On Relation Between P-Matrices and Regularity of Interval Matrices

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Abstract We explore new results between P-matrix property and regularity of interval matrices. In particular, we show that an interval matrix is regular in and only if some special matrices constructed from its center and radius matrices are P-matrices. We also investigate the converse direction. We reduce the problem of checking P-matrix property to regularity of a special interval matrix. Based on this reduction, novel sufficient condition for a P-matrix property is derived, and its strength is inspected. We also state a new observation to interval P-matrices.

Keywords Interval matrix · P-matrix · Interval analysis · Linear complementarity

1 Introduction

Notation. The kth row of a matrix A is denoted as A_{k*}. The sign of a real r is defined as $\operatorname{sgn}(r) = 1$ if $r \geq 0$ and $\operatorname{sgn}(r) = -1$ otherwise; for vectors the sign is meant entrywise. For a vector y, the diagonal matrix with entries $y_1, \dots, y_n$ is denoted by D_y. Eventually, $e = (1, \dots, 1)^T$ stands for a vector of ones and $\rho(A)$ for the spectral radius of a matrix A.

Interval computation. An interval matrix is defined as

$$\mathbf{A} := \{A \in \mathbb{R}^{m \times n};\ \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}$ and $\overline{A}$, $\underline{A} \leq \overline{A}$, are given matrices. The midpoint and radius matrices are defined as

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

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The set of interval matrices of size $m \times n$ is denoted by $\mathbb{IR}^{m\times n}$. For definition of interval arithmetic see [8, 10], for instance.

We say that $\mathbf{A}$ is regular if every $A \in \mathbf{A}$ is nonsingular. Regularity of interval matrices is dealt with, e.g., in [5, 15, 16]. In particular, Rohn [16] presents forty equivalent characterizations. NP-hardness of checking regularity was proven by Poljak and Rohn [12, 13]. Sufficient conditions for checking regularity are surveyed in Rex and Rohn [14]. We recall the following one, due to Beeck [1].

Theorem 1 (Beeck [1]) *If* $\rho(|(A_c)^{-1}|A_\Delta) < 1$*, then* $\mathbf{A}$ *is regular.*

P-matrices. A square matrix is a P-matrix if all its principal minors are positive. P-matrices play an important role in linear complementarity problems [9, 22]

$$q + Mx \geq 0, \quad x \geq 0, \quad (q + Mx)^T x = 0.$$

Such a complementarity problem has a unique solution for each q if and only if M is a P-matrix. Since linear complementarity problems appear in so many situations (quadratic programming, bimatrix games, equilibria in specific economies, etc.), P-matrix property is of high importance.

Unfortunately, the problem of checking whether a given matrix is a P-matrix is known to be co-NP-hard [3, 7]. That is why diverse polynomially recognizable subclasses of P-matrices were studied; see [11, 24] and the references therein. Some of them are:

- positive definite matrices;
- M-matrices ($a_{ij} \leq 0\ \forall i, j$ and $A^{-1} \geq 0$);
- B-matrices ($\sum_{k=1}^n a_{ik} > 0$ and $\frac{1}{n}\sum_{k=1}^n a_{ik} > a_{ij}$ for $j \neq i$);
- H-matrices with positive diagonal entries (A is an H-matrix if $\langle A\rangle$ is an M-matrix, where $\langle A\rangle_{ii} = |a_{ii}|$ and $\langle A\rangle_{ij} = -|a_{ij}|, i \neq j$).

The related problem how to generate P-matrices was considered in [18, 24].

The following characterization of P-matrices is due to Fiedler and Pták [4].

Theorem 2 (Fiedler and Pták [4]) *A matrix* $A \in \mathbb{R}^{n\times n}$ *is a P-matrix if and only if for each vector* $x \neq 0$ *there is* i *such that* $x_i(Ax)_i > 0$*.*

The following relations between regularity of interval matrices and P-matrices are by Rohn [15].

Theorem 3 (Rohn [15]) *An interval matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *is regular if and only if for each* $y \in \{\pm 1\}^n$ *the matrix* $A_c - D_y A_\Delta$ *is nonsingular and* $(A_c - D_y A_\Delta)^{-1}(A_c + D_y A_\Delta)$ *is a P-matrix.*

Theorem 4 (Rohn [15]) *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *be regular. Then* $A_1^{-1}A_2$ *is a P-matrix for each* $A_1, A_2 \in \mathbf{A}$*.*

The following reduction of P-matrix property to interval matrix regularity comes from [19, 21].

Theorem 5 (Rump [21]) *Let $A \in \mathbb{IR}^{n\times n}$ with $A - I$ and $A + I$ nonsingular. Then A is a P-matrix if and only if $[(A-I)^{-1}(A+I) - I,\ (A-I)^{-1}(A+I) + I]$ is regular.*

Similar problem with convex combinations of rows or columns instead of full interval matrices was discussed in [6].

2 Results

Lemma 1 *Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ with A_c nonsingular. Then $\mathbf{A}$ is regular if and only if $I - A_c^{-1}R$ is a P-matrix for each $R \in [-A_\Delta, A_\Delta]$.*

Proof "Only if." Follows from Theorem 4 by choosing $A_1 := A_c$.

"If." Let $A \in \mathbf{A}$ be singular and denote $R := A_c - A \in [-A_\Delta, A_\Delta]$. Then there is $x \neq 0$ such that $Ax = (A_c - R)x = 0$, from which $(I - A_c^{-1}R)x = 0$. Therefore $I - A_c^{-1}R$ is singular and cannot be a P-matrix. □

Theorem 6 *Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ with A_c nonsingular. Then $\mathbf{A}$ is regular if and only if $I - A_c^{-1}D_yA_\Delta D_z$ is a P-matrix for each $y, z \in \{\pm 1\}^n$.*

Proof "Only if." Follows from Lemma 1.

"If." Suppose to the contrary that $\mathbf{A}$ is not regular. By Lemma 1, there is $R \in [-A_\Delta, A_\Delta]$ such that $I - A_c^{-1}R$ is not a P-matrix. Hence $I - R^TA_c^{-T}$ is not a P-matrix as well. By Theorem 2, there is $x \neq 0$ such that $x_i((I - R^TA_c^{-T})x)_i \leq 0$ for each i. Equivalently, $x_i^2 \leq x_i(R^TA_c^{-T}x)_i$ for each i. Define $y := \operatorname{sgn}(A_c^{-T}x)$ and $z := \operatorname{sgn}(x)$. Then

$$x_i^2 \leq x_i(R^TA_c^{-T}x)_i \leq x_i(z_i|R|^T|A_c^{-T}x|)_i \leq x_i(D_zA_\Delta^T|A_c^{-T}x|)_i = x_i(D_zA_\Delta^TD_yA_c^{-T}x)_i$$

for each i. Thus, $x_i((I - D_zA_\Delta^TD_yA_c^{-T})x)_i \leq 0$ for each i. This means that $I - D_zA_\Delta^TD_yA_c^{-T}$ is not a P-matrix, and also $I - A_c^{-1}D_yA_\Delta D_z$ is not a P-matrix. A contradiction. □

Remark. Since P-property is not changed by multiplying from the left and from the right by D_z, we can formulate the theorem also as follows: Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ with A_c nonsingular. Then $\mathbf{A}$ is regular if and only if $I - D_zA_c^{-1}D_yA_\Delta$ is a P-matrix for each $y, z \in \{\pm 1\}^n$.

Contrary to the characterization of regularity in Theorem 3, we have to use both diagonal matrices D_y and D_z. The following example illustrates it. Let

$$\mathbf{A} = \begin{pmatrix} [1,2] & [-1,1] \\ 1 & [1,2] \end{pmatrix}.$$

This interval matrix is not regular since it contains the all-one matrix. On the other hand, all matrices of the form $I - A_c^{-1}D_yA_\Delta$, $y \in \{\pm 1\}^n$, or of the form $I - A_c^{-1}A_\Delta D_z$, $z \in \{\pm 1\}^n$, are P-matrices.

Theorem 7 *Let $A \in \mathbb{R}^{n\times n}$. If $\alpha > 0$ is sufficiently small, then $P := \alpha A$ is a P-matrix if and only if $[(I-P)^{-1} - I,\ (I-P)^{-1} + I]$ is regular.*

Proof "If." By Theorem 4, regularity of $\mathbf{M} := [(I-P)^{-1} - I,\ (I-P)^{-1} + I]$ implies that $M_c^{-1}\underline{M}$ is a P-matrix. This matrix, however, reads $M_c^{-1}\underline{M} = (I - P)((I-P)^{-1} - I) = I - (I-P) = P$.

"Only if." By Theorem 6, have to verify that $I - (I-P)D_y I D_z$ is a P-matrix for each $y, z \in \{\pm 1\}^n$. Obviously, is it sufficient to verify matrices $I - (I-P)D_y$, $y \in \{\pm 1\}^n$, only. Without loss of generality suppose that $y = (-e^T, e^T)^T$, where the number of minus ones is k. Then $I - (I-P)D_y = PD_y + (I - D_y)$ has the form of

$$\left(\begin{array}{c|c} - & + \\ \hline - & + \end{array}\right) + \left(\begin{array}{c|c} 2I_k & 0 \\ \hline 0 & 0 \end{array}\right).$$

By the column linearity of determinants (applied on the first k columns), we can express the determinant of this matrix as

$$\sum_{J\subseteq\{1,\dots,k\}} 2^{|J|}(-1)^{k-|J|}\alpha^{n-|J|}\det(A_J), \tag{1}$$

where A_J denotes the principal submatrix of A obtained by removing the rows and columns indexed by J. So, as $\alpha \to 0$, the dominant term in the summation is that for $J = \{1, \dots, k\}$ and it draws

$$2^k\alpha^{n-k}\det(A_J).$$

Since A is a P-matrix, this term is positive, as well as the whole summation. Thus, $I - (I-P)D_y$ has the positive determinant. Its principal minors are positive for the same reasons. Therefore, $I - (I-P)D_y$ is a P-matrix. □

Remark 1 (*Estimation of α*) Here we estimate from below the sufficient value of α. This value should be small enough to ensure that (1) is positive, where $k > 0$ (case $k = 0$ holds trivially). That is,

$$\sum_{J\subseteq\{1,\dots,k\}} 2^{|J|}(-1)^{k-|J|}\alpha^{k-|J|}\det(A_J) > 0.$$

This will be satisfied if

$$2^k\det(A_{\{1,\dots,k\}}) > \sum_{J\subsetneq\{1,\dots,k\}} 2^{|J|}\alpha^{k-|J|}\det(A_J).$$

Denote

$$m_1 = \min_{J \subsetneq \{1,\dots,k\}} \det(A_J),$$
$$m_2 = \max_{J \subsetneq \{1,\dots,k\}} \det(A_J).$$

Now, we can write a stronger inequality

$$\begin{aligned} 2^k m_1 &> m_2 \sum_{J \subsetneq \{1,\dots,k\}} 2^{|J|} \alpha^{k-|J|} \\ &= m_2(\alpha + 2)^k - m_2 2^k. \end{aligned}$$

From this, we have

$$(\alpha + 2)^k < 2^k (1 + m_1/m_2),$$

or,

$$\alpha < -2 + 2\sqrt[k]{1 + m_1/m_2}.$$

Due to overestimations, it suffices to take

$$\alpha := -2 + 2\sqrt[n]{1 + m_1/m_2}.$$

This value can be further simplified. By using concavity of log function and $e^x \geq x + 1$, we have

$$\begin{aligned} -2 + 2\sqrt[n]{1 + m_1/m_2} &= -2 + 2\exp\left(\frac{1}{n}\log(1 + m_1/m_2)\right) \\ &\geq -2 + 2\exp\left(\frac{1}{n}\big((1 - m_1/m_2)\log 1 + (m_1/m_2)\log 2\big)\right) \\ &= -2 + 2\exp\left(\frac{1}{n}(m_1/m_2)\log 2\right) \\ &\geq -2 + 2 + \frac{2}{n}(m_1/m_2)\log 2 = \frac{2}{n}(m_1/m_2)\log 2. \end{aligned}$$

The minimal and maximal determinants m_1 and m_2 can be estimated as follows. By Hadamard's inequality, we have

$$m_2 \leq \prod_{i=1}^{n} \|A_{i*}\|_2.$$

To estimate m_1 is a more involved task. For any nonsingular matrix $M \in \mathbb{R}^{n\times n}$, its determinant (and also sub-determinant) is bounded by

$$\det(M) = \det(M^{-1})^{-1} \geq \rho(M^{-1})^{-n} \geq \sigma_{\max}(M^{-1})^{-n} = \sigma_{\min}(M)^n.$$

This bound, however, can be very conservative. Anyway, we arrive at the possible value of

$$\alpha := \frac{2\log 2}{n} \cdot \frac{\sigma_{\min}(M)^n}{\prod_{i=1}^n \|A_{i*}\|_2}.$$

2.1 Sufficient Conditions for P-Matrices

Characterizations of P-matrix property from the previous section enables us to derive new sufficient conditions.

Theorem 8 *The matrix $A \in \mathbb{R}^{n\times n}$ is a P-matrix provided $A - I$ and $A + I$ are nonsingular and*

$$\rho(|(A+I)^{-1}(A-I)|) < 1. \tag{2}$$

Proof Let $A - I$ and $A + I$ be nonsingular. By Theorem 5, A is a P-matrix if and only if $[(A-I)^{-1}(A+I) - I,\ (A-I)^{-1}(A+I) + I]$ is regular. By employing the Beeck sufficient condition for regularity (Theorem 1), we arrive at the final form. □

Obviously, this condition is incomparable with positive definiteness. Moreover, it is also incomparable with M-matrix and H-matrix conditions. For example, the matrix

$$\begin{pmatrix} 46 & -19 \\ -33 & 14 \end{pmatrix}$$

is an M-matrix (and thus also H-matrix), but the condition (2) is not satisfied since the spectral radius is greater than 1.084 (verified by `versoft` [17]). On the other hand, the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 10 \end{pmatrix}$$

is neither an M-matrix nor an H-matrix, but (2) is satisfied with the spectral radius less than 0.955.

Theorem 9 *The matrix $A \in \mathbb{R}^{n\times n}$ is a P-matrix provided for $I - \alpha A$ is nonsingular and $\rho(|I - \alpha A|) < 1$ for some $\alpha > 0$.*

Proof It follows again from the Beeck condition applied to $[(I-\alpha A)^{-1}-I,\ (I-\alpha A)^{-1}+I]$ and using Theorem 7. □

The latter condition is not new in the essence. If $\rho(|I-\alpha A|)<1$, then $I-|I-\alpha A|$ is an M-matrix, so also $I-|I-\alpha A|-\mathrm{diag}(I-\alpha A)+\mathrm{diag}(|I-\alpha A|)$ is an M-matrix. The matrix $I-|I-\alpha A|-\mathrm{diag}(I-\alpha A)+\mathrm{diag}(|I-\alpha A|)$ is the comparison matrix of $I-(I-\alpha A)=\alpha A$, so αA is an H-matrix. Moreover, αA has positive diagonal since otherwise if $(\alpha A)_{ii}\leq 0$ for some i, then $|I-\alpha A|_{ii}\geq 1$ and so $\rho(|I-\alpha A|)\geq 1$. Therefore, the sufficient condition is weaker than checking if A is an H-matrix.

2.2 *Interval P-Matrices*

An interval matrix $A\in\mathbb{IR}^{n\times n}$ is called *an interval P-matrix* if each $A\in\mathbf{A}$ is a P-matrix [2, 7, 20]. A more general concept of P-matrix sets was investigated by Song and Gowda [23]. The following characterization of interval P-matrices is due to Białas and Garloff [2], see also [7].

Theorem 10 (Białas and Garloff [2]) $\mathbf{A}\in\mathbb{IR}^{n\times n}$ *is an interval P-matrix if and only if* $A_c-D_zA_\Delta D_z$ *is a P-matrix for each* $z\in\{\pm1\}^n$.

As a direct consequence we have:

Corollary 1 *Let* $\mathbf{A}\in\mathbb{IR}^{n\times n}$ *such that* $A_c=D$ *is diagonal. Then* $\mathbf{A}$ *is an interval P-matrix if and only if* $\underline{A}$ *is a P-matrix.*

Proof We have that $\mathbf{A}\in\mathbb{IR}^{n\times n}$ is an interval P-matrix if and only if for each $z\in\{\pm1\}^n$ the matrix $A_c-D_zA_\Delta D_z=D-D_zA_\Delta D_z$ is a P-matrix. This matrix is a P-matrix if and only if $D_zDD_z-A_\Delta=D-A_\Delta=\underline{A}$ is. □

Even though the assumption $A_c=D$ is strong, it might possibly help for checking interval P-matrix property. In a similar way, interval linear equation are often preconditioned such that the midpoint matrix becomes an identity matrix since this case is much easier to solve.

Another special case, reducing the interval P-matrix property to P-property of $\underline{A}$ only, is the following.

Corollary 2 *Let* $\mathbf{A}\in\mathbb{IR}^{n\times n}$ *such that* $A_\Delta=D$ *is diagonal. Then* $\mathbf{A}$ *is an interval P-matrix if and only if* $\underline{A}$ *is a P-matrix.*

Proof We have that $\mathbf{A}\in\mathbb{IR}^{n\times n}$ is an interval P-matrix if and only if for each $z\in\{\pm1\}^n$ the matrix $A_c-D_zA_\Delta D_z=A_c-D_zDD_z=A_c-D=\underline{A}$ is a P-matrix. □

While Theorem 10 presents a reduction of interval to real P-matrix property, in the theorem below, we show a direct reduction to an elementary formula.

Theorem 11 $\mathbf{A} \in \mathbb{IR}^{n \times n}$ *is an interval P-matrix if and only if*

$$\det(D_{e-|y|} + D_{|y|} A_c D_{|z|} - D_y A_\Delta D_z) > 0 \tag{3}$$

for each $y, z \in \{0, \pm 1\}^n$ *such that* $|y| = |z|$.

Proof "Only if". This is obvious since $D_{e-|y|} + D_{|y|} A_c D_{|z|} - D_z A_\Delta D_z$ is a block diagonal matrix with entries either ones, or a principal submatrix of some $A \in \mathbf{A}$. Due to P-matrix property, this principal minor is positive.

"If". We use the result from Rohn [16] that an interval matrix $\mathbf{M} \in \mathbb{IR}^{k \times k}$ has all determinants positive, that is, $\det(M) > 0\ \forall M \in \mathbf{M}$, if and only if $\det(M_c - D_y A_\Delta D_z) > 0$ for all $y, z \in \{\pm 1\}^k$. Now, $\mathbf{A}$ is an interval P-matrix if and only if for each $A \in \mathbf{A}$, each minor of A is positive. A minor of A can be expressed as $\det(D_{e-s} + D_s A D_s)$ for some $s \in \{0, 1\}^n$. Thus, we have to show that for each $s \in \{0, 1\}^n$, all determinants of $D_{e-s} + D_s \mathbf{A} D_s$ are positive. By the above reasoning, this is equivalent to $\det(D_{e-s} + D_s A_c D_s - D_y D_s A_\Delta D_s D_z) > 0$ for all $y, z \in \{\pm 1\}^n$. When $s_i = 0$, the values of y_i and z_i play no role, so we can set $s = |y|$ and arrive at the resulting form of (3). □

Theorem 12 *The number of determinants in* (3) *is* 5^n.

Proof By the binomial formula, the number of determinants in (3) is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k 2^k = \sum_{k=0}^{n} \binom{n}{k} 4^k 1^{n-k} = (4+1)^n = 5^n,$$

where k denotes the number of nonzero entries of y (or z), $\binom{n}{k}$ gives the number of vectors in $\{0, \pm 1\}^n$ with k nonzero entries, and 2^k counts the number of possibilities for y (and z) when the number of nonzero entries is k. □

3 Conclusion

We reviewed relations between P-matrix property and regularity of interval matrices. We also proposed some new observations and links. In particular, a reduction of interval matrix regularity to P-property and vice versa. As a consequence, new sufficient conditions for P-matrices were stated.

Some new open problems arised as well, e.g., determining a sharper estimation of α from Remark 1. Efficient utilization of Corollary 1 for interval P-matrix property checking is a challenging problem, too.

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