The term “inverse problem” has no acknowledged mathematical definition; its meaning relies on notions from physics. One assumes that there is a known mapping

$$T : \mathbb{U} \rightarrow \mathbb{W},$$

which models a physical law or a physical device. Here, \(\mathbb{U}\) is a set of “causes” and \(\mathbb{W}\) is a set of “effects.” The computation of an effect \(T(u)\) for a given cause \(u\) is called a direct problem. Finding a cause \(u \in \mathbb{U}\), which entails a given effect \(w \in \mathbb{W}\), is called an inverse problem. Solving an inverse problem thus means to ask for the solution of an equation \(T(u) = w\).

Maybe a certain effect \(w \in \mathbb{W}\) is desirable and one is looking for some \(u \in \mathbb{U}\) to produce it. An inverse problem of this kind is called a control problem. In the following, it will be assumed that an effect is actually observed and that its cause has to be found. An inverse problem of this kind is called an identification problem. It arises, when an interesting physical quantity is not directly amenable to measurements, but can only be derived from observations of its effects. There are numerous examples of identification problems in science and engineering like:

- **Inverse gravimetry**: Given its mass density distribution (cause), the gravitational force (effect) exercised by a body can be computed (direct problem). The inverse problem is to derive the mass density distribution from measured gravitational forces. An application is the detection of oil or gas reservoirs in geological prospecting.
- **Transmission tomography**: Given the density distribution of tissue (cause), the intensity loss (effect) of an X-ray traversing it can be computed (direct problem). Inversely, one tries to find the tissue’s density from measured intensity losses of X-rays traversing it. Applications exist in diagnostic radiology and nondestructive testing.
- **Elastography**: From known material properties (cause), the displacement field (effect) of an elastic body under external forces can be computed (direct
problem). The inverse problem is to derive material properties from observed displacements. An application is the medical imaging of soft tissue.

- **Seismic tomography**: Knowing the spatial distribution of mechanical characteristics (cause) of the earth’s crust and mantle, seismic energy propagation (effect) from controlled sound sources or earthquakes can be computed (direct problem). Inversely, one seeks to find mechanical characteristics from records of seismic data. A geophysical application is to map the earth’s interior.

Very often, the solution $u^*$ of an inverse problem $T(u) = w$ depends in an extremely sensitive manner on $w$, two very similar effects having very different causes. This would not be a principal problem, if $w$ was known exactly. However, in practice, one never knows $w$ perfectly well but can only construct some approximation $\hat{w}$ of it from a limited number of generally inexact measurements. Then the true cause $u^* \in \mathbb{U}$ defined by $T(u) = w$ cannot even be found to good approximation, no matter the available computing power. This is so because $w$ is not known, and, by the aforementioned sensitivity, solving $T(u) = \hat{w}$ would produce a completely wrong answer $\hat{u}$, far away from $u^*$. There is no way to miraculously overcome this difficulty, unless we are lucky and have some additional information of the kind “the true cause $u^*$ has property $P$.” If this is the case, then we might replace the problem of solving $T(u) = \hat{w}$ by the different problem of finding a cause $\tilde{u}$ within the set of all candidates from $\mathbb{U}$ also having property $P$, such that some distance between $T(\tilde{u})$ and $\hat{w}$ is minimized. If the solution $\tilde{u}$ of this replacement problem depends less sensitively on the effect $\hat{w}$ than the solution $u^*$ of the original problem depends on $w$ but converges to $u^*$ if $\hat{w}$ converges to $w$, then one speaks of a regularization of the original inverse problem.

**Scope**

Many excellent books on inverse problems have been written; we only mention [Kir96, EHN96], and [Isa06]. These books rely on functional analysis as an adequate mathematical tool for a unified approach to analysis and regularized solution of inverse problems. Functional analysis, however, is not easily accessible to non-mathematicians. It is the first goal of the present book to provide an access to inverse problems without requiring more mathematical knowledge than is taught in undergraduate math courses for scientists and engineers. From abstract analysis, we will only need the concept of functions as vectors. Function spaces are introduced informally in the course of the text, when needed. Additionally, a more detailed but still condensed introduction is given in Appendix B. We will not deal with regularization theory for operators. Instead, inverse problems will first be discretized and described approximately by systems of algebraic equations and only then will be regularized by setting up a replacement problem, which will always be a minimization problem. A second goal is to elaborate on the single steps to be taken when solving an inverse problem: discretization, regularization, and practical
solution of the regularized optimization problem. Rather than being as general as possible, we want to work out these steps for model problems from the fields of inverse gravimetry and seismic tomography. We will not delve into details of numerical algorithms, though, when high-quality software is readily available on which we can rely for computations. For the numerical examples in this book, the programming environment Matlab ([Mat14]) was used as well as were C programs from [PTVF92].

Content

We start in Chap. 1 by presenting four typical examples of inverse problems, already illustrating the sensitivity issue. We then formalize inverse problems as equations in vector spaces and also formalize their sensitivity as “ill-posedness.” In the chapter’s last two sections, we have a closer look on problems from inverse gravimetry and seismic tomography. We define specific model problems that we will tackle and solve in later chapters. Special attention will be payed to the question whether sought-after parameters (causes) can be uniquely identified from observed effects, at least in the hypothetical case of perfect observations.

All model problems introduced in Chap. 1 are posed in function spaces: effects and causes are functions. Chapter 2 is devoted to the discretization of such problems, that is, to the question of how equations in function spaces can approximately be turned into equations in finite-dimensional spaces. Discretization is a prerequisite for solving inverse problems on a computer. A first section on spline approximation describes how general functions can be represented approximately by a finite set of parameters. We then focus on linear inverse problems. For these, the least squares method, investigated in Sect. 2.2, is a generally applicable discretization method. Two alternatives, the collocation method and the method of Backus-Gilbert, are presented in Sects. 2.3 and 2.4 for special but important classes of inverse problems, known as Fredholm equations of the first kind. Two of our model problems belong to this class. Even more specific are convolutional equations, which play a significant role in inverse gravimetry but also in other fields like communications engineering. Convolutional equations can be discretized in the Fourier domain, which leads to very efficient solution algorithms. This is discussed in Sect. 2.5. In the nonlinear case, discretizations are tailored to the specific problem to be solved. We present discretizations of two nonlinear model problems from inverse gravimetry and seismic tomography in Sect. 2.6.

The last two chapters investigate possibilities for a regularized solution of inverse problems in finite-dimensional spaces (i.e., after discretization). In Chap. 3, we treat the linear case, where the discretized problem takes the form of a linear system of equations $Ax = b$. More generally, one considers the linear least squares problem of minimizing $\|b - Ax\|_2$, which is equivalent to solving $Ax = b$, if the latter does have a solution, and is still solvable, if the system $Ax = b$ no longer is. In the first two sections, the linear least squares problem is analyzed, with an emphasis on
quantifying the sensitivity of its solution with respect to the problem data, consisting of the vector $b$ and of the matrix $A$. A measure of sensitivity can be given, known as condition number in numerical analysis, which allows to estimate the impact of data perturbations on the solution of a linear least squares problem. If the impact is too large, a meaningful result cannot be obtained, since data perturbations can never be completely avoided. The least squares problem then is not solvable practically. Regularization tries to set up a new problem, the solution of which is close to the actually desired one and which can be computed reliably. The general ideas behind regularization are outlined in Sect. 3.3. The most popular regularization technique, Tikhonov regularization, is explained and analyzed in Sect. 3.4, including a discussion of how a regularized solution can be computed efficiently. Tikhonov regularization requires the choice of a so-called regularization parameter. The proper choice of this parameter is much debated. In Sect. 3.5, we present only a single method to make it, the discrepancy principle, which is intuitively appealing and often successful in practice. For alternative choices, we refer only to the literature. Tikhonov regularization can also be used to solve problems derived by the Backus-Gilbert method or for convolutional equations transformed to the Fourier domain. These topics are treated in Sects. 3.7 and 3.8. Very interesting alternatives to Tikhonov regularization are iterative regularization methods, which are attractive for their computational efficiency. Two of these methods, the Landweber iteration and the conjugate gradient method, are described in Sects. 3.9 and 3.10. It will be found that the Landweber iteration can be improved by two measures which relate it back to Tikhonov regularization. One of these measures, a coordinate transform, can also be taken to improve the conjugate gradient method. Technical details about this coordinate transform are given in Sect. 3.6, which describes a transformation of Tikhonov regularization to some standard form. This is also of independent interest, since Tikhonov regularization in standard form is the easiest to analyze.

Regularization of nonlinear problems is studied in Chap. 4. In Sect. 4.1, nonlinear Tikhonov regularization is treated abstractly, whereas in Sect. 4.2, this method is applied to a model problem from nonlinear inverse gravimetry. Nonlinear Tikhonov regularization leads to an optimization problem, namely, a nonlinear least squares problem. Section 4.3 discusses various possibilities for solving nonlinear least squares problem numerically. All of these methods require the computation of gradients, which can mean a prohibitive computational effort, if not done carefully. The “adjoint method,” presented in Sect. 4.4, can sometimes drastically reduce the numerical effort to obtain gradients. The adjoint method is presented in the context of a nonlinear problem from seismic tomography, which is then solved in Sect. 4.5. A final section presents one example of a nonlinear iterative regularization method, the “inexact Newton-CG method.” The usefulness of this method is illustrated for a problem of inverse gravimetry.

Appendix A lists required material from linear algebra and derives the singular value decomposition of a matrix. Appendix B gives a condensed introduction into function spaces, i.e., into abstract vector spaces containing functions as elements. Appendix C gives an introduction into the subject of (multidimensional) Fourier transforms and their numerical implementation, including the case of
non-equidistant sample points. Finally, Appendix D contains a technical proof outsourced from Chap. 3, which shows the regularizing property of the conjugate gradient method applied to linear least squares problems.

Acknowledgments

I am much indebted to my editor, Dr. Clemens Heine, from Birkhäuser-Springer publishers for his promotion of this book. I am greatly thankful to my friend and colleague, Professor Stefan Schäffler, for his critical advice when reading my manuscript and for having supported my work for well over 20 years.

Munich, Germany

Mathias Richter
Inverse Problems
Basics, Theory and Applications in Geophysics
Richter, M.
2016, XII, 240 p. 52 illus., 32 illus. in color., Softcover
ISBN: 978-3-319-48383-2
A product of Birkhäuser Basel