Over the past fifteen years, motivated by regularity problems in evolution equations, there has been tremendous progress in the analysis of Banach space-valued functions and processes. For so-called UMD spaces in particular, central areas of harmonic analysis, such as the theory of Fourier multipliers and singular integrals, were extended to operator-valued kernels acting on Bochner spaces, and basic estimates of stochastic analysis, including the Itô isometry and the Burkholder–Davis–Gundy inequalities, were generalised to Banach space-valued processes.

As it was long known that extensions of such sophisticated scalar-valued estimates are not possible for all Banach spaces, these results depended on essential progress in the geometry of Banach spaces during the 70s and 80s. The theory of Burkholder and Bourgain on UMD spaces became the foundation on which the recent theory we wish to report on was built; just as important are results of Kwapień, Maurey, and Pisier on type and cotype, since they link the structure of the Banach space to estimates for random sums which replace to some extent the fundamental orthogonality relations in Hilbert spaces.

For most classical Banach spaces, the UMD, type and cotype properties are readily available and therefore the results of vector-valued Analysis can be applied to many situations of interest in the theory of partial differential equations; they have already proved their value by providing sharp regularity estimates for parabolic problems. Our aim is to give a detailed and careful presentation of these topics that is useful not only as a reference book but can be used also selectively as a basis for advanced courses and seminars.

This project ranges over a broad spectrum of Analysis and includes Banach space theory, operator theory, harmonic analysis and stochastic analysis. For this reason we have divided it into three parts. The present volume develops the theory of Bochner integration, Banach space-valued martingales and UMD spaces, and culminates in a treatment of the Hilbert transform, Littlewood–Paley theory and the vector-valued Mihlin multiplier theorem.

Volume II will present a thorough study of the basic randomisation techniques and the operator-theoretic aspects of the theory, such as $R$-
boundedness, vector-valued square functions and radonifying operators, as well as a detailed treatment of the relevant probabilistic Banach space notions such as type, cotype, $K$-convexity and properties related to contraction principles. These techniques will allow us to present the theory of $H^\infty$-functional calculus for sectorial operators and work out the main examples. This sets the stage for our final aim, a presentation of the theory of singular integral operators with operator-valued kernels and its applications to maximal regularity for deterministic and stochastic parabolic evolution equations, which will be the subject matter of Volume III.

The central theme in all volumes is the identification of the Banach spaces to which the key estimates of classical harmonic and stochastic analysis can be extended as those with the fundamental UMD property. The very definition behind this abbreviation is the unconditionality of martingale differences, a primarily probabilistic notion, and a number of different characterisations are formulated in purely probabilistic terms. However, this same property is also equivalent to the boundedness of the vector-valued Hilbert transform, the Littlewood–Paley inequality for vector-valued Fourier integrals, and several other estimates in the realm of classical harmonic analysis.

Each of these aspects of UMD spaces makes a substantial body of theory in its own right, and one could certainly produce respectable treatments of large parts of this material with a “clean” probabilistic or analytic flavour. However, rather than striving for such “purity”, our aim is to emphasise the rich connections between the two worlds and the unity of the subject. For example, while martingales are traditionally regarded as a topic in Probability, we define and discuss them on σ-finite measure spaces from the beginning, so that they are immediately applicable to Analysis on the Euclidean space $\mathbb{R}^d$ without the need of auxiliary truncations or decompositions into probability spaces. Moreover, it is important to observe that even if we (or the reader) wanted to concentrate on the analytic side of UMD spaces only, we could hardly present a complete picture without an occasional reference to the probabilistic notions, at least at the present state of knowledge. For instance, although we know that both the Hilbert transform boundedness and the Littlewood–Paley inequality are equivalent to the UMD property, and therefore to each other, the only known way of proving the equivalence of these two analytic notions passes through the probabilistic UMD. There are numerous other such examples, and new frontiers of the theory have shown over and over again that it is the probabilistic definition of UMD spaces that lies at the centre and connects everything together.

So much said about the unity of Analysis and Probability (in Banach spaces), we should acknowledge the existence of a third side of the triangle, which is barely touched by the present treatise, namely: Geometry (of Banach spaces). Our choice of topics is not meant in any way to downplay the importance of this huge topic, both in its own right and in relation to analytic and
probabilistic questions, but rather to admit our limits and to leave the proper account of the geometric connections for other treatments.

* This book can be studied in a variety of ways and for different motivations. The principal, but not the only, audience that we have in mind consists of researchers who need and use Analysis in Banach spaces as a tool for studying other problems, in particular the regularity of evolution equations mentioned above. Until now, the contents of this extensive and powerful toolbox have been mostly scattered around in research papers, or in some cases monographs addressed to readerships with a rather different background from our focus, and we feel that collecting this diverse body of material into a unified and accessible presentation fills a gap in the existing literature. Indeed, we regard ourselves as part of this audience, and we have written the kind of book that we would have liked to have for ourselves when working through this theory for the first time.

Aside from this, parts of the book may also offer an interesting angle to the classical analysis of scalar-valued functions, which is certainly covered as a special case, and seldom required as a prerequisite or used as a building block for the Banach space-valued theory. For a classical harmonic analyst, the approach that we take, say, to the $L^p$-boundedness of the Hilbert transform, is possibly exotic, but not necessarily substantially more difficult than more traditional treatments in the scalar-valued case.

There are a couple of technical features of the book worth mentioning. Most of the time, we are quite explicit with the constants appearing in our estimates, and we especially try to keep track of the dependence on the main parameters involved. Thus, rather than saying that a particular bound “only depends on the UMD constant $\beta_{p,X}$”, we prefer to write out, say, $(\beta_{p,X})^2$, or whichever function of $\beta_{p,X}$ appears from the calculation. We often go to the extent of writing, say, “2000” instead of “$c$, where $c$ is a numerical constant”, although we also might write “2000” instead of “1764”, when there is no reason to believe that the latter constant, although given by a particular computation, would be anywhere close to optimal. Indeed, except for a few select places, we make no claim that our explicit constants cannot be improved; however, in many places, we have made an effort to present the best (order of) bounds currently available by the existing methods. We hope that making this explicit documentation might spur some interest towards research on such quantitative issues.

We also pay more attention than many texts to the impact of the underlying scalar field (real or complex) on the results under consideration. While this is largely irrelevant for many questions, it does play a role in some others,
and we try to be quite explicit in pointing out the differences when they do occur, hopefully without insisting too much on this point when they do not.

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