Chapter 4
Sequences and Series of Functions

4.1 Uniform Convergence of Sequences of Functions

Suppose that $f_n : \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}$, $S \subset \mathbb{T}$.

Definition 4.1 We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ *converges pointwise*, i.e., at each point, to a function f defined on S if

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{for all} \quad t \in S.$$

We often write $\lim_{n\to\infty} f_n = f$ pointwise on S or $f_n \to f$ pointwise on S.

Example 4.2 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Consider

$$f_n(t) = t^n, \quad t \in [0, 1].$$

We have

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} t^n = \begin{cases} 0 & \text{for} \quad t \in [0, 1) \\ 1 & \text{for} \quad t = 1. \end{cases}$$

If

$$f(t) = \begin{cases} 0 & \text{for} \quad t \in [0, 1) \\ 1 & \text{for} \quad t = 1, \end{cases}$$

then $f_n \to f$ pointwise on $[0, 1]$.

Example 4.3 Let $\mathbb{T} = 2^{\mathbb{N}_0}$. If

$$f_n(t) = t\left(1 + e^{-nt}\right),$$

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then

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} t\left(1+e^{-nt}\right) = t.$$

If we set $f(t) = t$, then $f_n \to f$ pointwise on $\mathbb{T}$.

Example 4.4 If $f_n(t) = \frac{n+1}{n+t^2}$, then

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \frac{n+1}{n+t^2} = 1.$$

If $f(t) = 1$, $t \in \mathbb{T}$, then $f_n \to f$ pointwise on $\mathbb{T}$.

Exercise 4.5 Let

$$f_n(t) = \frac{nt^2}{n+t}, \quad f(t) = t^2.$$

Prove that $f_n \to f$ pointwise on $\mathbb{T}$.

Definition 4.6 We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ *converges uniformly* on S to a function f defined on S if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(t) - f(t)| < \varepsilon \quad \text{for all} \quad n > N \quad \text{and all} \quad t \in S.$$

Example 4.7 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Consider

$$f_n(t) = t^n, \quad 0 \le t \le a, \quad 0 < a < 1.$$

Note that

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} t^n = 0 \quad \text{for all} \quad t \in \mathbb{T}\setminus\{1\}.$$

Let $\varepsilon > 0$ be arbitrarily chosen. We choose $N \in \mathbb{N}$ such that

$$N > \frac{\log\varepsilon}{\log a}.$$

Hence, $a^N < \varepsilon$. Then, for every $n > N$, we have

$$a^n \le a^N < \varepsilon$$

and

$$|t^n - 0| = t^n \le a^n < \varepsilon.$$

Therefore, $f_n \to 0$ uniformly on $[0, a]$.

Example 4.8 Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider

$$f_n(t) = \sqrt{t^2 + \frac{1}{n^2}}.$$

If we take $f(t) = t$ and let $\varepsilon > 0$ be arbitrarily chosen, then

$$\begin{aligned}
|f_n(t) - f(t)| &= \left| \sqrt{t^2 + \frac{1}{n^2}} - t \right| \\
&= \frac{\left| \left(\sqrt{t^2 + \frac{1}{n^2}} - t \right) \left(\sqrt{t^2 + \frac{1}{n^2}} + t \right) \right|}{\sqrt{t^2 + \frac{1}{n^2}} + t} \\
&= \frac{t^2 + \frac{1}{n^2} - t^2}{\sqrt{t^2 + \frac{1}{n^2}} + t} \\
&= \frac{1}{n^2 \left(\sqrt{t^2 + \frac{1}{n^2}} + t \right)} \\
&\le \frac{1}{n^2 \frac{1}{n}} \\
&= \frac{1}{n}.
\end{aligned}$$

If we take $N = \frac{1}{\varepsilon}$, then, for every $n > N$, we have

$$\frac{1}{n} < \frac{1}{N} = \varepsilon \quad \text{and} \quad |f_n(t) - f(t)| < \varepsilon \quad \text{for any} \quad t \in \mathbb{T}.$$

Hence, $\{f_n\}_{n \in \mathbb{N}}$ is uniformly convergent to f on $\mathbb{T}$.

Example 4.9 Let $\mathbb{T} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. Consider

$$f_n(t) = \frac{nt}{2 + n^3 t^3}, \quad t \in \mathbb{T}.$$

If $f(t) = 0$, then $f_n \to f$ pointwise on $\mathbb{T}$. Assume that $\{f_n\}_{n \in \mathbb{N}}$ is uniformly convergent to f on $\mathbb{T}$. We take $0 < \varepsilon < \frac{1}{3}$. Then there exists $N = N(\varepsilon)$ such that for every $n > N$, we have

$$|f_n(t)| < \varepsilon \quad \text{for any} \quad t \in \mathbb{T}.$$

In particular, when $n > N$ and $t = \frac{1}{n}$, we get

$$\left| f_n\left(\frac{1}{n}\right) \right| < \varepsilon,$$

which is a contradiction because $f_n\left(\frac{1}{n}\right) = \frac{1}{3}$. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to f on $\mathbb{T}$.

Exercise 4.10 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Check if the following sequences are uniformly convergent on $\mathbb{T}$.

1. $f_n(t) = e^{-(t-3n)^2}$,
2. $f_n(t) = \frac{t}{4+2n^2t^2}$,
3. $f_n(t) = \frac{1}{2+3nt}$.

Solution 1. uniformly convergent to 0 on $\mathbb{T}$,
2. uniformly convergent to 0 on $\mathbb{T}$,
3. not uniformly convergent to 0 on $\mathbb{T}$.

Theorem 4.11 *If $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f on $D \subset \mathbb{T}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on D if and only if*

$$\lim_{n\to\infty} \sup_{t\in D} |f_n(t) - f(t)| = 0. \tag{4.1}$$

Proof 1. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on D. Then, for every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ so that $n > N$ implies

$$|f_n(t) - f(t)| < \varepsilon \quad \text{for any} \quad t \in D.$$

Hence,

$$\sup_{t\in D} |f_n(t) - f(t)| < \varepsilon \quad \text{for any} \quad n > N.$$

2. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f on D and (4.1) holds.Then, for every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ so that $n > N$ implies

$$\sup_{t\in D} |f_n(t) - f(t)| < \varepsilon.$$

Hence, for any $n > N$, we have

$$|f_n(t) - f(t)| < \varepsilon \quad \text{for any} \quad t \in D.$$

The proof is complete. □

Example 4.12 Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $f_n(t) = \frac{1}{n+t^2}$ and $f(t) = 0$, $t \in \mathbb{T}$. We have that $f_n \to f$ pointwise on $\mathbb{T}$. Also,

$$\sup_{t\in\mathbb{T}} |f_n(t) - f(t)| = \sup_{t\in\mathbb{T}} \frac{1}{n+t^2} = \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, using Theorem 4.11, it follows that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $\mathbb{T}$.

Example 4.13 Let $\mathbb{T} = \mathbb{N}$, $f_n(t) = \frac{nt}{nt+1}$, $f(t) = 1$, $t \in \mathbb{T}$. We have that $f_n \to 1$ pointwise on $\mathbb{T}$. Also,

$$\begin{aligned} |f_n(t) - f(t)| &= \left|\frac{nt}{nt+1} - 1\right| \\ &= \left|-\frac{1}{nt+1}\right| \\ &= \frac{1}{nt+1}, \\ \sup_{t\in\mathbb{T}} |f_n(t) - f(t)| &= \sup_{t\in\mathbb{T}} \frac{1}{nt+1} \\ &= \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, using Theorem 4.11, it follows that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $\mathbb{T}$.

Example 4.14 Let $\mathbb{T} = \mathbb{Z}$. We will investigate for uniform convergence of the sequence $\left\{f_n(t) = \frac{n+1}{n+t^2}\right\}_{n\in\mathbb{N}}$ on $D_1 = (-4, 4)$ and $D_2 = [1, \infty)$. Let $f(t) = 1$. Note that $f_n \to f$ pointwise on $\mathbb{Z}$. Moreover,

$$|f_n(t) - f(t)| = \left|\frac{n+1}{n+t^2} - 1\right| = \left|\frac{1-t^2}{n+t^2}\right|.$$

1. If $t \in D_1$, then

$$\begin{aligned} |f_n(t) - f(t)| &= \frac{1-t^2}{n+t^2} =: g(t), \\ g^{\Delta}(t) &= \frac{-(\sigma(t)+t)(n+t^2) - (1-t^2)(\sigma(t)+t)}{\left(n+(t+1)^2\right)\left(n+t^2\right)} \\ &= -\frac{(n+1)(2t+1)}{\left(n+(t+1)^2\right)(n+t^2)} \\ &\le 0 \quad \text{if} \quad t \ge 0, \\ g^{\nabla}(t) &= \frac{-(\rho(t)+t)(n+t^2) - (1-t^2)(\rho(t)+t)}{\left(n+(t-1)^2\right)\left(n+t^2\right)} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{(n+1)(2t-1)}{\left(n+(t-1)^2\right)(n+t^2)} \\
&\geq 0 \quad \text{if} \quad t \leq 0.
\end{aligned}
$$

Therefore, the function $\frac{1-t^2}{n+t^2}$ has a maximum at $t = 0$. Note that

$$
\frac{1-t^2}{n+t^2}\bigg|_{t=0} = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.
$$

Hence, using Theorem 4.11, it follows that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on D_1.

2. If $t \in D_2$, then

$$
|f_n(t) - f(t)| = \frac{t^2-1}{n+t^2}.
$$

Assume that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on D_2. Then, for $0 < \varepsilon < \frac{1}{2}$, there exists $N = N(\varepsilon)$ so that $n > N$ implies

$$
\frac{t^2-1}{n+t^2} < \varepsilon \quad \text{for any} \quad t \in D_2.
$$

We take $t = n+2 \in D_2$. Then

$$
\frac{(n+2)^2-1}{n+(n+2)^2} < \frac{1}{2},
$$

so

$$
\frac{n^2+4n+3}{2n^2+4n+4} < \frac{1}{2},
$$

so

$$
n^2+4n+3 < n^2+2n+2,
$$

so

$$
2n+1 < 0,
$$

which is a contradiction. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly to f on D_2.

Exercise 4.15 Let

$$
\mathbb{T} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup 2^{\mathbb{N}_0}.
$$

Using Theorem 4.11, investigate the sequence

$$\left\{ f_n(t) = \frac{n+t}{nt+1} \right\}_{n\in\mathbb{N}}$$

for uniform convergence on $D_1 = [0, 1]$ and $D_2 = 2^{\mathbb{N}_0}$.

Solution The sequence is not uniformly convergent on D_1, and it is uniformly convergent on D_2.

Theorem 4.16 *If the function sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f on $D \subset \mathbb{T}$, then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to f on D if and only if for an arbitrary sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \in D$, we have*

$$\lim_{n\to\infty} (f_n(t_n) - f(t_n)) = 0. \tag{4.2}$$

Proof 1. Necessity. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent on D. Then, using Theorem 4.11, we have

$$\sup_{t\in D} |f_n(t) - f(t)| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, for any sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \in D$, we have

$$|f_n(t_n) - f(t_n)| \le \sup_{t\in D} |f_n(t) - f(t)| \to 0 \quad \text{as} \quad n \to \infty,$$

i.e., (4.2) holds.

2. Sufficiency. Assume that for any sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \in D$, (4.2) holds and the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly to f on D. Hence, there exists $\varepsilon_0 > 0$ such that for any $N > 0$, there exist $n > N$ and $t \in D$ so that

$$|f_n(t) - f(t)| \ge \varepsilon_0.$$

For $N_1 = 1$, there exist $n_1 > 1$ and $t_{n_1} \in D$ such that

$$|f_{n_1}(t_{n_1}) - f(t_{n_1})| \ge \varepsilon_0.$$

For $N_2 = n_1$, there exist $n_2 > n_1$ and $t_{n_2} \in D$ such that

$$|f_{n_2}(t_{n_2}) - f(t_{n_2})| \ge \varepsilon_0,$$

and so on. Thus, we get a sequence $\{t_{n_k}\}_{k\in\mathbb{N}}$, $t_{n_k} \in D$, such that

$$|f_{n_k}(t_{n_k}) - f(t_{n_k})| \ge \varepsilon_0,$$

which leads to a contradiction due to (4.2).

The proof is complete. □

Example 4.17 Consider $f_n(t) = nt(1-t)^n$ on $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. We have that $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to 0 on D. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to 0 on D. Then, applying Theorem 4.16 for $t_n = \frac{1}{n}$, we have

$$f_n\left(\frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n \not\to 0,$$

which is a contradiction. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to 0 on D.

Example 4.18 Consider $f_n(t) = \frac{1}{1+nt}$ on $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. We have that $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to 0 on D. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to 0 on D. Then, using Theorem 4.16 for $t_n = \frac{1}{n}$, we have

$$f_n\left(\frac{1}{n}\right) = \frac{1}{2} \not\to 0,$$

which is a contradiction. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to 0 on D.

Example 4.19 Consider $f_n(t) = 1 - (1-t^2)^n$ on $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. We have that $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to 1 on D. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to 1 on D. Then, using Theorem 4.16 for $t_n = \frac{1}{n}$, we have

$$f_n\left(\frac{1}{n}\right) - 1 = -\left(1 - \frac{1}{n^2}\right)^n \not\to 0,$$

which is a contradiction. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to 1 on D.

Exercise 4.20 Consider $f_n(t) = \frac{nt+2}{3+4n^2t^2}$ on $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Using Theorem 4.16, prove that $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to 0 on D.

Theorem 4.21 *Let $D \subset \mathbb{T}$. If $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, are rd-continuous and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to $f : D \to \mathbb{R}$ on D, then f is rd-continuous on D and*

$$\int_a^b f(t)\Delta t = \lim_{n\to\infty} \int_a^b f_n(t)\Delta t$$

for every $[a, b] \subset D$.

Proof Since $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to f on D, for any given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$|f_M(t) - f(t)| < \frac{\varepsilon}{3} \quad \text{for all} \quad t \in D.$$

First, assume $t_0 \in D$ is left-dense. Because f_M is rd-continuous on D, there exists $\delta > 0$ such that

$$|f_M(t') - f_M(t'')| < \frac{\varepsilon}{3} \quad \text{for any} \quad t', t'' \in (t_0 - \delta, t_0).$$

If $t_n \to t_0^-$, $n \to \infty$, $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ such that $m, n > N$ imply $t_m, t_n \in (t_0 - \delta, t_0)$ and

$$|f_M(t_n) - f_M(t_m)| < \frac{\varepsilon}{3}. \tag{4.3}$$

Hence, for $m, n > N$, we have

$$\begin{aligned} |f(t_n) - f(t_m)| =& |f_M(t_n) - f(t_n) - f_M(t_n) - f_M(t_m) + f_M(t_m) - f(t_m)| \\ \le& |f_M(t_n) - f(t_n)| + |f_M(t_n) - f_M(t_m)| \\ &+ |f_M(t_m) - f(t_m)| \\ <& \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ =& \varepsilon. \end{aligned} \tag{4.4}$$

Hence, the left-sided limit of f in t_0 exists and is finite. Second, assume that $t_0 \in D$ is right-dense. Then f_n is continuous in t_0. Thus, there exists $N \in \mathbb{N}$ such that $m, n > N$ imply $t_m, t_n \in (t_0 - \delta, t_0 + \delta)$ and (4.3) holds. Therefore, (4.4) holds and f is continuous in t_0. Thus, f is rd-continuous on D. Hence, f is integrable on every $[a, b] \subset D$. For every $n > M$, we have

$$\begin{aligned} \left| \int_a^b f_n(t)\Delta t - \int_a^b f(t)\Delta t \right| &= \left| \int_a^b (f_n(t) - f(t))\Delta t \right| \\ &\le \int_a^b |f_n(t) - f(t)|\Delta t \\ &< \frac{\varepsilon}{3}(b-a), \end{aligned}$$

which completes the proof. □

Theorem 4.22 *Suppose that the function sequence*

$$\{f_n\}_{n\in\mathbb{N}}, \quad f_n : [a, b] \to \mathbb{R}, \quad n \in \mathbb{N},$$

satisfies the following conditions.

1. f_n, $n \in \mathbb{N}$, *is differentiable on* $[a, b]$, *and its derivative* f_n^Δ *is rd-continuous on* $[a, b]$,

2. *f_n converges pointwise to f on $[a, b]$,*
3. *$\left\{f_n^{\Delta}\right\}_{n\in\mathbb{N}}$ is uniformly convergent to g on $[a, b]$.*

Then f is differentiable on $[a, b]$ and

$$f^{\Delta}(t) = g(t) \ \textit{for any} \ \ t \in [a, b].$$

Proof By Theorem 4.21, we have that g is rd-continuous on $[a, b]$. Therefore, g is integrable on $[a, b]$. Hence, using Theorem 4.21, we get

$$\begin{aligned}\int_a^t g(s)\Delta s &= \lim_{n\to\infty} \int_a^t f_n^{\Delta}(s)\Delta s \\ &= \lim_{n\to\infty} (f_n(t) - f_n(a)) \\ &= f(t) - f(a) \quad \text{for any} \quad t \in [a, b].\end{aligned}$$

The left-hand side of the above formula is differentiable, so the right-hand side is also differentiable, and this leads to

$$f^{\Delta}(t) = g(t) \quad \text{for all} \quad t \in [a, b],$$

completing the proof. □

Theorem 4.23 (Dini Theorem) *Assume that the function sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n : [a, b] \to \mathbb{R}$, converges pointwise to the function f on $[a, b]$. If the conditions*

1. *f_n, $n \in \mathbb{N}$, are rd-continuous on $[a, b]$,*
2. *f is rd-continuous on $[a, b]$,*
3. *for any given $t \in [a, b]$, $\{f_n(t)\}_{n\in\mathbb{N}}$ is monotone with respect to $n \in \mathbb{N}$*

hold, then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to f on $[a, b]$.

Proof Suppose that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to f on $[a, b]$. Then there exists $\varepsilon_0 > 0$ such that for any given $N \in \mathbb{N}$, there exist $n > N, t \in [a, b]$, implying

$$|f_n(t) - f(t)| \geq \varepsilon_0. \tag{4.5}$$

For $N = 1$, there exist $n_1 > 1$, $t_1 \in [a, b]$, such that

$$|f_{n_1}(t_1) - f(t_1)| \geq \varepsilon_0.$$

For $N = n_1$, there exist $n_2 > n_1$, $t_2 \in [a, b]$, such that

$$|f_{n_2}(t_2) - f(t_2)| \geq \varepsilon_0,$$

and so on. For $N = n_k$ there exist $n_{k+1} > n_k$, $t_{k+1} \in [a, b]$, such that

$$|f_{n_{k+1}}(t_{k+1}) - f(t_{k+1})| \geq \varepsilon_0.$$

Hence, we obtain a point sequence $\{t_k\}_{k\in\mathbb{N}}$, $t_k \in [a, b]$. This sequence has a convergent subsequence. Let $\{t_{k_l}\}_{l\in\mathbb{N}}$ be a convergent subsequence of the sequence $\{t_k\}_{k\in\mathbb{N}}$ and $t_{k_l} \to \xi$ as $l \to \infty$. We have that $\xi \in [a, b]$. Because $f_n(\xi) \to f(\xi)$ as $n \to \infty$, for the above ε_0, there exists $N \in \mathbb{N}$ such that

$$|f_N(\xi) - f(\xi)| < \frac{\varepsilon_0}{2}.$$

Suppose that ξ is left-dense and right-scattered. Then, for the above sequence $\{t_{k_l}\}_{l\in\mathbb{N}}$, $t_{k_l} \leq \xi$ and $t_{k_l} \to \xi$ as $l \to \infty$. Thus, for $\varepsilon_0 > 0$, as above, there exists $L \in \mathbb{N}$ such that $l > L$ implies

$$|\xi - t_{k_l}| < \varepsilon_0.$$

Since f_N and f are rd-continuous on $[a, b]$, we have that

$$\left|\left(f_N(t_{k_l}) - f(t_{k_l})\right) - (f_N(\xi) - f(\xi))\right| < \frac{\varepsilon_0}{2}.$$

Hence,

$$\begin{aligned}
\left|f_N(t_{k_l}) - f(t_{k_l})\right| &< \frac{\varepsilon_0}{2} + |f_N(\xi) - f(\xi)| \\
&< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} \\
&= \varepsilon_0.
\end{aligned}$$

Suppose that ξ is not left-dense and right-scattered. Then ξ is a point of continuity of f_N and f. Because $t_{k_l} \to \xi$, $l \to \infty$, there exists $L_1 \in \mathbb{N}$ such that $l > L_1$ implies

$$|f_N(t_{k_l}) - f(t_{k_l})| < \varepsilon_0.$$

By using the monotonicity condition, we get

$$|f_n(t_{k_l}) - f(t_{k_l})| \leq |f_N(t_{k_l}) - f(t_{k_l})| < \varepsilon_0$$

with $n > N$, $l > \max\{L, L_1\}$. So when n is sufficiently large, $n_l > N$ and $l > \max\{L, L_1\}$ are satisfied. Thus,

$$|f_{n_{k_l}(t_{k_l})} - f(t_{k_l})| < \varepsilon_0,$$

which contradicts to (4.5). This completes the proof. $\square$

4.2 Uniform Convergence of Series of Functions

Let $f_n : \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}$. We consider the infinite series

$$\sum_{n=1}^{\infty} f_n(t). \tag{4.6}$$

Definition 4.24 If the numerical series $\sum_{n=1}^{\infty} f_n(t_0)$, $t_0 \in \mathbb{T}$, is convergent, then t_0 is called a *point of convergence* of the function series (4.6). The set D of all points of convergence of the series (4.6) is called its *domain of convergence*. If a domain of convergence of a function series is not empty, then the function series is called *pointwise convergent* on its domain of convergence. We define the *sum function* $F = \sum_{n=1}^{\infty} f_n$ on D.

Definition 4.25 The function

$$F_n(t) = f_1(t) + f_2(t) + \cdots + f_n(t), \quad n \in \mathbb{N},$$

is called the *partial sum* or, more precisely, the nth partial sum of the function series (4.6).

Definition 4.26 If the partial sum sequence $\{F_n\}_{n\in\mathbb{N}}$ of the function series (4.6) is uniformly convergent to F on D, then we term that the function series (4.6) is *uniformly convergent* to F on D.

Example 4.27 Let

$$\mathbb{T} = (-1, 0] \cup \left\{ \frac{1}{n} : \; n \in \mathbb{N} \setminus \{1\} \right\},$$

where $(-1, 0]$ is the real-valued interval. Consider the series

$$\sum_{l=1}^{\infty} t^{l-1}. \tag{4.7}$$

Note that the series (4.7) is pointwise convergent on $\mathbb{T}$ to $F(t) = \frac{1}{1-t}$. Moreover,

$$F_n(t) = \sum_{l=1}^{n} t^{l-1} = \frac{1-t^n}{1-t}$$

and

$$F(t) - F_n(t) = \frac{1}{1-t} - \frac{1-t^n}{1-t} = \frac{t^n}{1-t}.$$

Assume that the series (4.7) is uniformly convergent to F on $\mathbb{T}$. Take $\varepsilon > 0$ arbitrarily. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\left|\frac{t^n}{1-t}\right| < \varepsilon \quad \text{for all} \quad t \in \mathbb{T}.$$

If $t \in [0, 1)$, then

$$\frac{t^n}{1-t} \to \infty \quad \text{as} \quad t \to 1$$

and

$$\sup_{t\in[0,1)} \frac{t^n}{1-t} \geq 1.$$

Therefore,

$$\sup_{t\in[0,1)} \frac{t^n}{1-t} \not\to 0 \quad \text{as} \quad n \to \infty.$$

Hence, the series (4.7) is not uniformly convergent to F on $\mathbb{T}$.

Example 4.28 Let $\mathbb{T} = \mathbb{Z}$. Consider the series

$$\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{t^2 + l}.$$

By the Leibniz criterion, we have that this series is pointwise convergent on $\mathbb{T}$. Moreover,

$$\begin{aligned}\sup_{t\in\mathbb{T}} |F_n(t) - F(t)| &\leq \sup_{t\in\mathbb{T}} \frac{1}{t^2+n+1} \\ &\leq \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore, the considered series is uniformly convergent on $\mathbb{T}$.

Theorem 4.29 (Cauchy Criterion for Uniform Convergence of a Function Series) *The function series $\sum_{n=1}^{\infty} f_n$ converges uniformly on D if and only if for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|f_{n+1}(t) + \cdots + f_m(t)| < \varepsilon \tag{4.8}$$

for all $m, n \in \mathbb{N}$ satisfying $m > n$ and every point $t \in D$.

Proof 1. Necessity. Suppose that the function series $\sum_{n=1}^{\infty} f_n$ converges uniformly on D and its sum function is F. Then, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\left|F(t) - \sum_{k=1}^{n} f_k(t)\right| < \frac{\varepsilon}{2} \quad \text{for all} \quad t \in D.$$

Hence, for all $m > n > N$ and all $t \in D$, we have

$$\begin{aligned}
|f_{n+1}(t) + \cdots + f_m(t)| &= \left|\sum_{k=1}^{m} f_k(t) - \sum_{k=1}^{n} f_k(t)\right| \\
&= \left|\sum_{k=1}^{m} f_k(t) - F(t) - \sum_{k=1}^{n} f_k(t) + F(t)\right| \\
&\leq \left|\sum_{k=1}^{m} f_k(t) - F(t)\right| + \left|F(t) - \sum_{k=1}^{n} f_k(t)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

2. Sufficiency. Suppose that for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that (4.8) holds for all $m > n > N$ and all $t \in D$. Fix $t_0 \in D$. Then the numerical series $\sum_{n=1}^{\infty} f_n(t_0)$ satisfies the Cauchy criterion for convergence of a numerical series. Thus, the numerical series $\sum_{n=1}^{\infty} f_n(t_0)$ is convergent. Because $t_0 \in D$ was arbitrarily chosen, we conclude that the series $\sum_{n=1}^{\infty} f_n$ is pointwise convergent on D. Let F be its sum function. Choose n for

$$\left|\sum_{k=1}^{m} f_k(t) - \sum_{k=1}^{n} f_k(t)\right| < \frac{\varepsilon}{2} \quad \text{for all} \quad t \in \mathbb{T}.$$

If $m \to \infty$, then we get

$$\left|F(t) - \sum_{k=1}^{n} f_k(t)\right| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for all} \quad t \in D.$$

Therefore, $\sum_{k=1}^{\infty} f_k$ converges uniformly to F on D.

This completes the proof. □

Corollary 4.30 (Necessary Condition for Uniform Convergence of a Function Series) *A necessary condition for the series $\sum_{n=1}^{\infty} f_n$ to converge uniformly on D is that $f_n \to 0$ uniformly on D as $n \to \infty$.*

Example 4.31 Let $\mathbb{T} = \left\{\frac{1}{\sqrt[3]{n}} : n \in \mathbb{N}\right\} \cup \{0\}$. Consider the series

$$\sum_{k=1}^{\infty} \frac{t^3}{(1+t^3)^k}.$$

Thus,

$$\begin{aligned}\sum_{k=n+1}^{3n} \frac{t^3}{(1+t^3)^k} &= \frac{t^3}{(1+t^3)^{n+1}} + \frac{t^3}{(1+t^3)^{n+2}} + \cdots + \frac{t^3}{(1+t^3)^{3n}} \\ &> \frac{3nt^3}{(1+t^3)^{3n}}.\end{aligned}$$

Let $\varepsilon = \frac{3}{e^3}$. For any $N \in \mathbb{N}$, we choose $m = 3n$, $n > N$, and $t_n = \frac{1}{\sqrt[3]{n}} \in \mathbb{T}$, so that

$$\begin{aligned}\sum_{k=n+1}^{3n} \frac{t^3}{(1+t^3)^k} &> \frac{3n\frac{1}{n}}{\left(1+\frac{1}{n}\right)^{3n}} \\ &= \frac{3}{\left(1+\frac{1}{n}\right)^{3n}} \\ &> \frac{3}{e^3} = \varepsilon.\end{aligned}$$

Then, using Theorem 4.29, we conclude that the considered series is nonuniformly convergent on $\mathbb{T}$.

Theorem 4.32 (Weierstraß M -Test for Uniform Convergence of a Function Series) *Suppose that every term f_n of the function series $\sum_{n=1}^{\infty} f_n$ satisfies*

$$|f_n(t)| \leq a_n \ \textit{for all} \ n \in \mathbb{N} \ \textit{and all} \ t \in D,$$

and the numerical series $\sum_{n=1}^{\infty} a_n$ is convergent. Then the function series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D.

Proof Since the numerical series $\sum_{k=1}^{\infty} a_k$ is convergent, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m > n > N$ imply

$$a_{n+1} + a_{n+2} + \cdots + a_m < \varepsilon.$$

Hence,

$$\begin{aligned}\left|\sum_{k=n+1}^{m} f_k(t)\right| &\leq \sum_{k=n+1}^{m} |f_k(t)| \\ &\leq \sum_{k=n+1}^{m} a_k \\ &< \varepsilon.\end{aligned}$$

Then, using Theorem 4.29, we conclude that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D. □

Example 4.33 Let $\mathbb{T} = \mathbb{P}_{1,2} = \bigcup_{k=0}^{\infty}[3k, 3k+1]$. Consider the series

$$\sum_{n=1}^{\infty} \frac{t^2}{1+n^5t^4}.$$

We have $f_n(t) = \frac{t^2}{1+n^5t^4}$ and

$$\begin{aligned} f_n(t) &\leq \frac{t^2}{n^{\frac{5}{2}}t^2} \\ &= \frac{1}{n^{\frac{5}{2}}}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ is convergent, by Theorem 4.32, we conclude that the considered series is uniformly convergent on $\mathbb{T}$.

Example 4.34 Let $\mathbb{T} = [-3, 0] \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, where $[-3, 0]$ is the real-valued interval. Consider the series

$$\sum_{n=1}^{\infty} \frac{(t+1)\sin^2(nt)}{n\sqrt{n+1}}.$$

Here, $f_n(t) = \frac{(t+1)\sin^2(nt)}{n\sqrt{n+1}}$. We have

$$\begin{aligned} |f_n(t)| &= \left|\frac{(t+1)\sin^2(nt)}{n\sqrt{n+1}}\right| \\ &= \frac{|t+1|\sin^2(nt)}{n\sqrt{n+1}} \\ &\leq \frac{|t|+1}{n\sqrt{n+1}} \\ &\leq \frac{4}{n\sqrt{n+1}} \quad \text{for all} \quad t \in \mathbb{T}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{4}{n\sqrt{n+1}}$ is convergent, by Theorem 4.32, we conclude that the considered series is uniformly convergent on $\mathbb{T}$.

Example 4.35 Let $\mathbb{T} = \mathbb{Z}$. Consider the series

$$\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^4+1}.$$

Here, $f_n(t) = \frac{\cos(nt)}{n^4+1}$. We have

$$
\begin{aligned}
|f_n(t)| &= \left|\frac{\cos(nt)}{n^4+1}\right| \\
&= \frac{|\cos(nt)|}{n^4+1} \\
&\le \frac{1}{n^4+1}.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^4+1}$ is convergent, by Theorem 4.32, we conclude that the considered series is uniformly convergent on $\mathbb{T}$.

Exercise 4.36 Using Theorem 4.32, prove that the following series are uniformly convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+nt+t^2}$, $\mathbb{T} = \mathbb{Z}$,
2. $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{1+(2n+1)t}}$, $\mathbb{T} = \mathbb{N}$,
3. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{t}{n}$, $\mathbb{T} = 3^{\mathbb{N}_0}$,
4. $\sum_{n=1}^{\infty} \frac{e^{-n^2t^2}}{1+n^2}$, $\mathbb{T} = \mathbb{Z}$,
5. $\sum_{n=1}^{\infty} \frac{\sqrt{1-t^{2n}}}{2^n}$, $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$,
6. $\sum_{n=1}^{\infty} \frac{t^n}{n\sqrt{n}}$, $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$.

Theorem 4.37 (Abel Test) *Let the function sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ be monotone for each fixed $t \in D$ with respect to n and suppose that $\{f_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded on D, i.e.,*

$$|f_n(t)| \le M \quad \textit{for all} \quad t \in D \quad \textit{and} \quad n \in \mathbb{N}.$$

If the function series $\sum_{n=1}^{\infty} g_n$ is uniformly convergent on D, then the function series $\sum_{n=1}^{\infty} f_n g_n$ is uniformly convergent on $\mathbb{T}$.

Proof Since $\sum_{n=1}^{\infty} g_n(t)$ is uniformly convergent on D, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{k=n+1}^{m} g_k(t)\right| < \varepsilon$$

for all $m > n > N$ and all $t \in D$. By applying the Abel lemma, we obtain

$$
\begin{aligned}
\left|\sum_{k=n+1}^{m} f_k(t)g_k(t)\right| &\le \varepsilon\left(|f_{n+1}(t)| + 2|f_m(t)|\right) \\
&\le 3M\varepsilon
\end{aligned}
$$

for all $m > n > N$ and all $t \in D$. Hence, by Theorem 4.29, it follows that $\sum_{k=1}^{\infty} f_k g_k$ is uniformly convergent on D. $\square$

Example 4.38 Consider the series $\sum_{n=1}^{\infty} \frac{e^{-nt}}{n!}$ on $\mathbb{T} = \mathbb{N}$. We have that the sequence $\left\{e^{-nt}\right\}_{n\in\mathbb{N}}$ is monotone for each fixed $t \in \mathbb{T}$ with respect to n, and it is uniformly bounded by 1 on $\mathbb{T}$. Note that $\sum_{n=1}^{\infty} \frac{1}{n!}$ is a convergent numerical series. Hence, using Theorem 4.37, we conclude that the considered series is uniformly convergent on $\mathbb{T}$.

Example 4.39 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n(1+t^n)} \quad \text{on} \quad \mathbb{T}.$$

Note that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is a convergent numerical series. Also, the sequence $\left\{\frac{t^n}{1+t^n}\right\}_{n\in\mathbb{N}}$ is monotone on $\mathbb{T}$ with respect to n and uniformly bounded by 1 on $\mathbb{T}$. Hence, by Theorem 4.37, it follows that the considered series is uniformly convergent on $\mathbb{T}$.

Exercise 4.40 Using the Abel test, prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{\sqrt{n+t}\,\log\log(1+2\sqrt{n})}$$

is uniformly convergent on $\mathbb{T} = \mathbb{N}$.

Theorem 4.41 (Dirichlet Test) *Suppose that the function sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is monotone for each $t \in D$ with respect to n and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to* 0 *on D. In addition, assume that the partial sum sequence of $\sum_{n=1}^{\infty} g_n(t)$ is uniformly bounded on D, i.e.,*

$$\left|\sum_{k=1}^{n} g_k(t)\right| \le M \;\; \textit{for all} \;\; t \in D \;\; \textit{and} \;\; n \in \mathbb{N}.$$

Then $\sum_{n=1}^{\infty} f_n g_n$ is uniformly convergent on D.

Proof Since $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to 0 on D, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|f_n(t)| < \varepsilon \quad \text{for all} \quad t \in D.$$

Moreover, for all $m > n > N$, we have

$$\begin{aligned}\left|\sum_{k=n+1}^{m} g_k(t)\right| &= \left|\sum_{k=1}^{m} g_k(t) - \sum_{k=1}^{n} g_k(t)\right| \\ &\le \left|\sum_{k=1}^{m} g_k(t)\right| + \left|\sum_{k=1}^{n} g_k(t)\right| \\ &\le 2M.\end{aligned}$$

Using the Abel lemma, we get

$$\begin{aligned}\left|\sum_{k=n+1}^{m} f_k(t)g_k(t)\right| &\le 2M\left(|f_{n+1}(t)| + 2|f_m(t)|\right) \\ &< 6M\varepsilon.\end{aligned}$$

Hence, using Theorem 4.29, it follows that $\sum_{n=1}^{\infty} f_n g_n$ is uniformly convergent on D. □

Example 4.42 Let $\mathbb{T} = \mathbb{N}_0 \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+t^2}.$$

We set

$$f_n(t) = \frac{1}{n+t^2}, \quad g_n(t) = (-1)^n, \quad t \in D, \quad n \in \mathbb{N}.$$

Then the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is monotone for each $t \in D$ with respect to n, and it is uniformly convergent to 0 on D. Moreover,

$$\left|\sum_{k=1}^{n} g_k(t)\right| \le 1 \quad \text{for all} \quad t \in D, \quad n \in \mathbb{N}.$$

Hence, utilizing the Dirichlet test, we conclude that the considered series is uniformly convergent on D.

Example 4.43 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N} \setminus \{1\}\right\} \cup \{0\}$. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{1+t+\cdots+t^{2n-1}}.$$

We set

$$f_n(t) = \frac{t^n}{1+t+\cdots+t^{2n-1}}, \quad g_n(t) = (-1)^{n-1}, \quad t \in D, \quad n \in \mathbb{N}.$$

Then

$$\begin{aligned} f_n(t) &< \frac{t^n}{1+t+\cdots+t^{n-1}} \\ &< \frac{t^n}{nt^n} \\ &= \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

i.e., the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to 0 on $\mathbb{T}$. Moreover,

$$\frac{t^{n+1}}{1+t+\cdots+t^{2n-1}+t^{2n+1}} < \frac{t^n}{1+t+t^2+\cdots+t^{2n-1}},$$

i.e., the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is monotone for each $t \in \mathbb{T}$ with respect to n. Note that

$$\left|\sum_{k=1}^{n} g_k(t)\right| \leq 1 \quad \text{for} \quad t \in \mathbb{T}, \quad n \in \mathbb{N}.$$

Hence, using the Dirichlet test, it follows that the considered series is uniformly convergent on $\mathbb{N}$.

Exercise 4.44 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Using the Dirichlet Test, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n}}{2n-1}$$

is uniformly convergent on $\mathbb{T}$.

By Theorem 4.21, the following result is clear.

Theorem 4.45 *If every term f_n of the function series $\sum_{n=1}^{\infty} f_n$ is rd-continuous and $\sum_{n=1}^{\infty} f_n$ is uniformly convergent to f on D, then f is rd-continuous and*

$$\int_a^b f(t)\Delta t = \sum_{n=1}^{\infty} \int_a^b f_n(t)\Delta t \ \textit{for every} \ [a,b] \subset D.$$

By Theorem 4.22, we obtain the following result.

Theorem 4.46 *Suppose that the function series $\sum_{n=1}^{\infty} f_n$ satisfies*

1. *f_n is differentiable and its derivative function f_n^{Δ} is rd-continuous on $[a,b]$,*

2. $\sum_{n=1}^{\infty} f_n$ *converges pointwise to* f *on* $[a, b]$,
3. $\sum_{n=1}^{\infty} f_n^{\Delta}$ *is uniformly convergent to* g *on* $[a, b]$.

Then $f = \sum_{n=1}^{\infty} f_n$ *is differentiable on* $[a, b]$ *and* $f^{\Delta} = g$ *for all* $t \in [a, b]$.

By Theorem 4.23, we get the following result.

Theorem 4.47 (Dini Theorem) *Assume that the function series* $\sum_{n=1}^{\infty} f_n$ *is pointwise convergent to its sum function* f *on* $[a, b]$. *If the conditions*

1. f_n, $n \in \mathbb{N}$, *are rd-continuous on* $[a, b]$,
2. f *is rd-continuous on* $[a, b]$,
3. *for any given* $t \in [a, b]$, *the function series* $\sum_{n=1}^{\infty} f_n(t)$ *is either a positive term series or a negative term series*

hold, then $\sum_{n=1}^{\infty} f_n$ *is uniformly convergent to* f *on* $[a, b]$.

4.3 Advanced Practical Problems

Problem 4.48 Let

$$f_n(t) = t^2 \left(t + 1 + e^{-n^2 t^2} \right), \quad f(t) = t^3 + t^2.$$

Prove that $f_n \to f$ pointwise on $\mathbb{T}$.

Problem 4.49 Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Check if the following sequences are uniformly convergent on $\mathbb{T}$.

1. $f_n(t) = ne^{-nt^2}$,
2. $f_n(t) = nte^{-nt^2}$,
3. $f_n(t) = \frac{1}{t^2+nt+1}$.

Solution 1. Uniformly convergent to 0 on $\mathbb{T}$,
2. uniformly convergent to 0 on $\mathbb{T}$,
3. uniformly convergent to 0 on $\mathbb{T}$.

Problem 4.50 Let $\mathbb{T} = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup 2^{\mathbb{N}_0}$. Using Theorem 4.11, investigate the sequence

$$\left\{ f_n(t) = \frac{nt^2}{n+t} \right\}_{n \in \mathbb{N}}$$

for uniform convergence on $D_1 = [1, \infty)$ and $D_2 = [0, 2]$.

Solution The sequence is not uniformly convergent on D_1, and it is uniformly convergent on D_2.

Problem 4.51 Consider $f_n(t) = 2 + \frac{n^2t^2+nt+2}{3+n^3t^3+n^2t^2+nt}$ on $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Using Theorem 4.16, prove that this sequence is not uniformly convergent to 2 on D.

Problem 4.52 Using Theorem 4.32, prove that the following series are uniformly convergent.

1. $\sum_{n=1}^{\infty} \frac{t}{(1+nt)(1+(n+1)t)}$, $\mathbb{T} = 2^{\mathbb{N}_0}$,
2. $\sum_{n=1}^{\infty} \frac{(\pi-t)\cos^2(nt)}{\sqrt[5]{n^7+1}}$, $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$,
3. $\sum_{n=1}^{\infty} \arctan \frac{t}{t^2+n^3}$, $\mathbb{T} = \mathbb{Z}$,
4. $\sum_{n=1}^{\infty} \frac{t^3\sin^2(nt)}{2+n^3t^6}$, $\mathbb{T} = \mathbb{N}$,
5. $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin \frac{n^3t}{n^2+1}$, $\mathbb{T} = 2^{\mathbb{N}_0}$,
6. $\sum_{n=1}^{\infty} \frac{\arctan(2n^2t)}{\sqrt[3]{n^7+n+t}}$, $\mathbb{T} = \mathbb{N}$.

Problem 4.53 Using the Abel test, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^t}$$

is uniformly convergent on $\mathbb{T} = \mathbb{N}$.

Problem 4.54 Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Using the Dirichlet test, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2t^2+n}$$

is uniformly convergent on $\mathbb{T}$.

4.4 Notes and References

In this chapter, the concept of function series and sequences is extended to time scales. Necessary and sufficient conditions and several criteria for uniform convergence of function series and sequences are presented. Several analytical properties of function series and function sequences on general time scales are given. All results in this chapter are taken from Pang and Wang [36].

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