text{Segre}", "\sim"'] & G \subset \mathbb{P}\operatorname{End}(U) \arrow[r, dashed, "\text{Plücker}"] & \mathbb{P}\begin{pmatrix} \Lambda^2U^* \otimes S^2U \\ \oplus \\ S^2U^* \otimes \Lambda^2 U \end{pmatrix} \\
\mathbb{P}_1^- = \mathbb{P}(U^*) \arrow[rr, hook, "\text{Veronese}"] & & \mathbb{P}(S^2U^*) = \Lambda_- \arrow[u, hook]
\end{tikzcd}$$

where the Plücker embedding is dashed, because it takes lines to points.

**Problem 2.22 (Hodge Star)** Under the conditions of Problem 2.20, verify that for every nondegenerate quadratic form $g$ on $V$, the linear operator $* : \Lambda^2 V \to \Lambda^2 V$, $\omega \mapsto \omega^*$, is well defined by the formula

$$\omega_1 \wedge \omega_2^* = \Lambda^2\widetilde{g}(\omega_1, \omega_2) \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \quad \forall\, \omega_1, \omega_2 \in \Lambda^2 V,$$

where $e_1$, $e_2$, $e_3$, $e_4$ is a $g$-orthonormal basis of $V$. Check that, up to a scalar complex factor of modulus one, the star operator does not depend on the choice of orthonormal basis. Describe the eigenspaces of the star operator and indicate their place in the diagram from Problem 2.21.

**Problem 2.23 (Grassmannian Exponential)** Let $V$ be a vector space over a field $\Bbbk$ of *arbitrary* characteristic. The *Grassmannian exponential* is defined for decomposable $\omega \in \Lambda^{2m}$ by the assignment $e^{\omega} \overset{\text{def}}{=} 1 + \omega$. For an arbitrary even-degree homogeneous Grassmannian polynomial $\zeta \in \Lambda^{2m}V$, we write $\zeta = \sum \omega_i$, where all $\omega_i$ are decomposable, and put $e^{f} \overset{\text{def}}{=} \prod e^{\omega_i}$. Verify that this product depends neither on an ordering of factors nor on the choice of expression[23] $\zeta = \sum \omega_i$. Prove that the exponential map $\Lambda^{\text{even}} V \hookrightarrow \Lambda^{\text{even}} V$, $\zeta \mapsto e^{\zeta}$, is an injective homomorphism of the additive group of even-degree Grassmannian polynomials to the multiplicative group of even-degree Grassmannian polynomials with unit constant term. Show that over a field of characteristic zero, $\partial_\alpha e^{\zeta} = e^{\zeta} \wedge \partial_\alpha \zeta$ for all $\alpha \in V^*$, and $e^{\zeta} = \sum_{k \geqslant 0} \frac{1}{k!} \zeta^{\wedge k}$.

**Problem 2.24** Let $V$ be a finite-dimensional vector space. Show that the subspaces

$$\mathcal{I}_{\text{sym}} \cap (V \otimes V) \subset V \otimes V \quad \text{and} \quad \mathcal{I}_{\text{skew}} \cap (V^* \otimes V^*) \subset V^* \otimes V^*,$$

which generate the ideals of the commutativity and skew-commutativity relations[24] $\mathcal{I}_{\text{sym}} \subset \mathsf{T}V$, $\mathcal{I}_{\text{skew}} \subset \mathsf{T}V^*$, are the annihilators of each other under the perfect pairing between $V \otimes V$ and $V^* \otimes V^*$ provided by the complete contraction.

**Problem 2.25 (Koszul and de Rham Complexes)** Let $e_1, e_2, \dots, e_n$ be a basis of a vector space $V$ over a field $\Bbbk$ of characteristic zero. Write $x_i$ and $\xi_i$ for the images of the basis vector $e_i$ in the symmetric algebra $SV$ and the exterior algebra $\Lambda V$ respectively. Convince yourself that there are well-defined linear operators

$$d \stackrel{\text{def}}{=} \sum_\nu \xi_\nu \otimes \frac{\partial}{\partial x_\nu} : \Lambda^k V \otimes S^m V \to \Lambda^{k+1} V \otimes S^{m-1} V,$$

$$\partial \stackrel{\text{def}}{=} \sum_\nu \frac{\partial}{\partial \xi_\nu} \otimes x_\nu : \Lambda^k V \otimes S^m V \to \Lambda^{k-1} V \otimes S^{m+1} V,$$

acting on decomposable tensors by the rules

$$d : \omega \otimes f \mapsto \sum_\nu \frac{\partial \omega}{\partial \xi_\nu} \otimes x_\nu \cdot f,$$

$$\partial : \omega \otimes f \mapsto \sum_\nu \xi_\nu \wedge \omega \otimes \frac{\partial f}{\partial x_\nu}.$$

Prove that neither operator depends on the choice of basis in $V$ and that both operators have zero squares, $d^2 = 0 = \partial^2$. Verify that their *s-commutator* $d\partial + \partial d$ acts on $\Lambda^k V \otimes S^m V$ as a homothety $(k+m) \cdot \mathrm{Id}$. Describe the *homology spaces* $\ker d / \operatorname{im} d$ and $\ker \partial / \operatorname{im} \partial$.

[23] Note that the decomposition of a Grassmannian polynomial into a sum of decomposable monomials is highly nonunique.

[24] See Sect. 2.3.1 on p. 26 and Sect. 2.3.3 on p. 29.

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