Chapter 2
Boolean Descriptions

2.1 On the Necessity of Distinctions

> You'll soon know it all, as planned,
> When you've learnt the science of reduction,
> And everything's proper classification.
>
> *Johann Wolfgang von Goethe (2003, p. 78)*

According to the British mathematician George Spencer Brown, the most fundamental of all cognitive operations is to draw distinctions. In his seminal book *Laws of Form* (Spencer Brown 1969), he developed a binary arithmetic based upon distinctions and principles of symbolic logic.[10] He begins his book with the key statement (Spencer Brown 1969, p. 1):

> We take as given the idea of distinction and the idea of indication, and that we cannot make an indication without drawing a distinction.

To make a distinction is to divide the universe of discourse into two parts (be it mathematical, cognitive, linguistic, perceptual, etc.). Every distinction has a purpose but no *a priori* meaning: "There can be no distinction without motive, and there can be no motive unless contents are seen to differ in value" (Spencer Brown 1969, p. 1). As Suzanne Langer (1978, p. 273) observed, "our world 'divides into facts' because we so divide it". Distinctions create frames of reference, necessary for any kind of description.

The search for regularities is the principal concern of all scientific inquiry. Regularities can be found if and only if we suppress irrelevant features. What is relevant and what is irrelevant is not determined by some law of nature but by convention—or by our interests, by our cognitive faculties, by evolution, or by the pattern recognition devices used by experimentalists.

All concepts of empirical science refer to observations obtained by pattern recognition protocols which ignore many features and concentrate on those which we

[10]Banaschewski (1977) showed that Spencer Brown's concept of a primary algebra is exactly the theory of join and addition (i.e., symmetric difference) of Boolean algebras.

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consider as relevant. A pattern is something that somebody recognizes as a pattern. What is considered as relevant, and what as irrelevant, is not written down in some first principles but depends in a crucial way on the abstractions chosen by someone. Since the world is neither intrinsically divided into subsystems nor are there unprejudiced sense data or unbiased observations, observable patterns of the world do not exist by themselves. Any recognition of patterns and resulting classifications are based on the primitive act of *making a distinction*. Distinctions eliminate irrelevant features. It follows that *every classification is context-dependent and unavoidably entails a loss of information*.

2.2 Boolean Logic

The concept of a Boolean algebra has its roots in the algebra of logic, introduced by George Boole (1847, 1848) as a study of the laws of thought. He tried to explain how the mind processes thought (Boole 1849, p. 13):

> The object of Logic as a Science is to explain the laws of those mental operations by which ordinary Reasoning is conducted.

Boole's logic is based on a binary approach—the yes-no, true-false approach. His classic *The Laws of Thought* (Boole 1854) begin with these words:

> The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolic language of a Calculus, and upon this foundation to establish the science of Logic and construct its method.

Boole's idea that logic is the study of the laws of thought was taken over by Frege when he stated that the task of logic can be seen "as the investigation of the mind".[11] Similarly, Hilbert assumed that there is a correspondence between human thinking and the formal system of proof theory:[12]

> Logic is a discipline which expresses the structure of all our thought.

Boolean logic, originally developed by George Boole in the late 1830s, is a two-valued logic, based on the *law of the excluded middle*, asserting that every proposition is either true or false (Boole 1847, 1848, 1854). Alessandro Padoa emphasized that logic is a purely formal system which allows us to make inferences without specifying a particular interpretation:[13]

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[11] Frege (1918/1919, p. 74, editor's translation): "Neither logic nor mathematics has the task to inquire into the souls and conscious contents of individual human beings. One might rather consider it as their task to study the mind, not the minds."

[12] Hilbert (1931, p. 125, editor's translation): "Through the tertium non datur logic acquires full harmony; its theorems attain such simplicity and the system of its concepts such completeness as it ought to conform to the significance of a discipline expressing the structure of all our thinking."

[13] See Padoa (1901, pp. 309–365). English translation quoted from van Heijenoort (1967, p. 121).

> ...for what is necessary to the logical development of a deductive theory *is not the empirical knowledge of the properties of things*, but the *formal knowledge of relations between symbols*.

The importance of logic as a formal symbolic (not natural) language lies in the exclusion of ambiguities and implicit assumptions.

The propositional calculus of Boolean logic (also called classical logic) deals with propositions which can be studied irrespective of their truth. Here, a proposition is understood as an unambiguous sentence that is either true or false. It is then possible to examine how propositions are combined by means of sentential connectives.

- If A is a proposition, then the *negation* "not A", denoted by $A^\perp$, is also a proposition.
- If A, B are two propositions, then the *conjunction* "A and B" is also a proposition, denoted by $A \wedge B$.
- The proposition "A or B, or both" is called the *disjunction* (the *inclusive or*) and is denoted by $A \vee B$.
- The relation "A implies B" is written as $A \leq B$.
- If proposition A is logically equivalent to proposition B, one writes $A = B$.

All false propositions are considered equal and are identified by the falsehood symbol $\mathbb{0}$. The universally true proposition is denoted by the symbol $\mathbb{1}$. Classical logic is a *two-valued* logic which accepts the doctrine of the "law of the excluded middle" that every proposition is either true or false—*tertium non datur*.

A *Boolean algebra* is a set $\mathcal{B}$ of propositions $A, B, \ldots$ with an operation $^\perp$ of rank one, with two operations $\wedge$ and $\vee$ of rank two, and with two distinguished elements $\mathbb{0}$ and $\mathbb{1}$, such that the following axioms hold:

$$\text{(i)} \quad \left(A^\perp\right)^\perp = A \text{ for all } A \in \mathcal{B}, \tag{2.1a}$$

$$\text{(ii)} \quad \wedge \text{ and } \vee \text{ are each associative and commutative,} \tag{2.1b}$$

$$\text{(iii)} \quad \wedge \text{ and } \vee \text{ are distributive with respect to one another,} \tag{2.1c}$$

$$\text{(iv)} \quad (A \vee B)^\perp = A^\perp \wedge B^\perp, (A \wedge B)^\perp = A^\perp \vee B^\perp, \tag{2.1d}$$

$$\text{(v)} \quad A \vee A = A \wedge A = A, \tag{2.1e}$$

$$\text{(vi)} \quad \mathbb{1} \wedge A = A, \quad \mathbb{0} \vee A = A. \tag{2.1f}$$

In Boolean (classical) logic the following axioms hold:

$$\left(A^\perp\right)^\perp = A \quad \text{(law of double negation)} \tag{2.2a}$$

$$A \leq B \text{ if and only if } B^\perp \leq A^\perp \quad \text{(law of contraposition)} \tag{2.2b}$$

$$(A \vee B)^\perp = A^\perp \wedge B^\perp \tag{2.2c}$$

$$\text{and} \quad (A \wedge B)^\perp = A^\perp \vee B^\perp \quad \text{(de Morgan's laws)} \tag{2.2d}$$

$$(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C) \tag{2.2e}$$

$$\text{and} \quad (A \wedge B) \vee C = (A \vee C) \wedge (B \vee C) \quad \text{(distributive laws)} \tag{2.2f}$$

$$A \wedge A^{\perp} = \mathbb{0} \quad \text{(law of contradiction)} \tag{2.2g}$$

$$A \vee A^{\perp} = \mathbb{1} \quad \text{(law of excluded middle)} \tag{2.2h}$$

2.3 On the Biological Basis of Binary Distinctions

> We have found a strange footprint on the shores of the unknown.
> We have devised profound theories, one after another,
> to account for its origins. At last, we have succeeded in
> reconstructing the creature that made the footprint.
> And lo! It is our own.
>
> *Arthur Stanley Eddington (1920, p. 201)*

In contemporary science the results of experiments have to be expressed in terms of an unambiguous *bivalent language* in the sense that every statement about an empirical fact has to be *either true or false*. Since binary yes/no or true/false distinctions have become a dominating principle of scientific reasoning, two-valued classical Boolean logic has a special status in our system of knowledge. But its outstanding simplicity proves in no way that it is given *a priori*. Rather, it is plausible that our innate preference for two-valued logic has a biological basis. Our apparently self-evident so-called "laws of thought" may be related to the neurobiological architecture of our brains.

The central nervous system is in an essential way based on the all-or-none character of neural events. As a result of biological evolution, animal and human neural networks operate with *binary choices*, presumably developed in the struggle for survival. Strictly binary decisions eliminate vagueness by reducing the available information. In this way conflicting responses in the brain are avoided so that simple and fast decisions become possible.

Practically all biological systems use a threshold logic to achieve clear yes-no responses (compare McCulloch and Pitts 1943; Platt 1956; von Neumann 1951, p. 22). It seems that built-in threshold gates ("tertium-non-datur" devices) have evolutionary advantages. They lead to Boolean circuits based on AND, OR and NOT gates. Warren McCulloch and Walter Pitts (1943) showed that anything that can be defined completely and unambiguously in a finite number of words can be realized by a Boolean neural network.

John Platt (1956) suggested that our preference for the "laws of thought" of two-valued logic are a natural consequence of their evolutionary advantage for survival. A neural network realizes a threshold logic to achieve strictly binary decisions which suppress conflicting stimuli so that ambiguous responses are avoided and clear-cut instructions for action become feasible. As Platt (1956, p. 195) remarked, "survival selects the reproducible". On a similar vein, Szent-Györgi (1962, p. 11) stated:

> Primarily the human brain is an organ of survival. It was built by nature to search for food, shelter, and the like, to gain advantage—before addressing itself to the pursuit of truth.

Neural threshold gates explain why we preferably perceive and interpret the world filtered through a Boolean frame of reference.

2.4 Boolean Worldviews

2.4.1 Boolean Frames of Reference

Scientific knowledge is partly based on sense perception, partly on theoretical ideas, partly constructed by social processes, and partly informed by historical preconceptions. Therefore, the established body of scientific knowledge at any given time depends also on history and culture, and determines a system of beliefs. We cannot avoid that scientific knowledge is always influenced by extra-scientific factors. A *conceptual scheme* by which we define reality and within which we think and interpret our experiences is called a *worldview*. According to Feyerabend (1994, p. 152), a worldview is

> a collection of beliefs, attitudes and assumptions that involves the whole person, not only the intellect, has some kind of coherence and universality and imposes itself with a power far greater than the power of facts and fact-related theories.

From the very beginning, humankind has asked for and developed knowledge that can be considered as trustworthy. Knowledge always rests on assumptions and choices. It is never exhaustive, so we need to distinguish what is relevant from what is irrelevant in a certain situation. The idea that science is the only legitimate source of beliefs about reality has been criticized by Michael Polanyi (1958, p. 217):

> ...the scientists actually establish the current meaning of the term "science", determine what should be accepted as science, and establish also the current meaning of the term "scientist" and decide that they themselves and those designated by themselves as their successors should be recognized as such.

Moreover, Polanyi (1958, pp. 274f) argued that even among "scientists" intersubjective agreement is not sufficient to avoid serious errors or mistaken beliefs:

> Ordinary people were convinced of the fall of a meteorite, when an incandescent mass struck the earth with a crash of thunder a few yards away, and they tended to attach supernatural significance to it. The scientific committees of the French Academy disliked this interpretation so much that they managed, during the whole of the eighteenth century, to explain the facts away to their own satisfaction. ... We regard these acts of scepticism as unreasonable and indeed preposterous today, for we no longer consider the falling of meteorites ... to be incompatible with the scientific worldview. But other doubts, which we now sustain as reasonable on the grounds of our own scientific worldview, have once more only our beliefs in this view to warrant them. Some of these doubts may turn out one day to have been as wanton, as bigoted and dogmatic as those of which we have now been cured.

Even in modern science we have to take into account that knowledge is always influenced by biases and ideologies which exert a powerful influence—knowledge and belief cannot strictly be delineated. For these reasons William Cobern (1996, p. 584) disapproved the notion of a "scientific worldview" as misleading since a worldview

> provides a non-rational foundation for thought, emotion, and behavior. A worldview provides a person with presuppositions about what the world is really like and what constitutes valid and important knowledge about the world.

We call a domain, in which all that is knowable in principle is also simultaneously knowable, a *Boolean frame of reference* or a *Boolean context*. Niels Bohr (1949, p. 209) characterized the privileged role of Boolean, or classical, descriptions as follows:

> *However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms.* The argument is simply that by the word "experiment" we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics.

This requirement actually reflects scientific practice: the outcome of every experiment ever performed in physics, chemistry, biology or psychology can be described in a language based on classical Boolean logic.[14] For the description of an observed result of an experiment Bohr proposed to use (Bohr 1948, p. 317)

> the word phenomenon to refer exclusively to observations obtained under specified circumstances, including an account of the whole experiment.

2.4.2 Incompatible Boolean Descriptions

The fact that every outcome of a *single* experiment allows a description in terms of classical Boolean logic does not imply that the family of all feasible experiments can be combined into a single Boolean frame of reference. We adopt the following terminology:

> Two Boolean descriptions are said to be *compatible* if they can be embedded into a single Boolean reference frame.
> Two Boolean descriptions are said to be *incompatible* if they cannot be embedded into a single Boolean reference frame.

Since there are many mutually incompatible Boolean contexts, there are always many different mutually incompatible descriptions of one and the same universe of

[14] According to Bohr (1934, p. 53), our "interpretation of the experimental material rests essentially upon the classical concepts". Bohr's emphasis on the special role of classical physics is somewhat misleading. By a "classical description" Bohr means a description in terms of ordinary language. Important for Bohr's arguments is only that *facts* have to be described in a Boolean language, but not necessarily in terms of classical physics.

discourse. For this situation, Bohr coined the notion of *complementarity* to describe reality using incompatible classical concepts. Although no formal definition of complementarity can be found in his publications,[15] its spirit is closely related to that of incompatibility:[16]

> Two Boolean descriptions are said to be *complementary* if they cannot be embedded into a single Boolean description.

The idea that truths cannot contradict each other characterizes a Boolean worldview which, from a modern point of view, cannot adequately express our knowledge about the structure of the world. The binary "either-or" thinking of the Boolean worldview is a prejudice that prevents us from new ways of thinking. The development of quantum physics throughout the 20th century has shown that an all-encompassing description of the universe does not have a Boolean logical structure with the law of the excluded middle. Consequently, *the strictly Boolean worldview of classical science has to be given up*. This implies the necessity to use different, incompatible, sets of concepts for different contexts. This incompatibility does not just express a cognitive, or epistemic, limit, but is a fundamental features of the world.

In order to understand the world properly we need more than one Boolean frame of reference. Since science requires that empirical results must be expressed in terms of a *Boolean language*, an empirically meaningful framework for incompatible concepts requires a *locally Boolean structure* which must be *globally non-Boolean*.

This is possible by connecting mutually incompatible *locally Boolean* descriptions within a globally non-Boolean framework. The resulting structure is called partially Boolean. A partial Boolean algebra is a set of Boolean algebras pasted together such that their operations agree with each other where they overlap. In this way one obtains a manifold of locally Boolean descriptions, within which each Boolean algebra is a local chart of the manifold. This manifold generates a partial Boolean algebra in which all elements are connected. In general, the global structure of a partial Boolean algebra is not Boolean but allows a rigorous definition of the mutual compatibility of the elements of the Boolean manifold. For more details see Sects. 2.5 and 3.4.

A partial Boolean algebra provides a globally non-Boolean description with restricted sentential connectivity,[17] generated by patching together incompatible local Boolean descriptions in a smooth way—as geometric manifolds can be constructed from locally Euclidean spaces. In such an approach one does not introduce new connectives, the meaning of the Boolean logical connectives remains the same. As

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[15] Editor's note: For the history of the concept of complementarity and its origin in psychology and philosophy see Holton (1970).

[16] Editor's note: In fact, complementarity is a maximal form of incompatibility, as will be discussed later in this monograph (see Sects. 3.4 and 3.5).

[17] Strauss (1936a, 1936b, 1938, 1967, 1970, 1973) defined a logic of complementarity as a *two-valued logic with restricted sentential connectivity* whose propositional calculus is given by a partial Boolean algebra.

a consequence there are no problems to interpret partial Boolean descriptions. We have to acknowledge that we always perceive nature through Boolean filters, and that these filters are usually incompatible. All Boolean descriptions are equally valid in their own way, and all together are required for a complete picture.

2.4.3 Simultaneously Undecidable Propositions

The modern version of partial Boolean algebras was introduced informally by Ernst Specker (1960) in a parable about a princess, her admirers, and prophecy as a logic of propositions that are not simultaneously decidable.[18]

> At the Assyrian School of Prophets in Arba'ilu in the time of King Asarhaddon, there taught a seer from Nineva. He was a distinguished representative of his faculty (eclipses of the sun and moon) and aside from the heavenly bodies, his interest was almost exclusively in his daughter. His teaching success was limited, the subject proved to be dry, and required a previous knowledge of mathematics which was scarcely available. If he didn't find the student interest which he desired in class, he did find it elsewhere in overwhelming measure. His daughter had hardly reached a marriageable age when he was flooded with requests for her hand from students and young graduates. And though he didn't believe that he would always have her by his side, she was in any case still too young and her suitors in no way worthy. In order that they might convince themselves of their worthiness, he promised her to the one who could solve a "prediction problem" which he set. The suitor was taken before a table on which three little boxes stood in a row and was asked to say which boxes contained a gem and which didn't. But no matter how many tried, the task seemed impossible. In accordance with his prediction, each of the suitors was requested by the father to open two boxes which he had marked as both empty or both full. But it always turned out that one contained a gem and the other one didn't, and furthermore the stone was sometimes in the first box and sometimes in the second. But how should it be possible, given three boxes, neither to mark two as empty nor two as full? The daughter would have remained single until her father's death had she not followed the advice of a prophet's son and quickly opened two boxes, one of which was marked full and the other empty. Following the weak protest of her father that he had wanted two *other* boxes opened, she tried to open the third. But this proved impossible whereupon the father grudgingly admitted that the prediction was correct.

The calculus of propositions underlying this parable takes into account that propositions cannot be connected as in classical thinking.[19] The proper algebraic structure for this calculus is that of a partial Boolean algebra.[20]

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[18]English translation by A. Stairs as *The logic of propositions which are not simultaneously decidable*, reprinted by Hooker (1975, pp. 135–140). A more recent translation is by M.P. Seevinck: *The logic of non-simultaneously decidable propositions*, arxiv.org/abs/1103.4537. Editor's note: Specker presented a slightly revised version on March 27, 2000, in his seminar on *Quantum Logic and Hidden Parameters* at ETH Zurich. This version has not been translated into English so far.

[19]Editor's note: The particular version of Specker's parable is a complication of the usual pairwise incompatibility of measurements in quantum physics. It expresses that compatible measurements are jointly possible pairwise but not triplewise (see Liang et al. 2011, see also Sect. 3.5.4).

[20]Partial Boolean algebras have been used by Franz Kamber (1964) as the propositional calculus for quantum systems before Kochen and Specker (1965a, 1965b) proved that the partial Boolean

2.5 Boolean Classifications

The search for regularities is the point of departure of all scientific inquiry. Classifications and the recognition of patterns are based on the primitive act of *making a distinction* (Spencer Brown 1969). A distinction splits the world into two parts. Since the world is not intrinsically divided into subsystems, there is no unique way to make a distinction. That is, every classification is context-dependent and always entails a loss of information: it eliminates "irrelevant" information.

In the simplest case, a classification refers to the error-free grouping of finitely many objects into classes according to certain attributes they have in common. Any classification creates and isolates phenomena relevant to us. If we consider something else to be relevant, we need a different classification. The feasibility of a classification presupposes that there are distinct individual objects (like material or mathematical objects, electric signals, pictures, ideas) which can be characterized by well-defined attributes. Since there are no *a priori* distinctions, there is no unique way to classify.

Classifications can be based on *equivalence relations* which allow us (for a well-specified purpose) to merge objects with selected distinguishing characteristics together into *equivalence classes.* A binary relation $\sim$ on a nonempty set Ω is called an *equivalence relation* if it is reflexive, symmetric, and transitive:

$$\text{(i)}\quad \textit{reflexive}: \quad \mathcal{A} \sim \mathcal{A} \text{ for all } \mathcal{A} \in \Omega, \tag{2.3a}$$

$$\text{(ii)}\quad \textit{symmetric}: \quad \text{whenever } \mathcal{A} \sim \mathcal{B}, \text{ then } \mathcal{B} \sim \mathcal{A}, \tag{2.3b}$$

$$\text{(iii)}\quad \textit{transitive}: \quad \text{if } \mathcal{A} \sim \mathcal{B} \text{ and } \mathcal{B} \sim \mathcal{C}, \text{ then } \mathcal{A} \sim \mathcal{C}. \tag{2.3c}$$

If $\mathcal{A} \sim \mathcal{B}$, we say that they are equivalent under the equivalence relation $\sim$. If $\mathcal{A} \in \Omega$, then the set of all elements of Ω that are equivalent to $\mathcal{A}$ is called the *equivalence class* $[\mathcal{A}]_\sim$ of $\mathcal{A}$,

$$[\mathcal{A}]_\sim := \{\mathcal{B} \in \Omega \mid \mathcal{B} \sim \mathcal{A}\}. \tag{2.4}$$

Two equivalence classes either coincide or are disjoint:

$$\mathcal{A}, \mathcal{B} \in \Omega \quad \Rightarrow \quad \text{either} \quad [\mathcal{A}] = [\mathcal{B}] \quad \text{or} \quad [\mathcal{A}] \cap [\mathcal{B}] = \emptyset. \tag{2.5}$$

The strongest equivalence relation is $=$, the equality relation, producing the smallest possible equivalence classes.

Regularities can be found if and only if we suppress features that can be declared irrelevant in the context considered. This elimination of irrelevant features is necessary to classify objects according to attributes, and sometimes even to "produce" such attributes. That is, every classification is context-dependent and entails the loss of "irrelevant" information. Every classification can be realized by a battery

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structure arising in quantum theory cannot be embedded into a Boolean algebra. Compare Sect. 3.4 for more details.

of filters, i.e., devices which make binary decisions based on binary distinctions (Hammer 1969, p. 107):

> A filter is here defined as any device which accepts or passes certain elements in a set and rejects others. Thus a filter produces an ordered dichotomy in a set, the set of elements passed and its complement—those not passed.

The result of every classification can be expressed by a *finite* number of binary decisions. Accordingly, every exhaustive ideal classification can be represented by finitely many filters $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_n$. Filters can be represented algebraically by classification operators which we denote be $F_1, F_2, \ldots, F_n$ which can be combined as serial connections or as parallel connections.

The *serial connection* of two filters $\mathbb{F}_j$ and $\mathbb{F}_k$ is a filter which accepts objects that are accepted by both filters. By convention the serial connection of two filters characterized by the classification operators F_j and F_k is written *multiplicatively* and represented by the selection operator $F_j \wedge F_k$.

The *parallel connection* of two filters $\mathbb{F}_j$ and $\mathbb{F}_k$ is a filter which accepts the union of objects that are accepted by at least one of the component filters. By convention the parallel connection of two filters characterized by the classification operators F_j and G_k is written *additively* and represented by the selection operator $F_j \vee G_k$.

- A filter is said to be *ideal* if its repeated application has the same effect as its single application, i.e. if the selection operator F_j is idempotent,

$$F_j \wedge F_j = F_j. \tag{2.6a}$$

- Two filters F_j and F_k are called *compatible* if the result of the filter operation does not depend on the order, so that the corresponding selection operators commute,

$$F_j \wedge F_k = F_k \wedge F_j. \tag{2.6b}$$

- A system of n filters $F_1, F_2, \ldots, F_n$ is said to be *complete* if

$$F_1 \vee F_2 \vee \cdots \vee F_n = \mathbb{1}. \tag{2.6c}$$

  where $\mathbb{1}$ is the trivial filter that accepts every object.

Every system of finitely many *compatible* ideal filters generates a finite *Boolean classification* associated with the idempotent projectors $F_1, F_2, \ldots, F_n$. A Boolean classification is said to be *ideal* if it is exhaustive, reproducible and mutually exclusive:

$$\text{(i)} \quad \textit{exhaustive}: \quad F_1 \vee F_2 \vee \cdots \vee F_n = \mathbb{1}, \tag{2.7a}$$

$$\text{(ii)} \quad \textit{reproducible}: \quad F_j \wedge F_j = F_j, \tag{2.7b}$$

$$\text{(iii)} \quad \textit{mutually exclusive}: \quad F_j \wedge F_k = \mathbb{O} \text{ for } j \neq k. \tag{2.7c}$$

A *partial ordering* on the set of the projections of all ideal filters of a given context is defined by:

$$F_j \leq F_k \quad \text{if and only if} \quad F_j \wedge F_k = F_k \wedge F_j = F_j, \tag{2.8}$$

The zero operator $\mathbb{O}$ is the smallest projector, and the identity operator $\mathbb{1}$ is the largest projector. Every ideal Boolean classification generates a *Boolean algebra* $\{\mathcal{B}, \mathbb{1}, \mathbb{O}, \wedge, \vee, ^{\perp}\}$ of projections $F_1, F_2, \ldots, F_n$ with $F_j^2 := F_j \wedge F_j = F_j$, where the orthocomplementation $^{\perp}$ is defined by

$$F_j^{\perp} := \mathbb{1} - F_j. \tag{2.9}$$

The Boolean algebra $\{\mathcal{B}, \mathbb{1}, \mathbb{O}, \wedge, \vee, ^{\perp}\}$ defines a linear space $\{\mathfrak{C}, \mathbb{1}, \mathbb{O}, \cdot, +, ^*, \mathbb{C}\}$ over $\mathbb{C}$ which is a *commutative* *-*algebra* over $\mathbb{C}$ under the multiplication $\cdot$ and the addition $+$,

$$F_j F_k := F_j \cdot F_k := F_j \wedge F_k, \tag{2.10a}$$

$$F_j + F_k := F_j \vee F_k + F_j \wedge F_k. \tag{2.10b}$$

It is generated by the projections $F_1, F_2, \ldots, F_n$,

$$\mathfrak{C} := \left\{ \sum_{j=1}^{n} c_j F_j \,\middle|\, c_j \in \mathbb{C} \right\}. \tag{2.10c}$$

The involution $C \to C^*$ of $\mathfrak{C}$ is characterized by ($C, C_1, C_2 \in \mathfrak{C}, c \in \mathbb{C}$):

$$\left(C^*\right)^* = C, \qquad (cC)^* = c^* C^*, \tag{2.10d}$$

$$(C_1 + C_2)^* = C_1^* + C_2^*, \qquad (C_1 C_2)^* = C_1^* C_2^*. \tag{2.10e}$$

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