Preventive Maintenance of Consecutive Multi-unit Systems

Won Young Yun and Alfonsus Julanto Endharta

Abstract This chapter deals with a preventive maintenance problem for consecutive-k-out-of-n and connected-(r, s)-out-of-(m, n): F systems. The system failure probability of two types of multi-unit systems is analytically derived by utilizing the system failure paths. Dependence mechanisms between components in the systems are illustrated and the system failure paths under dependence models are shown. The expected cost rates of corrective maintenance, age preventive maintenance, and condition-based maintenance policies are estimated. The optimal maintenance policies to minimize the expected cost rates in two preventive maintenance models are obtained and three maintenance policies are compared by numerical examples.

Keywords Consecutive multi-unit system · Preventive maintenance · Expected cost rate

1 Introduction

A consecutive-k-out-of-n: F system is one type of multi-unit systems consisting of n components arranged linearly or circularly. The system fails if and only if there are at least k consecutive failed components [4, 6, 8]. This type of system structure is used the design of integrated circuits, microwave relay stations in telecommunications, oil pipeline systems, vacuum systems in accelerators, computer ring networks (k loop), and spacecraft relay stations.

The reliability of consecutive-k-out-of-n: F systems have been reviewed in several papers. The reliability of consecutive-k-out-of-n: F systems were studied first by Kontoleon [27]; however, the system name of consecutive-k-out-of-n:

W.Y. Yun · A.J. Endharta (✉)
Department of Industrial Engineering, Pusan National University,
Busan, South Korea
e-mail: endharta_aj@pusan.ac.kr

W.Y. Yun
e-mail: wonyun@pusan.ac.kr

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F system originated from Chiang and Niu [8]. A mirror image of this system structure, namely consecutive-k-out-of-n: G systems have been studied and the system works if and only if there are at least k consecutive working components [24, 28, 48].

A closed form of the reliability function of a linear consecutive-2-out-of-n: F system has been developed by Chang and Niu [8]. They considered a constant failure probability p of components and obtained the exact reliability value. Another reliability function of the similar system has been proposed by Bollinger and Salvia [4] and involved the term of $N(j, k, n)$ which was interpreted as the number of binary numbers with length n containing exactly j ones with at most $k - 1$ consecutive ones. A combinatorial approach was considered for $N(j, 3, n)$ and the recursive method for the general $N(j, k, n)$ was proposed by Derman et al. [10]. A spreadsheet table was utilized in deriving $N(j, k, n)$ by Bollinger [3]. A closed form of general $N(j, k, n)$ has been also derived [21, 23, 31].

Circular structure of this system was introduced and the system reliability was provided by Derman et al. [10]. A recursive model to estimate the reliability of circular-type systems was developed by Du and Hwang [13]. A closed-form expression of the reliability of circular consecutive-k-out-of-n: F systems for $n \leq 2k + 1$ and when the components were identical with constant failure probability has been derived by Zuo and Kuo [48].

Some papers calculated the dynamic system reliability of consecutive-k-out-of-n systems in which the time to component failures is a random variable and follows a certain lifetime distribution (for example, Exponential or Weibull distributions). The expressions of the system reliability, mean time to system failure (MTTF), and the rth moment of the lifetime of consecutive-k-out-of-n: F systems can be seen in Kuo and Zuo [29].

Most existing papers related to consecutive-k-out-of-n: F systems assumed that the failures of components are independent but some papers considered dependence between component failures. Shantikumar [42] considered the linear-type system with exchangeable components, which are identical but not independent and the failure rate of operating components increases when there is one or more component failures. Papastavridis [38] considered the circular-type system with dependent components. Papastavridis and Lambiris [39] introduced another dependence mechanism called $(k - 1)$-step Markov dependence. They assumed that the reliability of component i is dependent on the states of k consecutive components besides that component i. Fu [20] considered this mechanism in a practical problem of oil transportation by pumps. Load-sharing model is a special case of Markov dependence model with $(n - 1)$-step.

A system design problem can also be considered in consecutive-k-out-of-n: F systems. In the design phase for consecutive-k-out-of-n: F systems, we should determine the system structure optimally; for example, the system parameters, such as k, n, and the component reliability, p_i for $1 \leq i \leq n$. A system design problem for a circular consecutive-k-out-of-n: F system with $(k - 1)$-step Markov dependent components has been considered in Yun et al. [46] and the optimal k and n are optimized minimizing the expected cost rate when only corrective maintenance is considered. A branch and bound algorithm was developed and used in the estimation of the system reliability for cases where the system size is small. For the cases where the system size is large, a simulation method is developed. Later, a system design problem for linear and circular consecutive-k-out-of-n: F systems with load-sharing dependent components were considered in Yun et al. [47].

Another design problem is the optimal allocation problem of given reliability values to the positions or components of the system [29, 30]. Birnbaum's importance measure was utilized in the reliability importance of component i in a linear consecutive-k-out-of-n: F system [37]. The expression for the circular-type system was also provided by Griffith and Govindarajalu [22]. For the liner-type system, the importance measure increases as the component locates nearer to the center of the system and this situation occurs for cases where the system contains only i.i.d components. The general deduction is that the optimal design is to assign the least reliable component to position 1, the next least reliable component to position n, the most reliable component to position 2, the next most reliable component to position $n - 1$, and so on [12, 34]. However, for linear consecutive-k-out-of-n: F system, the mentioned optimal design cannot be generalized for all k values [35]. The optimal design is called as the invariant optimal design if the optimal solutions depend only on the ranking of the reliability values of components. Otherwise, it is called as the variant optimal design.

Another optimization problem is the maintenance problem. This problem aims to determine how and when the maintenance should be performed in order to avoid the system failures. Basic maintenance policies, such as age replacement, periodic replacement, and block replacement policies can be seen in Barlow and Proschan [1] or Nakagawa [32]. An algorithm has been developed to select the maintenance policy based on the critical component policy (CCP) for the consecutive-k-out-of-n: F systems [19]. Based on that study, the failed components are replaced if and only if the failed components are contained within the critical component sets. An age replacement model for linear and circular consecutive-k-out-of-n: F systems with load-sharing dependent components has been considered by Yun et al. [47] and a condition-based maintenance model for the linear-type systems has been considered by Endharta and Yun [15].

Salvia and Lasher [41] has introduced the multidimensional structure of the systems [29]. Two- or the three-dimensional system is a square or cubic grid of side n. Thus, the system contains n^2 or n^3 components and the system fails if and only if there is at least a square or a cube of k failed components. The more general two-dimensional system has been introduced by Boehme et al. [2] and later is called as a linear or circular connected-(r, s)-out-of-(m, n): F system. The system contains the components which are arranged into a rectangular pattern with m rows and n columns and it fails if and only if there is at least one grid of r rows and s columns comprising failed components only.

A recursive method for the reliability estimation of a linear connected-(r, s)-out-of-(m, n): F system has been constructed by Yamamoto and Miyakawa [43]. By using the method, the lower and upper bounds of the system reliability can be obtained. The component allocation and design problem showed that the existence of invariant design is very limited even in the case of superfluous consecutive systems, as well as the two-dimensional systems [25]. An age replacement model for a connected-(r, s)-out- of-(m, n): F system has been considered by Yun et al. [45], in which they developed a simulation procedure to estimate the expected cost rate, and genetic algorithm to obtain the optimal time interval. A condition-based maintenance model for one- and two-dimensional systems has been considered by Yun and Endharta [44], in which they assume a continuous monitoring and a simulation method is developed to obtain the near optimal solutions.

In this chapter, we review the maintenance problem in consecutive-k-out-of-n: F systems and connected-(r, s)-out-of-(m, n): F systems. Section 2 shows the multi-unit systems considered, such as consecutive-k-out-of-n and connected-(r, s)-out-of-(m, n): F systems. Section 3 presents the component dependence models and Sect. 4 presents the maintenance models (corrective maintenance, age-based preventive maintenance, and condition-based maintenance models). Section 5 shows the numerical examples, including comparison results of the expected cost rates of three maintenance policies. The study is concluded in Sect. 6.

2 Multi-unit Systems

In this section, consecutive-k-out-of-n and connected-(r, s)-out-of-(m, n): F systems are introduced and minimal cut sets of the systems are used to obtain the system reliability.

2.1 *Consecutive-k-Out-of-n: F Systems*

A system which fails if and only at least k of the n components fail is called a k-out-of-n: F system. A series and parallel systems are special case of this system, where the series system is a 1-out-of-n: F system and the parallel system is an n-out-of-n: F system. Another developed structure of this system is consecutive-k-out-of-n: F system. A consecutive-k-out-of-n: F system is a system which fails if and only if at least k consecutive components fail among n components.

There are two types of consecutive-k-out-of-n: F systems: linear and circular consecutive-k-out-of-n: F systems. Thus, the linear or circular consecutive-k-out-of-n: F systems consist of n components arranged linearly or circularly and fail if and only if at least k consecutive components fail. Linear and circular systems are illustrated in Fig. 1.

Definition 1 Cut sets and minimal cut sets [40]

A cut set K is a set of components which by failing causes the system to fail. A cut set is said to be minimal if it cannot be reduced without losing its status as a cut set.

The number of minimal cut sets for the system with linear form is $n - k + 1$ and the minimal cut sets are

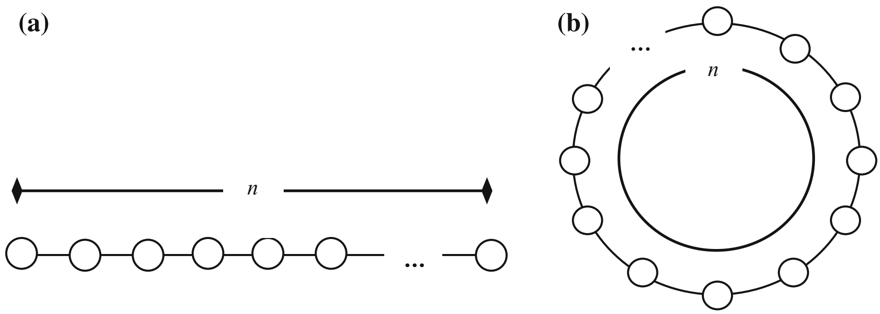


Fig. 1 **a** Linear and **b** circular consecutive-k-out-of-n: F systems

$$K_1 = \{1, 2, \ldots, k\}, K_2 = \{2, 3, \ldots, k+1\}, \ldots, K_{n-k+1} \\ = \{n-k+1, n-k+2, \ldots, n\}.$$

Since the component 1 and component n in the system with circular form are connected, there are n minimal cut sets for this system and they are

$$K_1 = \{1, 2, \ldots, k\}, K_2 = \{2, 3, \ldots, k+1\}, \ldots, K_n = \{n, 1, \ldots, k-1\}.$$

Based on this information, with the same number of components, n, the system with circular type tends to fail quicker than the system with linear type because the probability of system failure increases as the number of minimal cut set increases.

In order to estimate the system unreliability or the probability of the system failure, the system failure events must be known. Since the system consists of multiple components, one way to know the system failure events is to arrange the sequences of component failures in the system. Sequences from the beginning to the system failure events are the system failure paths.

The system state consists of the states of components, where the working component is 1 and failed component is 0. Thus, in the beginning, the system state is represented as a vector of ones with size n. The component fails one at a time and the sequences are called as paths. Accordingly, at the system failure event (the last step), the system state can be represented as a vector with size n, where there is at least k consecutive zeros (failed components), which are components in the minimal cut sets. The number of system failure paths is denoted as P. System failure paths are illustrated in Tables 1 and 2 for linear and circular consecutive 2-out-of-3: F systems, respectively.

Number of system failure paths P for the circular type can be reduced by assuming that the first failed component is Component 1. Thus, system failure paths in Table 2 changes into Table 3. Denote the reduced number of system failure paths in the circular-type system by p.

Table 1 System failure paths for linear consecutive 2-out of-3: F system

Path j	Step i						
	0		1		2		3
1	**1**11	→	0**1**1	→	001		
2	**1**11	→	01**1**	→	0**1**0	→	000
3	1**1**1	→	**1**01	→	001		
4	1**1**1	→	10**1**	→	100		
5	11**1**	→	**1**10	→	0**1**0	→	000
6	11**1**	→	1**1**0	→	100		

Table 2 System failure paths for circular consecutive 2-out of-3: F system

Path	Step				
	0		1		2
1	**1**11	→	0**1**1	→	001
2	**1**11	→	01**1**	→	010
3	1**1**1	→	**1**01	→	001
4	1**1**1	→	10**1**	→	100
5	11**1**	→	**1**10	→	010
6	11**1**	→	1**1**0	→	100

Table 3 Reduced number of paths for circular consecutive 2-out of-3: F system

Path	Step				
	0		1		2
1	**1**11	→	0**1**1	→	001
2	**1**11	→	01**1**	→	010

It is assumed that the components are identical and the failure times of the components follow an exponential distribution with rate λ. Thus, the system failure time distribution can be derived as follows. First, a lemma is introduced.

Lemma 1 [5]

Let Y_1, Y_2, …, Y_m be exponentially distributed random variables with failure rates h_1, h_2, …, h_m, and let $Z = \min(Y_1, Y_2, \ldots, Y_m)$. Then Z is also exponentially distributed with failure rate $\sum h_i$ and $\Pr\{Z = Z_i\} = h_i/\sum h_i$.

Lemma 1 provides the probability of one component fails among all working components in step i.

Lemma 1 gives the probability of selecting the component which will fail in step i among working components. The sum of failure rates of working components after ith failure and the failure rate of the ith failed component in path j are denoted as α_{ji} and β_{ji}, respectively.

Define T_j and T, respectively, as the time to complete path j and the time to system failure. The probability that the system follows path j, which is denoted as π_j, can be estimated as

$$\pi_j = \Pr\{T = T_j\} \tag{1}$$

Define X_i is the time between $(i-1)$th failure and ith failure in the system and X_{ji} as the time between $(i-1)$th failure and ith failure in path j, thus

$$\begin{aligned}\pi_j &= \Pr\{X_1 = X_{j1}, X_2 = X_{j2}, \ldots, X_{N_j} = X_{jN_j}\}\\ &= \Pr\{X_1 = X_{j1}\} \cdot \prod_{i=2}^{N_j} \Pr\{X_i = X_{ji} \big| X_1 = X_{j1}, X_2 = X_{j2}, \ldots, X_{i-1} = X_{j,i-1}\}\end{aligned} \tag{2}$$

Based on Lemma 1, we can verify that

$$\Pr\{X_i = X_{ji} \big| X_1 = X_{j1}, X_2 = X_{j2}, \ldots, X_{i-1} = X_{j,i-1}\} = \frac{\beta_{ji}}{\alpha_{ji}}.$$

Thus, the probability that the change of system states follows path j is

$$\pi_j = \prod_{i=0}^{N_j-1} \frac{\beta_{ji}}{\alpha_{ji}}. \tag{3}$$

Also based on Lemma 1, it can be seen that X_{ji} is exponentially distributed with rate α_{ji}. Thus, the distribution of T_j can be obtained through convolution of X_{ji}. Laplace transform of X_{ji} is

$$f^e_{X_{ji}}(s) = \frac{\alpha_{ji}}{\alpha_{ji} + s}$$

thus, Laplace transform for T_j is

$$f^e_j(s) = \prod_{i=0}^{N_j-1} \frac{\alpha_{ji}}{\alpha_{ji} s}$$

By using partial fraction, we can obtain

$$f^e_j(s) = \sum_{i=0}^{N_j-1} A_{ji} \frac{\alpha_{ji}}{\alpha_{ji} + s}$$

where $A_{ji} = \prod_{\substack{m=0\\ m\neq i}}^{N_j-1} \frac{\alpha_{jm}}{\alpha_{jm} - \alpha_{ji}}$, $i = 0, 1, \ldots, N_j - 1$.

Following the inversion of the Laplace transform, the p.d.f and c.d.f of T_j, respectively, are

$$f_j(t) = \sum_{i=0}^{N_j-1} A_{ji}\alpha_{ji}\mathrm{e}^{-\alpha_{ji}t} \tag{4}$$

and

$$\begin{aligned} F_j(t) &= \int_0^t f_j(x)\mathrm{d}x \\ &= 1 - \sum_{i=0}^{N_j-1} A_{ji}\mathrm{e}^{-\alpha_{ji}t} \end{aligned} \tag{5}$$

We can see that the distribution of the system failure time T is a mixture distribution of T_j. Therefore, the system failure probability at time t can be estimated as follows.

$$\begin{aligned} F(t) &= \sum_{j=1}^{P} \pi_j \cdot F_j(t) \\ &= 1 - \sum_{j=1}^{P}\sum_{i=0}^{N_j-1} \pi_j A_{ji}\mathrm{e}^{-\alpha_{ji}t} \end{aligned} \tag{6}$$

Note that p is the reduced number of system failure paths in the systems with circular type. For the system with circular type, the probability of the system failure $F(t)$ can be estimated by

$$\begin{aligned} F(t) &= n\sum_{j=1}^{p} \pi_j \cdot F_j(t) \\ &= 1 - n\sum_{j=1}^{p}\sum_{i=0}^{N_j-1} \pi_j A_{ji}\mathrm{e}^{-\alpha_{ji}t} \end{aligned} \tag{7}$$

Thus, the system failure probabilities of consecutive-k-out-of-n: F systems can be calculated by Eqs. 6 and 7.

2.2 *Connected-*(r, s)*-Out-of-*(m, n)*: F System*

Salvia and Lasher [41] introduced a multidimensional consecutive-k-out-of- n: F system and define that the two- or the three-dimensional system is a square or cubic grid of side n. The system fails if and only if there is at least a square or a cube of k consisting of failed components only. More generalized systems are linear and circular connected-(r, s)-out-of-(m, n): F systems [2]. This system contains

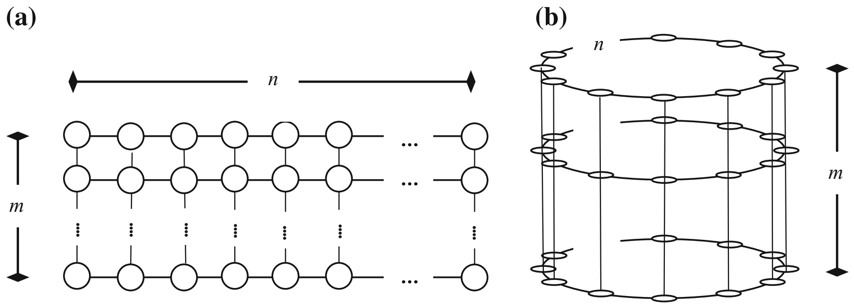


Fig. 2 **a** Linear and **b** circular connected-(r, s)-out-of-(m, n): F systems

components which are arranged into a rectangular pattern with m rows and n columns and it will fail if and only if there is at least a grid of r rows and s columns which consists of only failed components. The circular-type system forms a cylinder where the components in the first column connected to the components in the last column (see Fig. 2).

We denote the component in column a and row b by $\{a, b\}$. Thus the minimal cut sets for the linear type are

$$\begin{aligned} K_{ab} = \{&\{a,b\},\{a,b+1\},\ldots,\{a,s+b-1\},\{a+1,b\},\{a+1,b+1\},\ldots,\\ &\{a+1,s+b-1\},\ldots,\{r+a-1,b\},\{r+a-1,b+1\},\ldots,\\ &\{r+a-1,s+b-1\}\} \end{aligned}$$

for $a = 1, 2, \ldots, m - r + 1$ and $b = 1, 2, \ldots, n - s + 1$ and there are $(m - r + 1)(n - s + 1)$ minimal cut sets.

Since the components in the column 1 and column n are connected, there are $(m - r + 1)n$ minimal cut sets for the circular connected-(r, s)-out-of-(m, n): F system.

Because of the size of the system, the system failure probability for the general two-dimensional system: connected-(r, s)-out-of-(m, n): F system is difficult to obtain. Some studies derived the system reliability of two-dimensional systems with very small size (connected-(1, 2)-or-(1, 2)-out-of-(m, n): F system). In the other studies, the reliability value is estimated by simulation, approximation, or only the lower bound is obtained [25, 43–45].

3 Component Dependence Models

This section describes two component dependence mechanisms in consecutive k-out-of-n: F systems and shows the consequence in the system failure paths. Although the failures of components are dependent on each other, the system failure paths are independent on each other. Thus, it is easier to obtain the probability by exploiting the system failure paths. In this subsection, we consider the systems with identical components of which are identical and the failure rates are constant.

3.1 Systems with i.i.d Components

The failure rates of a working component in every step in each path in the system consisting of i.i.d components are identical and the value is denoted as λ. We define w_{ji} as the number of working components in step i in path j. The term α_{ji} and β_{ji} can be given as follows, respectively,

$$\alpha_{ji} = w_{ji}\lambda, \tag{8}$$

$$\beta_{ji} = \lambda. \tag{9}$$

In order to understand more easily, a small illustration is made. Table 4 shows the system failure paths and the corresponding α_{ji} and β_{ji} in step i in path j for linear consecutive-2-out-of-3: F system. Similarly, we can get the system failure paths for the circular-type system.

By substituting Eqs. (8) and (9) into Eq. (3), we obtain

Table 4 Paths for a linear consecutive-2-out-of-3: F system with i.i.d components

Path j	Step i												
	0				1				2				3
	State	α_{j0}	β_{j0}		State	α_{j1}	β_{j1}		State	α_{j2}	β_{j2}		State
1	**1**11	3λ	λ	–	0**1**1	2λ	λ	–	001				
2	**1**11	3λ	λ	–	01**1**	2λ	λ	–	010	λ	λ	–	000
3	1**1**1	3λ	λ	–	**1**01	2λ	λ	–	001				
4	1**1**1	3λ	λ	–	10**1**	2λ	λ	–	100				
5	11**1**	3λ	λ	–	**1**10	2λ	λ	–	0**1**0	λ	λ	–	000
6	11**1**	3λ	λ	–	1**1**0	2λ	λ	–	100				

$$\pi_j = \prod_{i=0}^{N_j - 1} \frac{1}{w_{ji}}. \tag{10}$$

Thus, we can estimate the system failure probability by substituting α_{ji}, β_{ji} and π_j into the system failure probability equation in the previous section.

3.2 Systems with (k − 1)-Step Markov Dependence

Another component dependence model for these systems is $(k-1)$-step Markov dependence model [39]. The reliability of component i depends on the states of $k-1$ components which are located beside that component. Load-sharing model is a special case of Markov dependence with $(n-1)$-step.

Failure rate of a component in the beginning time point denoted as λ_0 because all components are working. The failure rate of the working components becomes λ_i when there are i failed components preceding it. Table 5 shows the illustration for linear consecutive-2-out-of-3: F system with $(k-1)$-step Markov dependence model.

Thus, we can estimate the system failure probability by estimating the terms α_{ji}, β_{ji} and π_j and substituting into the system failure probability equation in the previous section.

3.3 Systems with Load-Sharing Components

If one component fails, the workload has to be shared by the remaining components. Thus, there will be an increased load share for each surviving component. In most conditions, increased workload causes a higher failure rate of the component

Table 5 Paths for a linear consecutive-2-out-of-3: F system with $(k-1)$-step Markov dependence

Path j	Step i												
	0				1				2				3
	State	α_{j0}	β_{j0}		State	α_{j1}	β_{j1}		State	α_{j2}	β_{j2}		State
1	**1**11	$3\lambda_0$	λ_0	–	0**1**1	$\lambda_1 + \lambda_0$	λ_1	–	001				
2	**1**11	$3\lambda_0$	λ_0	–	01**1**	$\lambda_1 + \lambda_0$	λ_0	–	0**1**0	λ_1	λ_1	–	000
3	1**1**1	$3\lambda_0$	λ_0	–	**1**01	$\lambda_1 + \lambda_0$	λ_0	–	001				
4	1**1**1	$3\lambda_0$	λ_0	–	10**1**	$\lambda_1 + \lambda_0$	λ_1	–	100				
5	11**1**	$3\lambda_0$	λ_0	–	**1**10	$2\lambda_0$	λ_0	–	0**1**0	λ_1	λ_1	–	000
6	11**1**	$3\lambda_0$	λ_0	–	1**1**0	$2\lambda_0$	λ_0	–	100				

Table 6 Paths for a linear consecutive-2-out-of-3: F system with load-sharing components

Path j	Step i												
	0				1				2				3
	State	α_{j0}	β_{j0}		State	α_{j1}	β_{j1}		State	α_{j2}	β_{j2}		State
1	**1**11	$3\lambda_0$	λ_0	–	0**1**1	$2\lambda_1$	λ_1	–	001				
2	**1**11	$3\lambda_0$	λ_0	–	01**1**	$2\lambda_1$	λ_1	–	0**1**0	λ_2	λ_2	–	000
3	1**1**1	$3\lambda_0$	λ_0	–	**1**01	$2\lambda_1$	λ_1	–	001				
4	1**1**1	$3\lambda_0$	λ_0	–	10**1**	$2\lambda_1$	λ_1	–	100				
5	11**1**	$3\lambda_0$	λ_0	–	**1**10	$2\lambda_1$	λ_1	–	0**1**0	λ_2	λ_2	–	000
6	11**1**	$3\lambda_0$	λ_0	–	1**1**0	$2\lambda_1$	λ_1	–	100				

and many practical studies with mechanical and computer systems showed that the failure rates of working components strongly relates to the workload. For the case of long belt conveyors used in open-cast mining, even if a roll station (for example, motor) of the belt conveyor fails, the conveyor does not stop and tt only stops (fails) when k consecutive roll stations fail [47].

Failure rate of a component in the beginning of time is λ_0 because all components are working. After the first component fails, $n - 1$ working components must carry the same workload. Thus, these $n - 1$ working components have a higher failure rate, which is λ_1. In general, when there are i failed components, the $n - i$ working components will have a failure rate λ_i (where $\lambda_0 < \lambda_1 < \ldots < \lambda_{n-k+1}$).

The failure rate of a working component where there are i failed components and given load L is denoted as $\lambda_i(L)$. There are i failed components in step i for all paths. As we defined in the previous subsection that w_{ji} is the number of working components in step i in path j, we can calculate the term α_{ji} and β_{ji}, respectively, and Table 6 shows the illustration for linear consecutive-2-out-of-3: F system with load-sharing components. In similar way, we can get the system failure paths for circular-type systems.

$$\alpha_{ji} = w_{ji}\lambda_i(L), \tag{11}$$

$$\beta_{ji} = \lambda_i(L). \tag{12}$$

By substituting Eqs. (11) and (12) into Eq. (3), we can obtain the probability that the system follows path j, π_j,

$$\pi_j = \prod_{i=0}^{N_j-1} \frac{1}{w_{ji}}. \tag{13}$$

Thus, we can estimate the system failure probability by substituting α_{ji}, β_{ji} and π_j into the system failure probability equation in the previous section.

In this subsection, we considered two dependence models but components fail one by one. However, in addition to components failing one by one, there may be other causes to result in the system failure. Common cause failures are ones of these factors. Common cause failures describe single events which cause multiple component failures [29]. These events can be external events, such as storms, floods, lightning, seismic activities, maintenance errors, other human intervention errors, and sudden environmental change, or internal events, such as the failures of other components. Studies on k-out-of-n systems with common cause failures include Chung [9], Jung [26], Chari [7] and Dhillon and Anude [11]. This dependence model can be also applied to consecutive k-out-of-n systems.

4 Maintenance Models

In this section, the maintenance problem for the consecutive-k-out-of-n: F systems are studied. The basic maintenance policies, such as corrective maintenance and age preventive maintenance policies are considered. A condition-based maintenance policy is also studied. The performance evaluation of these policies for the consecutive-k-out-of-n: F systems can be seen in detail in Endharta [14], Endharta et al. [17] and Endharta and Yun [16].

4.1 Corrective Maintenance

We assume that the system is maintained only at the system failure time. The expected time to system failure for this policy is

$$E[T] = \sum_{j=1}^{P} \pi_j \cdot E[T_j]. \tag{14}$$

Because the time to failure in path j includes the time between failures X_{ji} and the time between failures follows an exponential distribution with failure rate α_{ji}, the expected time to system failure in path j can be estimated as

$$E[T_j] = \sum_{i=0}^{N_j-1} E[X_{ji}] = \sum_{i=0}^{N_j-1} \frac{1}{\alpha_{ji}}. \tag{15}$$

Thus, the expected time to system failure ET can be written as

$$E[T] = \sum_{j=1}^{P} \sum_{i=0}^{N_j-1} \frac{\pi_j}{\alpha_{ji}}. \tag{16}$$

The expected number of failures at system failure can be estimated as

$$E[N] = E\left[N^{\mathrm{SF}}\right]$$
$$= \sum_{j=1}^{P} \pi_j \cdot N_j \tag{17}$$

In this maintenance model, the cost terms which are considered are the cost spent for the corrective maintenance, c_{CM}, and the cost for component replacement, c_{R}. The expected cost rate of corrective replacement model is

$$EC_{\mathrm{CM}} = \frac{c_{\mathrm{CM}} + c_{\mathrm{R}} \cdot E[N]}{E[T]}. \tag{18}$$

Substituting Eqs. (16) and (17), the equation becomes

$$EC_{\mathrm{CM}} = \frac{c_{\mathrm{CM}} + c_{\mathrm{R}}\left(\sum_{j=1}^{P} \pi_j \cdot N_j\right)}{\sum_{j=1}^{P} \sum_{i=0}^{N_j-1} \frac{\pi_j}{\alpha_{ji}}}. \tag{19}$$

4.2 Age-Based Preventive Maintenance

In age replacement model, the system is maintained at age time T_{A} after its installation or at failure, whichever occurs first. When the system fails before $T_{\mathrm{A}}(T < T_{\mathrm{A}})$, the system is maintained correctively at the system failure. Otherwise, when there is no system failure before $T_{\mathrm{A}}(T > T_{\mathrm{A}})$, the system is maintained preventively at time T_{A}. Therefore, the expected time to renewal cycle, which is the expected time to system failure or to the preventive maintenance time, can be estimated as

$$\begin{aligned}
E[T(T_{\mathrm{A}})] &= \int_0^{T_{\mathrm{A}}} t \cdot d\,\Pr\{T \le t\} + T_{\mathrm{A}} \Pr\{T > T_{\mathrm{A}}\} \\
&= \int_0^{T_{\mathrm{A}}} t \cdot d\left(\sum_{j=1}^{P} \pi_j \cdot F_j(t)\right) + T_{\mathrm{A}}\left(1 - \sum_{j=1}^{P} \pi_j \cdot F_j(t)\right) \\
&= \int_0^{T_{\mathrm{A}}} \sum_{j=1}^{P} \sum_{i=0}^{N_j-1} \pi_j A_{ji} \alpha_{ji} t e^{-\alpha_{ji} t} \mathrm{d}t + T_{\mathrm{A}} \sum_{j=1}^{P} \sum_{i=0}^{N_j-1} \pi_j A_{ji} e^{-\alpha_{ji} t}
\end{aligned} \tag{20}$$

where $A_{jl} = \prod_{\substack{m=0 \\ m \neq l}}^{i-1} \frac{\alpha_{jm}}{\alpha_{jm} - \alpha_{jl}}$.

The failed components are replaced at maintenance time and the expected number of failed components includes the expected number of components failed at system failure and the expected number of components failed at time T_{A},

$$\begin{aligned} E[N(T_{\mathrm{A}})] &= E[N^{\mathrm{SF}}] + E[N^{T_{\mathrm{A}}}] \\ &= \sum_{j=1}^{P} \pi_j N_j F_j(T_{\mathrm{A}}) + \sum_{j=1}^{P} \pi_j E\left[N_j^{T_{\mathrm{A}}}\right] \end{aligned} \tag{21}$$

Note that X_{ji} is the time between $(i-1)$th failure and ith failure in path j and X_{ji} follows an exponential distribution with parameter α_{ji}. Thus, we define T_{ji} as the time to ith failure in path j such that

$$T_{jm} = X_{j1} + X_{j2} + \ldots + X_{jm}.$$

The following term is necessary for the estimation, for $a \leq b$,

$$\begin{aligned} \Pr\{T_{ji} > b, T_{j,i-1} < a\} &= \Pr\{T_{j,i-1} + X_{ji} > b, T_{j,i-1} < a\} \\ &= \Pr\{X_{ji} > b - a \big| T_{j,i-1} < a\} \cdot \Pr\{T_{j,i-1} < a\} \\ &= \int_0^a \Pr\{X_{ji} > b - y\} \cdot d \Pr\{T_{j,i-1} < y\} \\ &= \int_0^a \mathrm{e}^{-\alpha_{ji}(b-y)} \sum_{l=0}^{i-1} A_{jl}\alpha_{jl}\mathrm{e}^{-\alpha_{jl}y}\mathrm{d}y \\ &= \sum_{l=0}^{i-1} \frac{A_{jl}\alpha_{jl}}{\alpha_{ji} - \alpha_{jl}} \left(\mathrm{e}^{-\alpha_{jl}a - \alpha_{ji}(b-a)} - \mathrm{e}^{-\alpha_{ji}b}\right) \end{aligned} \tag{22}$$

where $A_{jl} = \prod_{\substack{m=0 \\ m \neq l}}^{i-1} \frac{\alpha_{jm}}{\alpha_{jm} - \alpha_{jl}}$.

Then, the expected number of components failed at time T_{A} in path j, $E[N_j^{T_{\mathrm{A}}}]$, needs to be derived.

$$\begin{aligned} E[N_j^{T_{\mathrm{A}}}] = 0 \cdot \Pr\{T_{j1} > T_{\mathrm{A}}\} + 1 \cdot \Pr\{T_{j2} > T_{\mathrm{A}}, T_{j1} < T_{\mathrm{A}}\} + \ldots + (N_{j-1}) \\ \cdot \Pr\{T_{jN_j} > T_{\mathrm{A}}, T_{jN_j-1} < T_{\mathrm{A}}\} \end{aligned}$$

We generalize the above equation into

$$E\left[N_j^{T_{\mathrm{A}}}\right] = \sum_{i=1}^{N_j-1} i \cdot \Pr\{T_{j,i+1} > T_{\mathrm{A}}, T_{j,i} < T_{\mathrm{A}}\} \tag{23}$$

Substituting Eq. (22) into Eq. (23), we obtain

$$E\left[N_j^{T_\mathrm{A}}\right] = \sum_{i=1}^{N_j-1}\sum_{l=0}^{i-1}\frac{iA_{jl}\alpha_{jl}}{\alpha_{ji}-\alpha_{jl}}\left(\mathrm{e}^{-\alpha_{ji}T_\mathrm{A}} - \mathrm{e}^{-\alpha_{jl}T_\mathrm{A}}\right). \tag{24}$$

In this maintenance model, the cost terms which are considered are the cost for the corrective maintenance, c_{CM}, cost for preventive maintenance at time T_A, c_{PM}, and the cost for component replacement, c_R. The expected cost rate under age-based preventive maintenance model is

$$EC_A(T_\mathrm{A}) = \frac{c_{\mathrm{CM}}F(T_\mathrm{A}) + c_{\mathrm{PM}}(1-F(T_\mathrm{A})) + c_\mathrm{R}\cdot\left(\sum_{j=1}^{P}\pi_j N_j F_j(T_\mathrm{A}) + \sum_{j=1}^{P}\pi_j E\left[N_j^{T_\mathrm{A}}\right]\right)}{E[T(T_\mathrm{A})]}, \tag{25}$$

where $F(t) = 1 - \sum_{j=1}^{P}\sum_{i=0}^{N_j-1}\pi_j A_{ji}\mathrm{e}^{-\alpha_{ji}t}, F_j(t) = 1 - \sum_{i=0}^{N_j-1}A_{ji}\mathrm{e}^{-\alpha_{ji}t},\ E[T(T_\mathrm{A})] = \int_0^{T_\mathrm{A}}\sum_{j=1}^{P}\sum_{i=0}^{N_j-1}\pi_j A_{ji}\alpha_{ji}t\mathrm{e}^{-\alpha_{ji}t}\mathrm{d}t + T_\mathrm{A}\sum_{j=1}^{P}\sum_{i=0}^{N_j-1}\pi_j A_{ji}\mathrm{e}^{-\alpha_{ji}T_\mathrm{A}}$, and $E\left[N_j^{T_\mathrm{A}}\right] = \sum_{i=1}^{N_j-1}\sum_{l=0}^{i-1}\frac{iA_{jl}\alpha_{jl}}{\alpha_{ji}-\alpha_{jl}}\left(\mathrm{e}^{-\alpha_{ji}T_\mathrm{A}} - \mathrm{e}^{-\alpha_{jl}T_\mathrm{A}}\right)$.

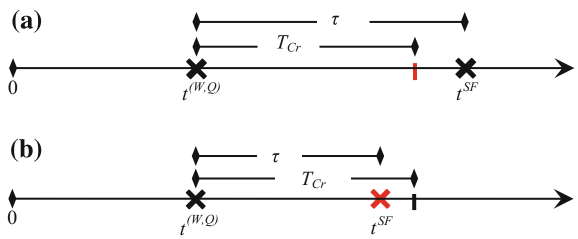
4.3 Condition-Based Maintenance

In this subsection, a condition-based maintenance policy proposed by Endharta and Yun [15] is considered. In this maintenance policy, the system is maintained when a certain system condition is satisfied. Since the system failure occurs when at least consecutive k components fail or all components in the minimal cut set fail, a condition-based maintenance policy is developed based on the number of working components in minimal cut sets and the number of minimal cut sets with specific number of working components.

a. **Condition-based maintenance with continuous monitoring**

In this model, we assume that that the system is monitored continuously and the condition about components in the minimal cut sets can be known at any time. The

Fig. 3 Illustration of the condition-based maintenance policy. **a** PM occurs **b** CM occurs



system will be maintained preventively at a certain time point T_{Cr} after a specified condition is satisfied. The condition is that there are Q or more minimal cut sets having W or less working components. If the system fails before reaching the preventive time point, the system will be correctively maintained. The maintenance is illustrated in Fig. 3 and several terms are used. The time when the condition is satisfied is represented by $t^{(W,Q)}$, the time of system failure is t^{SF}, and the step in which the condition is satisfied is represented by $s^{(W,Q)}$.

Define a random variable $\tau = t^{\mathrm{SF}} - t^{(W,Q)}$, representing the time difference between $t^{(W,Q)}$ and t^{SF}. If $\tau > T_{\mathrm{Cr}}$, the renewal cycle is ended by PM and the system is maintained preventively (see Fig. 3a). Otherwise, if $\tau < T_{\mathrm{Cr}}$, the renewal cycle is ended by CM and the system is maintained correctively (see Fig. 3b).

We know that until time $t^{(W,Q)}$ or at time to step $s^{(W,Q)}$, there is no system failure. Thus, the system failure distribution probability $F(t)$ in Eq. (7) changes into

$$F(t) = 1 - \sum_{j=1}^{P} \sum_{i=s_j^{(W,Q)}}^{N_j-1} \pi_j A_{ji} \mathrm{e}^{-\alpha_{ji} t} \tag{26}$$

where $s_j^{(W,Q)}$ is the step where the condition is satisfied in path j,

$$\pi_j = \prod_{i=s_j^{(W,Q)}}^{N_j-1} \frac{\beta_{ji}}{\alpha_{ji}} \quad \text{and} \quad A_{ji} = \prod_{\substack{m = s_j^{(W,Q)} \\ m \neq i}}^{N_j-1} \frac{\alpha_{jm}}{\alpha_{jm} - \alpha_{ji}}.$$

The expected time of a renewal cycle includes the time to $s^{(W,Q)}$, the expected time to system failure, and the expected time to T_{Cr} as follows:

$$\begin{aligned} E[T(T_{\mathrm{Cr}})] &= E[t^{(W,Q)}] + \int_0^{T_{\mathrm{Cr}}} t \cdot d\,\Pr\{T \le t\} + T_{\mathrm{Cr}} \Pr\{T > T_{\mathrm{Cr}}\} \\ &= E[t^{(W,Q)}] + \int_0^{T_{\mathrm{Cr}}} t \cdot dF(t) + T_{\mathrm{Cr}}(1 - F(T_{\mathrm{Cr}})) \\ &= E[t^{(W,Q)}] + \int_0^{T_{\mathrm{Cr}}} \sum_{j=1}^{P} \sum_{i=s_j^{(W,Q)}}^{N_j-1} \pi_j A_{ji} \alpha_{ji} t \mathrm{e}^{-\alpha_{ji} t} \mathrm{d}t + T_{\mathrm{Cr}} \sum_{j=1}^{P} \sum_{i=s_j^{(W,Q)}}^{N_j-1} \pi_j A_{ji} \mathrm{e}^{-\alpha_{ji} T_{\mathrm{Cr}}} \end{aligned} \tag{27}$$

where $E[t^{(W,Q)}] = \sum_{j=1}^{P} \sum_{i=0}^{s_j^{(W,Q)}-1} \frac{\pi_j}{\alpha_{ji}}$.

Denote $N^{(W,Q)}$ as the number of components failed at step $s^{(W,Q)}$ and $N_j^{(W,Q)}$ as the number of components failed at step $s_j^{(W,Q)}$ in path j. The components are replaced at maintenance time and the expected number of failed components includes the number of components failed at $s^{(W,Q)}$ and the expected number of components failed additionally until system failure and the expected number of components failed additionally until maintenance time T_{Cr},

$$\begin{aligned} E[N(T_{\mathrm{Cr}})] &= E[N^{(W,Q)}] + E[N^{\mathrm{SF}}] + E[N^{T_{\mathrm{Cr}}}] \\ &= \sum_{j=1}^{P} \pi_j \cdot N_j^{(W,Q)} + \sum_{j=1}^{P} \pi_j N_j F_j(T_{\mathrm{Cr}}) + \sum_{j=1}^{P} \pi_j E\left[N_j^{T_{\mathrm{Cr}}}\right] \\ &= \sum_{j=1}^{P} \pi_j \left(N_j^{(W,Q)} + N_j F_j(T_{\mathrm{Cr}}) + E\left[N_j^{T_{\mathrm{Cr}}}\right]\right) \end{aligned} \tag{28}$$

where $F_j(t) = 1 - \sum_{i=s_j^{(W,Q)}}^{N_j-1} A_{ji} \mathrm{e}^{-\alpha_{ji} t}$ and $E\left[N_j^{T_{\mathrm{Cr}}}\right] = \sum_{i=s_j^{(W,Q)}}^{N_j-1} \sum_{m=s_j^{(W,Q)}}^{i-1} \frac{i A_{jm} \alpha_{jm}}{\alpha_{ji} - \alpha_{jm}} (\mathrm{e}^{-\alpha_{ji} T_{\mathrm{Cr}}} - \mathrm{e}^{-\alpha_{jm} T_{\mathrm{Cr}}})$.

In this maintenance policy, we consider the cost for the corrective maintenance, c_{CM}, cost for preventive maintenance at time T_{Cr}, c_{PM}, and the cost for component replacement, c_{R}, in the expected cost rate. The expected cost rate under the condition-based maintenance model is

$$EC_{\mathrm{Cr}}(T_{\mathrm{Cr}}) = \frac{c_{\mathrm{CM}} F(T_{\mathrm{Cr}}) + c_{\mathrm{PM}}(1 - F(T_{\mathrm{Cr}})) + c_{\mathrm{R}} \cdot \sum_{j=1}^{P} \pi_j \left(N_j^{(W,Q)} + N_j F_j(T_{\mathrm{Cr}}) + E\left[N_j^{T_{\mathrm{Cr}}}\right]\right)}{E[T(T_{\mathrm{Cr}})]}, \tag{29}$$

where $F(t) = 1 - \sum_{j=1}^{P} \sum_{i=s_j^{(W,Q)}}^{N_j-1} \pi_j A_{ji} \mathrm{e}^{-\alpha_{ji} t}, F_j(t) = 1 - \sum_{i=s_j^{(W,Q)}}^{N_j-1} A_{ji} \mathrm{e}^{-\alpha_{ji} t}$, $E\left[N_j^{T_{\mathrm{Cr}}}\right] = \sum_{i=s_j^{(W,Q)}}^{N_j-1} \sum_{l=s_j^{(W,Q)}}^{i-1} \frac{i A_{jl} \alpha_{jl}}{\alpha_{ji} - \alpha_{jl}} (\mathrm{e}^{-\alpha_{ji} T_{\mathrm{Cr}}} - \mathrm{e}^{-\alpha_{jl} T_{\mathrm{Cr}}})$, and $E[T(T_{\mathrm{Cr}})] = \sum_{j=1}^{P} \sum_{i=0}^{s_j^{(W,Q)}-1} \frac{\pi_j}{\alpha_{ji}} + \int_0^{T_{\mathrm{Cr}}} \sum_{j=1}^{P} \sum_{i=s_j^{(W,Q)}}^{N_j-1} \pi_j A_{ji} \alpha_{ji} t \mathrm{e}^{-\alpha_{ji} t} \mathrm{d}t + T_{\mathrm{Cr}} \sum_{j=1}^{P} \sum_{i=s_j^{(W,Q)}}^{N_j-1} \pi_j A_{ji} \mathrm{e}^{-\alpha_{ji} T_{\mathrm{Cr}}}$.

Endharta and Yun [15] studied an special case with $W = 1$ and $Q = 1$, in which the system will be maintained preventively at certain time point after there is at least one minimal cut set having only one working component and proposed the equation to estimate the expected cost rate by utilizing the system failure paths.

b. **Condition-based maintenance with periodic inspection**

In practical situation, it may be impossible to know the system status, that is, whether components in the system fail or not. Thus, in this subsection, we consider inspection problem to know the system condition together. Endharta and Yun [15] considered a condition-based maintenance policy with periodic inspection.

We inspect the system periodically and if there are Q or more minimal cut sets having W or less working components, the system will be maintained immediately at inspection times. The condition can be represented as $x(n_W) \geq Q$, where $x(n_W)$ represents the number of minimal cut sets consisting of n_W working components

($n_W = 1, 2, \ldots, W$). In this policy, the decision variables are the minimum number of minimal cut sets, Q, the minimum number of the working components W in the minimal cut set, and the inspection interval T_I. Endharta and Yun [15] used a simulation method to obtain the near-optimal decision parameters which minimize the expected cost rate. The expected cost rate includes the system failure cost, preventive maintenance cost, replacement cost per component, and inspection cost.

5 Numerical Examples

In this subsection, we are going to compare three maintenance policies numerically. Several examples were studied and we selected some parts of existing numerical results. Tables 7 and 8 are parts of the numerical results from Endharta [14], Endharta et al. [17] and Endharta and Yun [16].

First, we consider linear consecutive-k-out-of-n: F systems with i.i.d components. Table 7 shows the expected cost rates for three maintenance policies. If the optimal interval in the age replacement model equals to infinite, the system should be maintained at system failures (corrective maintenance). Otherwise, if it equals to 0, the system should be maintained immediately at the time when the condition is satisfied.

Based on Table 7, the expected cost rates of consecutive-3-out-of-4 and 3-out-of-7: F systems are same, which means that three policies give same performance. However, because the optimal interval time is infinite, corrective maintenance policy is better than others for these particular systems. For consecutive-5-out-of-6 and 5-out-of-8: F systems, the expected cost rates of condition-based replacement policy are the smallest. Thus, for these particular systems, condition-based replacement policy should be applied and when there is one working component left in the minimal cut set, the system should be maintained immediately by replacing all failed components.

The example for the system with load-sharing dependence mechanism is shown in Table 8. Suppose that the relationship between the load and the component failure rate is as follows:

$$\lambda_i = a\left(\frac{L}{n-i}\right)^b \quad \text{where } a > 0 \text{ and } b > 0.$$

Filus [18] considered this function to determine the load size which maximizes the average asymptotic gain per unit time. Meeker and Escobar [36] also used this function in the accelerated life testing (ALT). For illustration, we use the following example with $L = 10$. In the numerical result for linear consecutive-5-out-of-8: F systems with load-sharing components (see Table 8), the optimal solution cannot be obtained due to the computational problem.

Table 7 Expected cost rates with various system parameters for linear consecutive-k-out-of-n: F systems with i.i.d components where $c_{\mathrm{CM}} = 2$, $c_{\mathrm{PM}} = 1$, $c_{\mathrm{R}} = 0.01$ and $\lambda = 0.01$

k	n	Corrective replacement			Age replacement				Condition-based replacement			
		$E[N]$	$E[T]$	EC_{CM}	$E[N]$	$E[T]$	EC_{A}	T_{A}^{*}	$E[N]$	$E[T]$	EC_{Cr}	T_{Cr}^{*}
3	4	3.50	158.333	0.0129	3.50	158.333	0.0129	∞	3.50	158.333	0.0129	∞
	7	4.54	97.619	0.0210	4.54	97.619	0.0210	∞	4.54	97.619	0.0210	∞
5	6	5.67	211.667	0.0097	5.54	199.366	0.0097	346.2	4.40	115.000	0.0091	0
	8	6.86	178.333	0.0116	6.86	178.333	0.0116	∞	5.11	94.881	0.0111	0

Table 8 Expected cost rates with various system parameters for linear consecutive-k-out-of-n: F systems with load-sharing components where $L = 10$, $c_{CM} = 2$, $c_{PM} = 1$, $c_R = 0.01$, $a = 0.01$ and $b = 0.5$

k	n	Corrective replacement			Age replacement				Condition-based replacement			
		$E[N]$	$E[T]$	EC_{CM}	$E[N]$	$E[T]$	EC_A	T_A^*	$E[N]$	$E[T]$	EC_{Cr}	T_{Cr}^*
3	4	3.50	22.8446	0.0891	3.39	21.8807	0.0889	42.3	2.17	11.9520	0.0855	0
	7	4.54	20.7643	0.0985	4.13	18.4381	0.0971	26.9	3.57	15.3824	0.0945	6.7
5	6	5.67	33.0659	0.0622	4.90	26.7673	0.0595	32.3	4.40	22.1565	0.0471	0
	8	6.86	33.5291	0.0617	NA	NA	NA	NA	5.11	21.7638	0.0483	0

Based on Table 8, condition-based maintenance policy outperforms the others because the expected cost rates are the smallest. For consecutive-3-out-of-4, 5-out-of-6, and 5-out-of-8: F systems, the system should be maintained immediately by replacing all failed components when there are one working component in the minimal cut set. For consecutive-3-out-of-7: F system, the system should be maintained by replacing all failed components at 6.7 time unit after there is one working component in the minimal cut set. If the system fails before reaching that time point, the system is maintained correctively by replacing all failed components.

For more numerical studies, refer to Endharta [14], Endharta et al. [17] and Endharta and Yun [16].

6 Conclusion and Future Works

This chapter considered the optimal maintenance problem for consecutive multi-unit systems, such as consecutive-k-out-of-n: F systems. Systems with i.i.d components, load-sharing dependence and $(k-1)$-step Markov dependence mechanisms are analyzed. Three maintenance policies: corrective maintenance, age preventive maintenance and condition-based maintenance policies were considered. The expected cost rates were obtained analytically by utilizing the system failure paths. The cost terms involved are the system failure cost, preventive maintenance cost, and component replacement cost.

The condition-based maintenance policy is based on the number of working components in minimal cut sets. When the system can be monitored continuously and the component condition can be known at any time, the system is maintained preventively at a certain time point after there are at least Q minimal cut sets with at most W working components. If the system fails before reaching the preventive maintenance time point, the system is correctively maintained by replacing all failed components. The closed-form equation for estimating the expected cost rate is obtained.

However, the system usually cannot be known without inspection. Thus, we considered a condition-based maintenance with periodic inspection and the system is maintained immediately at inspection times if there are at least Q minimal cut sets with at most W working components. We used simulation to estimate the expected cost rate and some metaheuristics to obtain the near optimal decision parameters: W, Q and inspection interval.

We compared the three maintenance policies numerically and knew the condition-based replacement policy outperforms other replacement policies based on the expected cost rate.

For further studies, the following topics may be promising ones in consecutive multi-unit systems;

1. Maintenance problem for consecutive-k-out-of-n: F systems with nonidentical components: We can consider various maintenance policies for the system with non-identical components.
2. Maintenance problem for consecutive-k-out-of-n: F systems with finite spare components: We consider spare parts provisioning problem and maintenance problem together.
3. Maintenance problem for consecutive-k-out-of-n: F systems with limited maintenance duration: When we maintain the system, we should select the failed components for replacement and sometimes we cannot replace all the failed components at maintenance times.
4. Maintenance problem for consecutive-k-out-of-n: F systems with non-periodic inspection intervals: Since the system fails significantly quicker as the time goes, non-periodic inspection intervals might be considered.
5. Dependence models in two and three dimensional cases?
6. Maintenance problem for toroidal-type systems (refer [33]: Toroidal-type system is a circular connected-(r, s)-out-of-(m, n): F system where the components in row 1 are connected to those in row m; thus, this system forms a torus.

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