Chapter 2
Exactly Realizable Trajectories

This chapter introduces the notion of exactly realizable trajectories. The necessary formalism is established in Sect. 2.1. After the definition of exactly realizable trajectories in Sect. 2.2, the linearizing assumption is introduced in Sect. 2.3. This assumption defines a class of nonlinear control systems which, to a large extent, behave like linear control systems. Combining the notion of an exactly realizable trajectory with the linearizing assumption allows one to extend some well known results about the controllability of linear systems to nonlinear control systems in Sects. 2.4 and 2.5. Output Realizability is discussed in Sect. 2.6, and Sect. 2.7 concludes with a discussion and outlook.

2.1 Formalism

This section introduces the formalism which is repeatedly used throughout the thesis. The main elements are two complementary projection matrices $\mathcal{P}$ and $\mathcal{Q}$. Projectors are a useful ingredient for a number of physical theories. Take, for example, quantum mechanics, which describes measurements as projections of the state (an element from a Hilbert space) onto a ray or unions of rays of the Hilbert space (Fick 1988; Cohen-Tannoudji et al. 2010). Also in non-equilibrium statistical mechanics, projectors have found widespread application to separate a subsystem of interest from its bath (Balescu 1975; Grabert 1982).

To the best of our knowledge, projectors have not been utilized in the context of control systems. Section 2.1.1 defines the projectors and Sect. 2.1.2 separates the controlled state equation in two equations. The first equation involves the control signal, while the second equation is independent of the control signal. This formalism provides a useful approach for analyzing general affine control systems. Appendix A.2 demonstrates how projectors arise in the context of overdetermined and underdetermined systems of linear equations.

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2.1.1 The Projectors $\mathcal{P}$ and $\mathcal{Q}$

Consider the affine control system with state dependent coupling matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\,\boldsymbol{u}(t)\,. \tag{2.1}$$

Define two complementary projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ in terms of the coupling matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))$ as

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x})\left(\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})\right)^{-1}\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,, \tag{2.2}$$

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,. \tag{2.3}$$

$\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ are $n \times n$ matrices which, in general, do depend on the state $\boldsymbol{x}$. Note that the $p \times p$ matrix $\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ has full rank p because of assumption Eq. (1.4) that $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ has full rank. Therefore, $\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ is a quadratic and non-singular matrix and its inverse exists. The projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ are also known as *Moore–Penrose projectors*. The rank of $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ is

$$\operatorname{rank}(\boldsymbol{\mathcal{P}}(\boldsymbol{x})) = p, \qquad \operatorname{rank}(\boldsymbol{\mathcal{Q}}(\boldsymbol{x})) = n - p. \tag{2.4}$$

Multiplying the n-component state vector $\boldsymbol{x}$ by the $n \times n$ matrix $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ yields an n-component vector $\boldsymbol{z} = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{x}$. However, because $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ has rank p, only p components of $\boldsymbol{z}$ are independent. Similar, only $n - p$ components of $\boldsymbol{y} = \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{x}$ are independent.

From the definitions Eqs. (2.2) and (2.3) follow the projector properties idempotence

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,, \quad \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,, \tag{2.5}$$

and complementarity

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{0}. \tag{2.6}$$

The projectors are symmetric,

$$\boldsymbol{\mathcal{P}}^T(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,, \quad \boldsymbol{\mathcal{Q}}^T(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,, \tag{2.7}$$

because the inverse of the symmetric matrix $\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ is symmetric. Furthermore, matrix multiplication from the right with the input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ yields the important relations

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x})\,, \quad \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \mathbf{0}. \tag{2.8}$$

Similarly, matrix multiplication from the left with the transposed input matrix $\mathcal{B}^T(\boldsymbol{x})$ yields

$$\mathcal{B}^T(\boldsymbol{x})\,\mathcal{P}(\boldsymbol{x}) = \mathcal{B}^T(\boldsymbol{x}), \quad \mathcal{B}^T(\boldsymbol{x})\,\mathcal{Q}(\boldsymbol{x}) = \boldsymbol{0}. \tag{2.9}$$

Some more properties of $\mathcal{P}$ and $\mathcal{Q}$ necessary for later chapters are compiled in Appendix A.3.

2.1.2 Separation of the State Equation

The projectors defined in Eqs. (2.2) and (2.3) are used to split up the controlled state equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \mathcal{B}(\boldsymbol{x}(t))\,\boldsymbol{u}(t). \tag{2.10}$$

Multiplying every term by $\boldsymbol{1} = \mathcal{P}(\boldsymbol{x}(t)) + \mathcal{Q}(\boldsymbol{x}(t))$, Eq. (2.10) can be written as

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{P}(\boldsymbol{x}(t))\,\boldsymbol{x}(t) + \mathcal{Q}(\boldsymbol{x}(t))\,\boldsymbol{x}(t)\right) &= \left(\mathcal{P}(\boldsymbol{x}(t)) + \mathcal{Q}(\boldsymbol{x}(t))\right)\boldsymbol{R}(\boldsymbol{x}(t)) \\ &\quad + \left(\mathcal{P}(\boldsymbol{x}(t)) + \mathcal{Q}(\boldsymbol{x}(t))\right)\mathcal{B}(\boldsymbol{x}(t))\,\boldsymbol{u}(t).\end{aligned} \tag{2.11}$$

Multiplying with $\mathcal{Q}(\boldsymbol{x}(t))$ from the left and using Eq. (2.8) yields an equation independent of the control signal $\boldsymbol{u}$,

$$\mathcal{Q}(\boldsymbol{x}(t))\left(\dot{\boldsymbol{x}}(t) - \boldsymbol{R}(\boldsymbol{x}(t))\right) = \boldsymbol{0}. \tag{2.12}$$

Equation (2.12) is called the *constraint equation*. Multiplying the controlled state equation (2.10) by $\mathcal{B}^T(\boldsymbol{x}(t))$ from the left yields

$$\mathcal{B}^T(\boldsymbol{x}(t))\,\dot{\boldsymbol{x}}(t) = \mathcal{B}^T(\boldsymbol{x}(t))\,\boldsymbol{R}(\boldsymbol{x}(t)) + \mathcal{B}^T(\boldsymbol{x}(t))\,\mathcal{B}(\boldsymbol{x}(t))\,\boldsymbol{u}(t). \tag{2.13}$$

Multiplying with $\left(\mathcal{B}^T(\boldsymbol{x}(t))\,\mathcal{B}(\boldsymbol{x}(t))\right)^{-1}$, which exists as long as $\mathcal{B}(\boldsymbol{x}(t))$ has full rank, from the left results in an expression for the vector of control signals $\boldsymbol{u}(t)$ in terms of the controlled state trajectory $\boldsymbol{x}(t)$,

$$\boldsymbol{u}(t) = \mathcal{B}^+(\boldsymbol{x}(t))\left(\dot{\boldsymbol{x}}(t) - \boldsymbol{R}(\boldsymbol{x}(t))\right). \tag{2.14}$$

The abbreviation

$$\mathcal{B}^+(\boldsymbol{x}) = \left(\mathcal{B}^T(\boldsymbol{x})\,\mathcal{B}(\boldsymbol{x})\right)^{-1}\mathcal{B}^T(\boldsymbol{x}) \tag{2.15}$$

is known as the *Moore–Penrose pseudo inverse* of the matrix $\mathcal{B}(\boldsymbol{x})$ (Campbell and Meyer Jr. 1991). See also Appendix A.2 how to express a solution to an overdetermined system of linear equations in terms of the Moore–Penrose pseudo inverse. With the help of $\mathcal{B}^{+}$, the projector $\mathcal{P}$ can be expressed as

$$\mathcal{P}(\boldsymbol{x}) = \mathcal{B}(\boldsymbol{x}) \left(\mathcal{B}^{T}(\boldsymbol{x}) \mathcal{B}(\boldsymbol{x})\right)^{-1} \mathcal{B}^{T}(\boldsymbol{x}) = \mathcal{B}(\boldsymbol{x}) \mathcal{B}^{+}(\boldsymbol{x}) . \tag{2.16}$$

Note that

$$\begin{aligned} \mathcal{B}^{+}(\boldsymbol{x}(t)) \dot{\boldsymbol{x}}(t) &= \left(\mathcal{B}^{T}(\boldsymbol{x}(t)) \mathcal{B}(\boldsymbol{x}(t))\right)^{-1} \mathcal{B}^{T}(\boldsymbol{x}(t)) \dot{\boldsymbol{x}}(t) \\ &= \left(\mathcal{B}^{T}(\boldsymbol{x}(t)) \mathcal{B}(\boldsymbol{x}(t))\right)^{-1} \mathcal{B}^{T}(\boldsymbol{x}(t)) \mathcal{P}(\boldsymbol{x}(t)) \dot{\boldsymbol{x}}(t) , \end{aligned} \tag{2.17}$$

such that expression (2.14) for the control involves only the time derivative $\mathcal{P}(\boldsymbol{x}(t)) \dot{\boldsymbol{x}}(t)$ and does not depend on $\mathcal{Q}(\boldsymbol{x}(t)) \dot{\boldsymbol{x}}(t)$.

In conclusion, every affine controlled state equation (2.10) can be split in two equations. The Eq. (2.14) involving $\mathcal{P}\dot{\boldsymbol{x}}$ determines the control signal $\boldsymbol{u}$ in terms of the controlled state trajectory $\boldsymbol{x}$ and its derivative. The constraint equation (2.12) involves only $\mathcal{Q}\dot{\boldsymbol{x}}$ and does not depend on the control signal. These relations are valid for any kind of control, be it an open or a closed loop control. The proposed separation of the state equation plays a central role in this thesis.

To illustrate the approach, the separation of the state equation is discussed with the help of two simple examples.

Example 2.1: Mechanical control system in one spatial dimension
The controlled state equation for mechanical control systems is (see Example 1.1),

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ R(x(t), y(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ B(x(t), y(t)) \end{pmatrix} u(t) . \tag{2.18}$$

The 2×1 coupling matrix is a vector which depends on the state vector $\boldsymbol{x}(t) = \left(x(t), y(t)\right)$,

$$\boldsymbol{B}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ B(x, y) \end{pmatrix}, \tag{2.19}$$

while its transpose is a row vector

$$\boldsymbol{B}^{T}(\boldsymbol{x}) = \left(0, B(x, y)\right) . \tag{2.20}$$

The computation of the Moore–Penrose pseudo inverse $\boldsymbol{B}^+$ involves the inner product

$$\boldsymbol{B}^T(\boldsymbol{x})\,\boldsymbol{B}(\boldsymbol{x}) = \left(0,\ B(x,y)\right)\begin{pmatrix} 0 \\ B(x,y) \end{pmatrix} = B(x,y)^2\,. \tag{2.21}$$

The pseudo inverse $\boldsymbol{B}^+$ of $\boldsymbol{B}$ is given by

$$\boldsymbol{B}^+(\boldsymbol{x}) = \left(\boldsymbol{B}^T(\boldsymbol{x})\,\boldsymbol{B}(\boldsymbol{x})\right)^{-1}\boldsymbol{B}^T(\boldsymbol{x}) = B(x,y)^{-2}\left(0,\ B(x,y)\right), \tag{2.22}$$

while the projectors $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ are given by

$$\begin{aligned}\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}} &= \boldsymbol{B}(\boldsymbol{x})\left(\boldsymbol{B}^T(\boldsymbol{x})\,\boldsymbol{B}(\boldsymbol{x})\right)^{-1}\boldsymbol{B}^T(\boldsymbol{x}) \\ &= \begin{pmatrix} 0 \\ B(x,y) \end{pmatrix} B(x,y)^{-2}\left(0,\ B(x,y)\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned} \tag{2.23}$$

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}} = \mathbf{1} - \boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.24}$$

Although the coupling vector $\boldsymbol{B}(\boldsymbol{x})$ depends on the state $\boldsymbol{x}$, the projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ are actually independent of the state. With $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$, the state $\boldsymbol{x}$ can be split up in two parts,

$$\boldsymbol{\mathcal{P}}\boldsymbol{x}(t) = \begin{pmatrix} 0 \\ y(t) \end{pmatrix}, \quad \boldsymbol{\mathcal{Q}}\boldsymbol{x}(t) = \begin{pmatrix} x(t) \\ 0 \end{pmatrix}. \tag{2.25}$$

Both parts are vectors with two components, but have only one non-vanishing component. The control signal can be expressed in terms of the controlled state trajectory $\boldsymbol{x}(t)$ as

$$\begin{aligned} u(t) &= \boldsymbol{B}^+(\boldsymbol{x}(t))\left(\dot{\boldsymbol{x}}(t) - \boldsymbol{R}(\boldsymbol{x}(t))\right) \\ &= B(x(t),y(t))^{-2}\left(0,\ B(x,y)\right)\left(\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} - \begin{pmatrix} y(t) \\ R(x(t),y(t)) \end{pmatrix}\right) \\ &= \frac{1}{B(x(t),y(t))}\left(\dot{y}(t) - R(x(t),y(t))\right). \end{aligned} \tag{2.26}$$

Note that the assumption of full rank for the coupling vector $\boldsymbol{B}$ implies that the function $B(x,y)$ does not vanish, i.e., $B(x,y) \neq 0$ for all values of x and y. Consequently, $u(t)$ as given by Eq. (2.26) is well defined for all times.

Example 2.2: Single input diagonal LTI system
Consider a diagonal 2×2 linear time-invariant (LTI) system for the state vector $\boldsymbol{x}(t) = \left(x_1(t), x_2(t)\right)^T$. Let both components be controlled by the same control signal $u(t)$,

$$\dot{x}_1(t) = \lambda_1 x_1(t) + u(t), \quad \dot{x}_2(t) = \lambda_2 x_2(t) + u(t). \tag{2.27}$$

The state matrix $\mathcal{A}$ and input matrix $\mathcal{B}$ are

$$\mathcal{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.28}$$

The constant projectors $\mathcal{P}$ and $\mathcal{Q}$ can be computed as

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathcal{Q} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.29}$$

The two projections of the state $\boldsymbol{x}(t)$ are

$$\boldsymbol{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \mathcal{P}\boldsymbol{x}(t) = \frac{1}{2}\begin{pmatrix} x_1(t) + x_2(t) \\ x_1(t) + x_2(t) \end{pmatrix} \tag{2.30}$$

and

$$\boldsymbol{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \mathcal{Q}\boldsymbol{x}(t) = \frac{1}{2}\begin{pmatrix} x_1(t) - x_2(t) \\ x_2(t) - x_1(t) \end{pmatrix}. \tag{2.31}$$

While both components of $\boldsymbol{y}(t)$ are non-zero, they are not linearly independent. The component $y_2(t)$ is redundant and is simply given by

$$y_2(t) = -y_1(t). \tag{2.32}$$

Similarly, the component $z_2(t)$ of vector $\boldsymbol{z}(t)$ is redundant because of

$$z_2(t) = z_1(t). \tag{2.33}$$

Example 2.2 shows that the projectors $\mathcal{P}$ and $\mathcal{Q}$ do not necessarily project onto single components of the state vector. If the projectors $\mathcal{P}(\boldsymbol{x}) = \mathcal{P}$ and $\mathcal{Q}(\boldsymbol{x}) = \mathcal{Q}$ are independent of the state $\boldsymbol{x}$, the parts $\boldsymbol{z} = \mathcal{P}\boldsymbol{x}$ and $\boldsymbol{y} = \mathcal{Q}\boldsymbol{x}$ are linear combinations of the original state components $\boldsymbol{x}$. If $\mathcal{P} = \mathcal{P}(\boldsymbol{x})$ and therefore also $\mathcal{Q}(\boldsymbol{x}) = \mathbf{1} - \mathcal{P}(\boldsymbol{x})$ depend on the state $\boldsymbol{x}$ itself, both parts $\boldsymbol{y}$ and $\boldsymbol{z}$ are nonlinear functions of the state $\boldsymbol{x}$. Only if the projectors are diagonal, constant, and appropriately ordered, $\mathcal{P}(\boldsymbol{x}) = \mathcal{P}_D$ and $\mathcal{Q}(\boldsymbol{x}) = \mathcal{Q}_D$, then the two parts $\boldsymbol{y}$ and $\boldsymbol{z}$ attain the particularly simple form

$$y = \mathcal{Q}_D x = \begin{pmatrix} 0, & \dots, & 0, & x_{p+1}, & \dots, & x_n \end{pmatrix}^T, \tag{2.34}$$

$$z = \mathcal{P}_D x = \begin{pmatrix} x_1, & \dots, & x_p, & 0, & \dots, & 0 \end{pmatrix}^T. \tag{2.35}$$

Only this form allows a clear interpretation which component of $\boldsymbol{x}$ belongs to which part. However, in any case, $\boldsymbol{y} = \mathcal{Q}(\boldsymbol{x})\boldsymbol{x}$ has exactly $n - p$ independent components because $\mathcal{Q}(\boldsymbol{x})$ has rank $n - p$, while $\boldsymbol{z} = \mathcal{P}(\boldsymbol{x})\boldsymbol{x}$ has p independent components because $\mathcal{P}(\boldsymbol{x})$ has rank p.

Projectors have only zeros and ones as possible eigenvalues. The diagonalization of an $n \times n$ projector $\mathcal{P}(\boldsymbol{x})$ with rank $(\mathcal{P}(\boldsymbol{x})) = p$ is always possible (Fischer 2013) and results in a diagonal $n \times n$ matrix with p entries of value one and $n - p$ entries of value zero on the diagonal. The transformation of the projectors $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{Q}(\boldsymbol{x})$ to their diagonal counterparts defines a transformation of the state $\boldsymbol{x}$. See Appendix A.4 how to construct this transformation. The transformation is nonlinear if $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{Q}(\boldsymbol{x})$ are state dependent. Expressed in terms of the transformed state, the projectors $\mathcal{P}$ and $\mathcal{Q}$ are constant, diagonal, and appropriately ordered. Consequently, they yield a state separation of the form Eqs. (2.34)–(2.35). Such a representation defines a normal form of an affine control system. For a specified affine control system, computations will usually be simpler after the system is transformed to its normal form. However, for computations with general affine control systems, it is dispensable to perform the transformation if $\boldsymbol{z}(\boldsymbol{x}) = \mathcal{P}(\boldsymbol{x})\boldsymbol{x}$ and $\boldsymbol{y}(\boldsymbol{x}) = \mathcal{Q}(\boldsymbol{x})\boldsymbol{x}$ are simply viewed as separate parts. This allows a coordinate-free treatment of affine control systems.

2.2 Exactly Realizable Trajectories

As demonstrated in Example 1.4, not every desired state trajectory $\boldsymbol{x}_d(t)$ can be realized by control. Here, we answer the question under which conditions a desired trajectory $\boldsymbol{x}_d(t)$ is exactly realizable.

Consider the controlled state equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \mathcal{B}(\boldsymbol{x}(t))\boldsymbol{u}(t), \tag{2.36}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2.37}$$

The notion of *exactly realizable trajectories* is introduced. A realizable trajectory is a desired trajectory $\boldsymbol{x}_d(t)$ which satisfies two conditions.

1. The desired trajectory $\boldsymbol{x}_d(t)$ satisfies the constraint equation

$$\mathcal{Q}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) = \boldsymbol{0}. \tag{2.38}$$

2. The initial value $\boldsymbol{x}_d(t_0)$ must equal the initial value $\boldsymbol{x}_0$ of the controlled state equation,

$$\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0. \tag{2.39}$$

The control solution for an exactly realizable trajectory is given by

$$\boldsymbol{u}(t) = \boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right), \tag{2.40}$$

with the Moore–Penrose pseudo inverse $p \times n$ matrix $\boldsymbol{\mathcal{B}}^+$ defined as

$$\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}) = \left(\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x})\right)^{-1}\boldsymbol{\mathcal{B}}^T(\boldsymbol{x}). \tag{2.41}$$

The notion of an exactly realizable trajectory allows the proof of the following statement.

If $\boldsymbol{x}_d(t)$ is an exactly realizable trajectory, i.e., if $\boldsymbol{x}_d(t)$ satisfies both conditions 1 and 2, then the state trajectory $\boldsymbol{x}(t)$ follows the desired trajectory $\boldsymbol{x}_d(t)$ exactly,

$$\boldsymbol{x}(t) = \boldsymbol{x}_d(t). \tag{2.42}$$

Using the control solution Eq. (2.40) in the controlled state equation (2.36) yields the following equation for the controlled state

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right). \tag{2.43}$$

Note that $\boldsymbol{\mathcal{B}}$ depends on the actual system state $\boldsymbol{x}(t)$ while $\boldsymbol{\mathcal{B}}^+$ depends on the desired trajectory $\boldsymbol{x}_d(t)$. The difference $\Delta\boldsymbol{x}(t)$ between the true state $\boldsymbol{x}(t)$ and the desired trajectory $\boldsymbol{x}_d(t)$ is defined as

$$\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_d(t). \tag{2.44}$$

Using the definition for $\Delta\boldsymbol{x}(t)$ and Eq. (2.43) results in an ordinary differential equation (ODE) for $\Delta\boldsymbol{x}(t)$,

$$\begin{aligned}\Delta\dot{\boldsymbol{x}}(t) = {} & \boldsymbol{R}(\Delta\boldsymbol{x}(t) + \boldsymbol{x}_d(t)) - \dot{\boldsymbol{x}}_d(t) \\ & + \boldsymbol{\mathcal{B}}(\Delta\boldsymbol{x}(t) + \boldsymbol{x}_d(t))\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right), \end{aligned} \tag{2.45}$$

$$\Delta\boldsymbol{x}(t_0) = \boldsymbol{x}(t_0) - \boldsymbol{x}_d(t_0). \tag{2.46}$$

Assuming $|\Delta\boldsymbol{x}(t)| \ll 1$ and expanding Eq. (2.45) in $\Delta\boldsymbol{x}(t)$ yields

$$\begin{aligned}\Delta\dot{\boldsymbol{x}}(t) = {} & \boldsymbol{R}(\boldsymbol{x}_d(t)) - \dot{\boldsymbol{x}}_d(t) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}_d(t))\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) \\ & + \nabla\boldsymbol{R}(\boldsymbol{x}_d(t))\Delta\boldsymbol{x}(t) + (\nabla\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d(t))\Delta\boldsymbol{x}(t))\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) \\ & + \mathcal{O}\left(\Delta\boldsymbol{x}(t)^2\right). \end{aligned} \tag{2.47}$$

Note that assuming $|\Delta \boldsymbol{x}(t)| \ll 1$ and subsequently expanding in $\Delta \boldsymbol{x}(t)$ does not result in a loss of generality of the final outcome. The expression $\nabla \boldsymbol{R}(\boldsymbol{x})$ denotes the Jacobian matrix of the nonlinearity $\boldsymbol{R}(\boldsymbol{x})$ with components

$$(\nabla \boldsymbol{R}(\boldsymbol{x}))_{ij} = \frac{\partial}{\partial x_j} R_i(\boldsymbol{x}), \; i, j \in \{1, \ldots, n\}. \tag{2.48}$$

The Jacobian $\nabla \boldsymbol{\mathcal{B}}(\boldsymbol{x})$ of $\boldsymbol{\mathcal{B}}$ is a third order tensor with components

$$(\nabla \boldsymbol{\mathcal{B}}(\boldsymbol{x}))_{ijk} = \frac{\partial}{\partial x_k} \mathcal{B}_{ij}(\boldsymbol{x}), \; i, k \in \{1, \ldots, n\}, \; j \in \{1, \ldots p\}. \tag{2.49}$$

In the first line of Eq. (2.47), one can recognize the projector $\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d(t)) \boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t)) = \boldsymbol{\mathcal{P}}(\boldsymbol{x}_d(t)) = \mathbf{1} - \boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t))$. Introducing the $n \times n$ matrix $\boldsymbol{\mathcal{T}}(\boldsymbol{x})$ with components

$$(\boldsymbol{\mathcal{T}}(\boldsymbol{x}))_{il} = \sum_{j=1}^{p} \sum_{k=1}^{n} \frac{\partial}{\partial x_l} \mathcal{B}_{ij}(\boldsymbol{x}) \mathcal{B}^+_{jk}(\boldsymbol{x}) (\dot{x}_k(t) - R_k(\boldsymbol{x})), \; i, l \in \{1, \ldots, n\}, \tag{2.50}$$

allows a rearrangement of Eq. (2.47) in the form

$$\Delta \dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t)) \big(\boldsymbol{R}(\boldsymbol{x}_d(t)) - \dot{\boldsymbol{x}}_d(t)\big) + (\nabla \boldsymbol{R}(\boldsymbol{x}_d(t)) + \boldsymbol{\mathcal{T}}(\boldsymbol{x}_d(t))) \Delta \boldsymbol{x}(t), \tag{2.51}$$

$$\Delta \boldsymbol{x}(t_0) = \boldsymbol{x}(t_0) - \boldsymbol{x}_d(t_0). \tag{2.52}$$

If $\boldsymbol{x}_d(t)$ is an exactly realizable trajectory, it satisfies the constraint equation (2.38) and the initial condition $\boldsymbol{x}_d(t_0) = \boldsymbol{x}(t_0)$, and Eq. (2.51) simplifies to the linear homogeneous equation for $\Delta \boldsymbol{x}(t)$,

$$\Delta \dot{\boldsymbol{x}}(t) = (\nabla \boldsymbol{R}(\boldsymbol{x}_d(t)) + \boldsymbol{\mathcal{T}}(\boldsymbol{x}_d(t))) \Delta \boldsymbol{x}(t) \tag{2.53}$$

$$\Delta \boldsymbol{x}(t_0) = \mathbf{0}. \tag{2.54}$$

Clearly, Eq. (2.53) has a vanishing solution

$$\Delta \boldsymbol{x}(t) \equiv \mathbf{0}. \tag{2.55}$$

In summary, it was proven that if the desired trajectory $\boldsymbol{x}_d(t)$ is an exactly realizable trajectory, then the state trajectory $\boldsymbol{x}(t)$ follows the desired trajectory exactly, i.e., $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$. Equation (2.12) in Sect. 2.1.2 proved already the converse: any controlled state trajectory $\boldsymbol{x}(t)$ satisfies the constraint equation. This allows the following conclusion:

The controlled state trajectory $\boldsymbol{x}(t)$ follows the desired trajectory $\boldsymbol{x}_d(t)$ exactly if and only if $\boldsymbol{x}_d(t)$ is an exactly realizable trajectory.

The notion of an exactly realizable trajectory leads to the following interpretation. Not every desired trajectory $\boldsymbol{x}_d(t)$ can be enforced in a specified controlled dynamical system. In general, the desired trajectory $\boldsymbol{x}_d(t)$ is that what you want, but is not what you get. What you get is an exactly realizable trajectory. Because the control signal $\boldsymbol{u}(t)$ consists of only p independent components, it is possible to find at most p one-to-one relations between state components and components of the control signal. Only p components of a state trajectory can be prescribed, while the remaining $n-p$ components are free. The time evolution of these $n-p$ components is given by the constraint equation (2.38). This motivates the name constraint equation. For an arbitrary desired trajectory $\boldsymbol{x}_d(t)$ to be exactly realizable, it has to be constrained by Eq. (2.38). There is still some freedom to choose which state components are actually prescribed, and which have to be determined by the constraint equation. Until further notice, we adopt the canonical view that the part $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$ is prescribed by the experimenter, while the part $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$ of the state vector is fixed by the constraint equation (2.38). This, however, is not the only possibility, and many more choices are possible. The part $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$ can be seen as an output for the control system which can be enforced exactly if $\boldsymbol{x}_d(t)$ is exactly realizable. Section 2.6 discusses the possibility to realize general desired outputs not necessarily given by $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$.

Chapter 3 investigates the relation of exactly realizable trajectories with optimal trajectory tracking. The control solution Eq. (2.40) is the solution to a certain optimal trajectory tracking problem. This insight is the starting point in Chap. 4 to obtain analytical approximations to optimal trajectory tracking of desired trajectories which are not exactly realizable.

The necessity to satisfy condition 2 of equal initial conditions leaves two possibilities. Either the system is prepared in the initial state $\boldsymbol{x}(t_0)=\boldsymbol{x}_0=\boldsymbol{x}_d(t_0)$, or the desired trajectory $\boldsymbol{x}_d(t)$ is designed such that it starts from the observed initial system state $\boldsymbol{x}_0$. In any case, the constraint equation (2.38), seen as an ODE for $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$, has to be solved with the initial condition $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_0)=\boldsymbol{\mathcal{Q}}\boldsymbol{x}_0$.

The control solution as given by Eq. (2.40) is an open loop control. As such, it does not guarantee a stable time evolution, and the controlled system does not necessarily follow the realizable trajectory in the presence of perturbations. The linear equation (2.53) encountered during the proof implies statements about the linear stability of realizable trajectories. A non-vanishing initial value $\Delta\boldsymbol{x}(t_0)=\Delta\boldsymbol{x}_0\neq\boldsymbol{0}$ constitutes a perturbation of the initial conditions of an exactly realizable trajectory. The control approach as proposed here is only a first step. For a specified exactly realizable trajectory, Eq. (2.53) has to be investigated to determine its linear stability properties. If the desired trajectory is linearly unstable, countermeasures in form of an additional feedback control, for example, have to be applied to guarantee a successful control. Stability of exactly realizable trajectories is not discussed in this thesis.

The concept of a realizable trajectory is elucidated with the help of some examples in the following.

Example 2.3: Controlled FHN model with invertible coupling matrix
Consider the controlled FHN model in the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x(t) + a_2 y(t) \\ R(x(t), y(t)) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}. \tag{2.56}$$

The constant coupling matrix $\mathcal{B}$ is identical to the identity,

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.57}$$

This system has two state components $x,\ y$, and two independent control signals $u_1,\ u_2$. The projector $\mathcal{P}$ is simply the identity, $\mathcal{P} = \mathbf{1}$, and $\mathcal{Q} = \mathbf{0}$ the zero matrix. The constraint equation (2.38) is trivially satisfied. Any desired trajectory $\boldsymbol{x}_d(t)$ is a realizable trajectory as long as initially, the desired trajectory equals the state trajectory,

$$\boldsymbol{x}_d(t_0) = \boldsymbol{x}(t_0). \tag{2.58}$$

Example 2.4: Mechanical control system in one spatial dimension
The control signal realizing a desired trajectory $\boldsymbol{x}_d(t)$ of a mechanical control system (see Examples 1.1 and 2.1 for more details) is

$$\begin{aligned} u(t) &= \boldsymbol{B}^{+}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) \\ &= \frac{1}{B(x_d(t), y_d(t))}\left(\dot{y}_d(t) - R(x_d(t), y_d(t))\right). \end{aligned} \tag{2.59}$$

The non-vanishing component of the constraint equation (2.38) for realizable desired trajectories simply becomes

$$\dot{x}_d(t) = y_d(t). \tag{2.60}$$

With a scalar control signal $u(t)$ only one state component can be controlled. According to our convention, this state component is

$$\mathcal{P}\boldsymbol{x}_d(t) = \begin{pmatrix} 0 \\ y_d(t) \end{pmatrix}. \tag{2.61}$$

The desired velocity over time $y_d(t)$ can be arbitrarily chosen apart from its initial value, which must be identical to the initial state velocity, $y_d(t_0) = y(t_0)$.

The corresponding position over time $x_d(t)$ is given by the constraint equation (2.60). Because Eq. (2.60) is a linear differential equation for $x_d(t)$, its solution in terms of the arbitrary velocity $y_d(t)$ is easily obtained as

$$x_d(t) = x_d(t_0) + \int_{t_0}^{t} d\tau y_d(\tau). \tag{2.62}$$

The initial desired position $x_d(t_0)$ has to agree with the initial state position, $x_d(t_0) = x(t_0) = x_0$. With the help of solution (2.62), the control (2.59) can be entirely expressed in terms of the prescribed velocity over time $y_d(t)$ as

$$\begin{aligned} u(t) = {} & \frac{1}{B\left(x_0 + \int_{t_0}^{t} d\tau y_d(\tau), y_d(t)\right)} \\ & \times \left(\dot{y}_d(t) - R\left(x_0 + \int_{t_0}^{t} d\tau y_d(\tau), y_d(t)\right)\right). \end{aligned} \tag{2.63}$$

Note that an exact solution to the nonlinear controlled state equation as well as to the control signal $\boldsymbol{u}(t)$ is obtained without actually solving any nonlinear equation. The context of a mechanical control system allows the following interpretation of our approach. The constraint equation (2.60) is the definition of the velocity of a point particle, and no external force R or control force Bu can change that definition. With only a single control signal u, position x and velocity y over time cannot be controlled independently from each other.

One might ask if it is possible to control position and velocity independently of each other by introducing an additional control signal. If both control signals act as forces, the controlled mechanical system with state space dimension $n = 2$ and control space dimension $p = 2$ is

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = {} & \begin{pmatrix} y(t) \\ R(x(t), y(t)) \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ B_1(x(t), y(t)) & B_2(x(t), y(t)) \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \end{aligned} \tag{2.64}$$

such that the 2×2 coupling matrix $\tilde{\boldsymbol{\mathcal{B}}}$ becomes

$$\tilde{\boldsymbol{\mathcal{B}}}(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 \\ B_1(x, y) & B_2(x, y) \end{pmatrix}. \tag{2.65}$$

However, the structure of $\tilde{\boldsymbol{\mathcal{B}}}$ reveals that it violates the condition of full rank. Indeed, for arbitrary functions $B_1 \neq 0$ and $B_2 \neq 0$, the rank of $\tilde{\boldsymbol{\mathcal{B}}}$ is $\operatorname{rank}\left(\tilde{\boldsymbol{\mathcal{B}}}(\boldsymbol{x})\right) = 1$ and therefore smaller than the control space dimension $p = 2$. The computation of the projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ as well as the computation of the control signal u requires the existence of the inverse of $\tilde{\boldsymbol{\mathcal{B}}}^T \tilde{\boldsymbol{\mathcal{B}}}$, which in turn requires $\tilde{\boldsymbol{\mathcal{B}}}$ to have full rank. Our approach cannot be applied to system (2.64) because both control signals u_1 and u_2 act on the same state component. The corresponding control forces are not independent of each other, but can be combined to a single control force $B_1 u_1 + B_2 u_2$.

The constraint equation (2.60) can also be regarded as an algebraic equation for the desired position over time $x_d(t)$. This is an example for a desired output different from the conventional choice Eq. (2.61). Eliminating the position from the control solution Eq. (2.59) yields

$$u(t) = \frac{1}{B(x_d(t), \dot{x}_d(t))} \left(\ddot{x}_d(t) - R(x_d(t), \dot{x}_d(t))\right). \tag{2.66}$$

Equation (2.66) is a special case of the so-called *computed torque formula*. This approach, also known as inverse dynamics, is regularly applied in robotics. For further information, the reader is referred to the literature about robot control (Lewis et al. 1993; de Wit et al. 2012; Angeles 2013).

Example 2.5: Activator-controlled FHN model

Consider the FHN model from Example 1.2,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x(t) + a_2 y(t) \\ R(x(t), y(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \tag{2.67}$$

with coupling vector $\boldsymbol{B} = \left(0,\ 1\right)^T$ and standard FHN nonlinearity $R(x, y) = y - \frac{1}{3}y^3 - x$. The projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ are readily computed as

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\mathcal{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.68}$$

Being given by $\mathcal{P}\boldsymbol{x}$, the desired activator component over time $y_d(t)$ can be prescribed. For the desired trajectory to be exactly realizable, the desired inhibitor $x_d(t)$ must be determined from the constraint equation

$$\mathcal{Q}\dot{\boldsymbol{x}}_d(t) = \mathcal{Q}\boldsymbol{R}(\boldsymbol{x}_d(t)). \tag{2.69}$$

Writing down only the non-vanishing component of Eq. (2.69) yields

$$\dot{x}_d(t) = a_1 x_d(t) + a_2 y_d(t) + a_0. \tag{2.70}$$

This linear differential equation for $x_d(t)$ with an inhomogeneity is readily solved in terms of the desired activator over time $y_d(t)$,

$$x_d(t) = \frac{a_0}{a_1}\left(e^{a_1(t-t_0)} - 1\right) + e^{a_1(t-t_0)} x_d(t_0) + a_2 \int_{t_0}^{t} d\tau e^{a_1(t-\tau)} y_d(\tau). \tag{2.71}$$

For the desired trajectory $\boldsymbol{x}_d(t)$ to be exactly realizable, it must agree with the initial state $\boldsymbol{x}(t_0)$ of the controlled system. The control is given by

$$u(t) = \dot{y}_d(t) - R(x_d(t), y_d(t)) = \dot{y}_d(t) - y_d(t) + \frac{1}{3} y_d(t)^3 + x_d(t). \tag{2.72}$$

Using the solution Eq. (2.71), the inhibitor variable $x_d(t)$ can be eliminated from the control signal. Consequently, the control can be expressed as a functional of the desired activator variable $y_d(t)$ and the initial desired inhibitor value $x_d(t_0)$ as

$$\begin{aligned} u(t) = \dot{y}_d(t) - y_d(t) + \frac{1}{3} y_d(t)^3 + \frac{a_0}{a_1}\left(e^{a_1(t-t_0)} - 1\right) \\ + e^{a_1(t-t_0)} x_d(t_0) + a_2 \int_{t_0}^{t} d\tau e^{a_1(t-\tau)} y_d(\tau). \end{aligned} \tag{2.73}$$

To evaluate the performance of the control, the control signal Eq. (2.73) is used in Eq. (2.67), and the resulting controlled dynamical system is solved numerically. The numerically obtained state trajectory is compared with the desired reference trajectory. The desired trajectory is chosen as

$$y_d(t) = \sin(20t)\cos(2t), \tag{2.74}$$

and the initial conditions are set to $x(t_0) = x_d(t_0) = y(t_0) = y_d(t_0) = 0$. As expected from Eq. (2.74), the controlled activator $y(t)$ oscillates wildly, see blue solid line in Fig. 2.1 left. The numerically obtained controlled inhibitor $x(t)$ (red dashed line) increases almost linearly. This behavior can easily be understood from the smallness of a_1 in Eq. (2.71) (see Example 1.2 for parameter values). Indeed, in the limit of vanishing a_1,

$$\lim_{a_1 \to 0} x_d(t) = a_0(t - t_0) + x_d(t_0) + a_2 \int_{t_0}^{t} d\tau y_d(\tau), \tag{2.75}$$

x_d increases linearly in time with coefficient a_0, while the integral term over a periodic function $y_d(t)$ with zero mean vanishes on average. The control signal $u(t)$, being proportional to $\dot{y}(t)$, oscillates as well, see Fig. 2.1 right. Comparing the differences between the controlled state components and its desired counterparts reveals agreement within numerical precision, see Fig. 2.2 left for the activator and Fig. 2.2 right for the inhibitor component, respectively. However, note that the error increases in time, which could indicate a developing instability. The control, being an open loop control, is potentially unstable. It often must be stabilized to guarantee a successful control. Stabilization of exactly realizable trajectories is not discussed in this thesis.

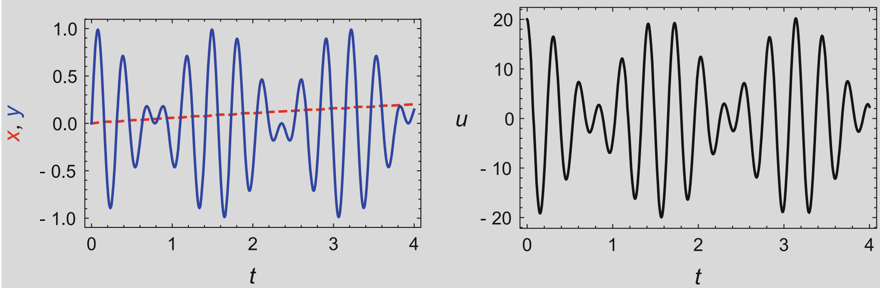

Fig. 2.1 Activator-controlled FHN model driven along an exactly realizable trajectory. The numerically obtained activator y (*blue solid line*) and inhibitor x (*red dashed line*) of the controlled system is shown *left*. The oscillating activator is prescribed according to Eq. (2.74), while the inhibitor cannot be prescribed and is given as the solution to the constraint equation (2.70). The control signal (*right*) oscillates as well because it is proportional to $\dot{y}_d$

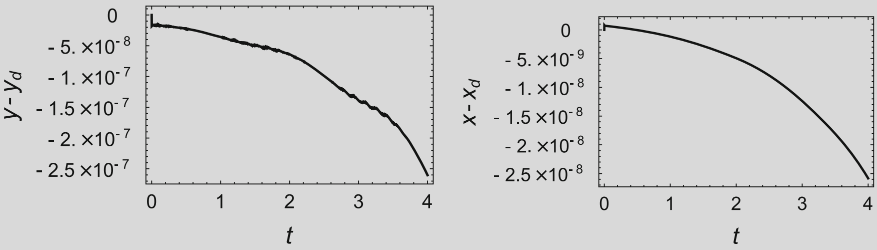

Fig. 2.2 Difference between desired and controlled state components in the activator-controlled FHN model. Plotting the difference between controlled and desired activator $y - y_d$ (*left*) and controlled and desired inhibitor $x - x_d$ (*right*) reveals agreement within numerical precision

Example 2.6: Inhibitor-controlled FHN model
Consider the same model as in Example 2.5 but with a coupling vector $\boldsymbol{B} = \left(1, 0\right)^T$ corresponding to a control acting on the inhibitor equation (see also Example 1.2).

The projectors are $\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The desired inhibitor over time $x_d(t)$ is prescribed while the activator component $y_d(t)$ must be determined from (the non-vanishing component of) the constraint equation,

$$\dot{y}_d(t) = R\left(x_d(t), y_d(t)\right) = y_d(t) - \frac{1}{3} y_d(t)^3 - x_d(t)\,. \tag{2.76}$$

The constraint equation is a nonlinear non-autonomous differential equation for $y_d(t)$. An analytical expression for the solution $y_d(t)$ in terms of the prescribed inhibitor trajectory $x_d(t)$ is not available. Equation (2.76) must be solved numerically.

Figure 2.3 shows the result of a numerical simulation of the controlled system with a desired inhibitor trajectory

$$x_d(t) = 4\sin(2t)\,, \tag{2.77}$$

and initial conditions $x_d(t_0) = x(t_0) = y_d(t_0) = y(t_0) = 0$. Comparing the desired activator and inhibitor trajectories with the corresponding controlled state trajectories in the bottom panels demonstrates a difference in the range of numerical precision over the whole time interval. Both state components (top left) as well as the control (top right) are oscillating.

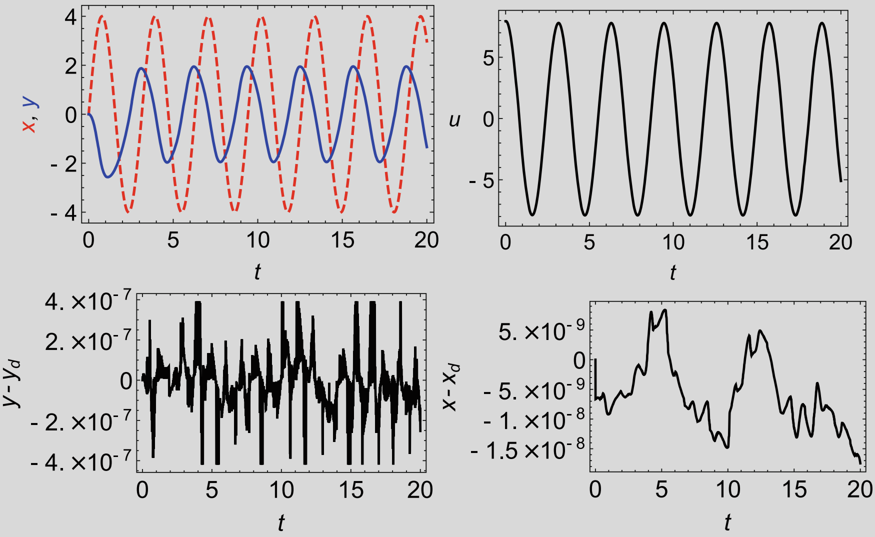

Fig. 2.3 Inhibitor-controlled FHN model driven along an exactly realizable trajectory. The numerically obtained solution of the controlled state is shown *top left*, and the control signal is shown *top right*. Comparing desired and controlled activator y_d and y (*bottom left*) as well as desired and controlled inhibitor x_d and x (*bottom left*) reveals a difference within numerical precision

2.3 Linearizing Assumption

An uncontrolled dynamical system requires solving

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)). \tag{2.78}$$

In contrast, control of exactly realizable trajectories requires only the solution of the constraint equation

$$\mathcal{Q}(\boldsymbol{x}(t))\left(\dot{\boldsymbol{x}}(t) - \boldsymbol{R}(\boldsymbol{x}(t))\right) = \boldsymbol{0}. \tag{2.79}$$

This opens up the possibility to solve a nonlinear control problem without actually solving any nonlinear equations. If the constraint equation is linear, the entire controlled system can be regarded, in some sense and to some extent, as being linear. Two conditions must be met for Eq. (2.79) to be linear. First of all, the projection matrices $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{Q}(\boldsymbol{x})$ should be independent of the state $\boldsymbol{x}$. This condition can be expressed as

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x})\left(\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})\right)^{-1}\boldsymbol{\mathcal{B}}^T(\boldsymbol{x}) = \text{const.} \tag{2.80}$$

or

$$\nabla\left(\boldsymbol{\mathcal{B}}(\boldsymbol{x})\left(\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})\right)^{-1}\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\right) = \mathbf{0}. \tag{2.81}$$

Note that this condition does not imply that the coupling matrix is independent of $\boldsymbol{x}$. Second, the nonlinearity $\boldsymbol{R}(\boldsymbol{x})$ must satisfy

$$\boldsymbol{\mathcal{Q}}\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{x} + \boldsymbol{\mathcal{Q}}\boldsymbol{b}, \tag{2.82}$$

with $n \times n$ matrix $\boldsymbol{\mathcal{A}}$ and n-component vector $\boldsymbol{b}$ independent of the state $\boldsymbol{x}$. Strictly speaking, the projector $\boldsymbol{\mathcal{Q}}$ in front of $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{b}$ is not really necessary. It is placed there to make it clear that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{b}$ do not contain any parts in the direction of $\boldsymbol{\mathcal{P}}$. Condition Eq. (2.80) combined with condition Eq. (2.82) constitute the *linearizing assumption*. Control systems satisfying the linearizing assumption behave, to a large extent, similar to truly linear control systems. A nonlinear control systems with scalar input $u(t)$ satisfying the linearizing assumption is sometimes said to be in companion form. A system in companion form is trivially feedback linearizable, see the discussion of feedback linearization in Chap. 1 and e.g. Khalil (2001).

Condition Eq. (2.82) is a strong assumption. It enforces $n - p$ components of $\boldsymbol{R}(\boldsymbol{x})$ to depend only linearly on the state. However, some important models of nonlinear dynamics satisfy the linearizing assumption. Among these are the mechanical control systems in one spatial dimension, see Examples 1.1 and 2.4, as well as the activator-controlled FHN model discussed in Example 2.5. In both cases, the control signal $\boldsymbol{u}(t)$ acts directly on the nonlinear part of the nonlinearity $\boldsymbol{R}$, such that condition Eq. (2.82) is satisfied. Furthermore, in both cases the coupling matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ is a coupling vector $\boldsymbol{B}(\boldsymbol{x}) = \left(0,\ B(x, y)\right)^T$ with only one non-vanishing component. This leads to constant projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$, and condition Eq. (2.80) is also satisfied. Another, less obvious example satisfying the linearizing assumption is the controlled SIR model.

Example 2.7: Linearizing assumption satisfied by the controlled SIR model

The controlled state equation for the SIR model was developed in Example 1.3. The nonlinearity $\boldsymbol{R}$ is

$$\boldsymbol{R}(\boldsymbol{x}(t)) = \left(-\beta\frac{S(t)\,I(t)}{N}, \beta\frac{S(t)\,I(t)}{N} - \gamma I(t), \gamma I(t)\right)^T, \tag{2.83}$$

while the coupling vector $\boldsymbol{B}$ explicitly depends on the state,

$$\boldsymbol{B}(\boldsymbol{x}(t)) = \frac{1}{N}\left(-S(t)\,I(t), S(t)\,I(t), 0\right)^T. \tag{2.84}$$

However, the projectors

$$\mathcal{P}(\boldsymbol{x}) = \mathcal{P} = \boldsymbol{B}(\boldsymbol{x})\left(\boldsymbol{B}^T(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\right)^{-1}\boldsymbol{B}^T(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.85)$$

$$\mathcal{Q}(\boldsymbol{x}) = \mathcal{Q} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad (2.86)$$

are independent of the state. Furthermore, the model also satisfies the linearizing assumption Eq. (2.80) because

$$\mathcal{Q}\boldsymbol{R}(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} 0 & -\gamma & 0 \\ 0 & -\gamma & 0 \\ 0 & 2\gamma & 0 \end{pmatrix} = \mathcal{Q}\mathcal{A}\boldsymbol{x}. \quad (2.87)$$

The constraint equation is a linear differential equation with three components,

$$\begin{pmatrix} \frac{1}{2}\left(\gamma I_d(t) + \dot{I}_d(t) + \dot{S}_d(t)\right) \\ \frac{1}{2}\left(\gamma I_d(t) + \dot{I}_d(t) + \dot{S}_d(t)\right) \\ -\gamma I_d(t) + \dot{R}_d(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (2.88)$$

of which one is redundant. Note that because the projectors $\mathcal{P}$ and $\mathcal{Q}$ are non-diagonal, the time derivatives of $I_d(t)$ and $S_d(t)$ are mixed in the constraint equation.

2.4 Controllability

A system is called controllable or state controllable if it is possible to achieve a transfer from an initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ at time $t = t_0$ to a final state $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$ at the terminal time $t = t_1$. Controllability is a condition on the structure of the dynamical system as given by the nonlinearity $\boldsymbol{R}$ and the coupling matrix $\mathcal{B}$. In contrast, for a given control system, trajectory realizability is a condition on the desired trajectory. While for linear control systems controllability is easily expressed in terms of a rank condition, the notion is much more difficult for nonlinear control systems. Section 2.4.1 discusses the Kalman rank condition for the controllability of LTI systems as introduced by Kalman (1959, 1960) in the early sixties. Section 2.4.3 derives a similar rank condition in the context of exactly realizable trajectories. Remarkably, this rank condition also applies to nonlinear systems satisfying the linearizing assumption from Sect. 2.3.

For general nonlinear systems, the notion of controllability must be refined and it is necessary to distinguish between controllability, accessibility, and reachability. Different and not necessarily equivalent notions of controllability exist, and it is said that there are as many notions of nonlinear controllability as there are researchers in the field. When applied to LTI systems, all of these notions reduce to the Kalman rank condition. Here, no attempt is given to generalize the notion of controllability to nonlinear systems which violate the linearizing assumption. The reader is referred to the literature (Slotine and Li 1991; Isidori 1995; Khalil 2001; Levine 2009).

2.4.1 Kalman Rank Condition for LTI Systems

Controllability for LTI systems was first introduced by Kalman (1959, 1960). An excellent introduction to linear control systems, including controllability, can be found in Chen (1998).

Consider the LTI system with n-dimensional state vector $\boldsymbol{x}(t)$ and p-dimensional control signal $\boldsymbol{u}(t)$,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \boldsymbol{\mathcal{B}}\boldsymbol{u}(t), \tag{2.89}$$

and initial condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2.90}$$

Here, $\boldsymbol{\mathcal{A}}$ is an $n \times n$ real constant matrix and $\boldsymbol{\mathcal{B}}$ an $n \times p$ real constant matrix. The system Eq. (2.89) is said to be controllable if, for any initial state $\boldsymbol{x}_0$ at the initial time $t = t_0$ and any final state $\boldsymbol{x}_1$ at the terminal time $t = t_1$, there exists an input that transfers $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$. The terminal condition for the state is

$$\boldsymbol{x}(t_1) = \boldsymbol{x}_1. \tag{2.91}$$

The definition of controllability requires only that the input $\boldsymbol{u}(t)$ be capable of moving any state in the state space to any other state in finite time. The state trajectory $\boldsymbol{x}(t)$ traced out in state space is not specified. Kalman showed that this definition of controllability is equivalent to the statement that the $n \times np$ *controllability matrix*

$$\boldsymbol{\mathcal{K}} = \left(\boldsymbol{\mathcal{B}}|\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}|\boldsymbol{\mathcal{A}}^2\boldsymbol{\mathcal{B}}|\cdots|\boldsymbol{\mathcal{A}}^{n-1}\boldsymbol{\mathcal{B}}\right) \tag{2.92}$$

has rank n, i.e., it satisfies the *Kalman rank condition*

$$\text{rank}(\boldsymbol{\mathcal{K}}) = n. \tag{2.93}$$

Since $n \leq np$, this condition states that $\boldsymbol{\mathcal{K}}$ has full row rank. Equation (2.93) is derived in the following.

2.4.2 Derivation of the Kalman Rank Condition

The solution $\boldsymbol{x}(t)$ to Eq. (2.89) with initial condition Eq. (2.90) and arbitrary control signal $\boldsymbol{u}(t)$ is

$$\boldsymbol{x}(t) = e^{\boldsymbol{\mathcal{A}}(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t} d\tau e^{\boldsymbol{\mathcal{A}}(t-\tau)}\boldsymbol{\mathcal{B}}\boldsymbol{u}(\tau). \tag{2.94}$$

See also Appendix A.1 for a derivation of the general solution to a forced linear dynamical system. The system is controllable if a control signal $\boldsymbol{u}$ can be found such that the terminal condition (2.91) is satisfied. Evaluating Eq. (2.94) at the terminal time $t = t_1$, multiplying by $e^{-\boldsymbol{\mathcal{A}}(t_1-t_0)}$, rearranging, and expanding the matrix exponential under the integral yields

$$e^{-\boldsymbol{\mathcal{A}}(t_1-t_0)}\boldsymbol{x}_1 - \boldsymbol{x}_0 = \sum_{k=0}^{\infty} \boldsymbol{\mathcal{A}}^k\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \frac{(t_0-\tau)^k}{k!}\boldsymbol{u}(\tau). \tag{2.95}$$

For the system to be controllable, it must in principle be possible to solve for the control signal $\boldsymbol{u}$. As a consequence of the Cayley–Hamilton theorem, the matrix power $\boldsymbol{\mathcal{A}}^i$ for any $n \times n$ matrix with $i \geq n$ can be written as a sum of lower order powers (Fischer 2013),

$$\boldsymbol{\mathcal{A}}^i = \sum_{k=0}^{n-1} c_{ik}\boldsymbol{\mathcal{A}}^k. \tag{2.96}$$

It follows that the infinite sum in Eq. (2.95) can be rearranged to include only terms with power in $\boldsymbol{\mathcal{A}}$ up to $\boldsymbol{\mathcal{A}}^{n-1}$. The sum on the right hand side (r.h.s.) of Eq. (2.95) can be simplified as

$$\begin{aligned}
&\sum_{k=0}^{\infty} \boldsymbol{\mathcal{A}}^k\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \frac{(t_0-\tau)^k}{k!}\boldsymbol{u}(\tau) \\
&= \sum_{k=0}^{n-1} \boldsymbol{\mathcal{A}}^k\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \frac{(t_0-\tau)^k}{k!}\boldsymbol{u}(\tau) + \sum_{i=n}^{\infty} \boldsymbol{\mathcal{A}}^i\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \frac{(t_0-\tau)^i}{i!}\boldsymbol{u}(\tau) \\
&= \sum_{k=0}^{n-1} \boldsymbol{\mathcal{A}}^k\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \frac{(t_0-\tau)^k}{k!}\boldsymbol{u}(\tau) + \sum_{i=n}^{\infty}\sum_{k=0}^{n-1} c_{ik}\boldsymbol{\mathcal{A}}^k\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \frac{(t_0-\tau)^i}{i!}\boldsymbol{u}(\tau) \\
&= \sum_{k=0}^{n-1} \boldsymbol{\mathcal{A}}^k\boldsymbol{\mathcal{B}} \int_{t_0}^{t_1} d\tau \left(\frac{(t_0-\tau)^k}{k!} + \sum_{i=n}^{\infty} c_{ik}\frac{(t_0-\tau)^i}{i!}\right)\boldsymbol{u}(\tau).
\end{aligned} \tag{2.97}$$

It follows that the sum in Eq. (2.95) can be truncated after n terms,

$$e^{-\boldsymbol{\mathcal{A}}(t_1-t_0)}\boldsymbol{x}_1 - \boldsymbol{x}_0 = \sum_{k=0}^{n-1} \boldsymbol{\mathcal{A}}^k \boldsymbol{\mathcal{B}} \boldsymbol{\beta}_k (t_1, t_0). \tag{2.98}$$

The $\boldsymbol{\beta}_k$ are $p \times 1$ vectors defined as

$$\boldsymbol{\beta}_k (t_1, t_0) = \int_{t_0}^{t_1} d\tau \left(\frac{(t_0 - \tau)^k}{k!} + \sum_{i=n}^{\infty} c_{ik} \frac{(t_0 - \tau)^i}{i!} \right) \boldsymbol{u} (\tau), \tag{2.99}$$

which depend on the initial and terminal time t_0 and t_1, respectively. These vectors are functionals of the control $\boldsymbol{u}$ and depend on the matrix $\boldsymbol{\mathcal{A}}$ through the expansion coefficients c_{ik}. Defining the $np \times 1$ vector

$$\boldsymbol{\beta} (t_1, t_0) = \begin{pmatrix} \boldsymbol{\beta}_0 (t_1, t_0) \\ \vdots \\ \boldsymbol{\beta}_{n-1} (t_1, t_0) \end{pmatrix}, \tag{2.100}$$

Equation (2.98) can be written in terms of $\boldsymbol{\beta}$ and Kalman's controllability $n \times np$ matrix $\boldsymbol{\mathcal{K}}$, Eq. (2.92), as

$$e^{-\boldsymbol{\mathcal{A}}(t_1-t_0)}\boldsymbol{x}_1 - \boldsymbol{x}_0 = \sum_{k=0}^{n-1} \boldsymbol{\mathcal{A}}^k \boldsymbol{\mathcal{B}} \boldsymbol{\beta}_k (t_1, t_0) = \boldsymbol{\mathcal{K}} \boldsymbol{\beta} (t_1, t_0). \tag{2.101}$$

Equation (2.101) is a linear equation for the vector $\boldsymbol{\beta} (t_1, t_0)$ with inhomogeneity $e^{-\boldsymbol{\mathcal{A}}(t_1-t_0)}\boldsymbol{x}_1 - \boldsymbol{x}_0$. For the system (2.89) to be controllable, every state point $\boldsymbol{x}_1$ must have a corresponding vector $\boldsymbol{\beta} (t_1, t_0)$. In other words, the linear map from $\boldsymbol{\beta} (t_1, t_0)$ to $\boldsymbol{x}_1$ must be *surjective*. This is the case if and only if the matrix $\boldsymbol{\mathcal{K}}$ has full row rank (Fischer 2013), i.e.,

$$\text{rank} (\boldsymbol{\mathcal{K}}) = n. \tag{2.102}$$

The Kalman rank condition Eq. (2.102) is a necessary and sufficient condition for the controllability of an LTI system.

A slightly different way to arrive at the same result is to solve (2.101) for the vector $\boldsymbol{\beta} (t_1, t_0)$. A solution in terms of the $n \times n$ matrix $\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^T$ is (see also Appendix A.2 how to solve an underdetermined system of equations)

$$\boldsymbol{\beta} (t_1, t_0) = \boldsymbol{\mathcal{K}}^T \left(\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^T\right)^{-1} \left(e^{-\boldsymbol{\mathcal{A}}(t_1-t_0)}\boldsymbol{x}_1 - \boldsymbol{x}_0\right). \tag{2.103}$$

The inverse of $\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^T$ does exist only if it has full rank, i.e., rank $\left(\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^T\right) = n$. This is the case if and only if the Kalman rank condition rank $(\boldsymbol{\mathcal{K}}) = n$ is satisfied.

Controllability has a number of interesting and important consequences. Two examples illustrate the concept and highlight one important consequence.

Example 2.8: Single input diagonal LTI system
We consider an LTI system with state and input matrix

$$\mathcal{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.104}$$

Kalman's controllability matrix is

$$\mathcal{K} = (\mathcal{B}|\mathcal{AB}) = \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}. \tag{2.105}$$

As long as $\lambda_1 \neq \lambda_2$, $\mathcal{K}$ has rank 2. If $\lambda_1 = \lambda_2$, the second row equals the first row, and $\mathcal{K}$ has rank 1. The system Eq. (2.104) is controllable as long as $\lambda_1 \neq \lambda_2$.

Example 2.9: Two pendulums mounted on a cart
Two pendulums mounted on a cart is a mechanical toy model for linear control systems (Chen 1998). As can be seen in Fig. 2.4, the positions of the masses m_1 and m_2 given by $\left(x_1, y_1\right)^T$ and $\left(x_2, y_2\right)^T$, respectively, are

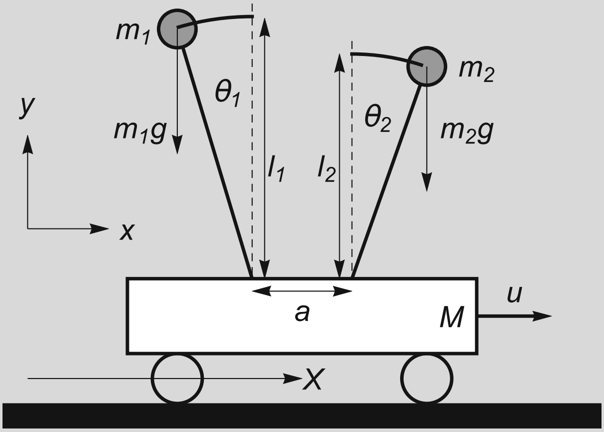

Fig. 2.4 Two inverted pendulums mounted on a cart. The control task is to keep both pendulums in the upright and unstable equilibrium position. The system is controllable as long as the pendulums are not exactly identical, i.e., as long as either their lengths ($l_1 \neq l_2$) or their masses are different ($m_1 \neq m_2$)

$$x_1(t) = X(t) - \frac{a}{2} + l_1 \sin(\theta_1(t)), \quad x_2(t) = X(t) + \frac{a}{2} + l_2 \sin(\theta_2(t)), \tag{2.106}$$

$$y_1(t) = l_1 \cos(\theta_1(t)), \quad y_2(t) = l_2 \cos(\theta_2(t)). \tag{2.107}$$

The cart can only move in the x direction without any motion in the y-direction. Its position is denoted by $X(t)$. The Lagrangian L equals the difference between kinetic energy T and potential energy V,

$$L = T - V = \frac{1}{2}m_1\left(\dot{x}_1^2(t) + \dot{y}_1^2(t)\right) + \frac{1}{2}m_2\left(\dot{x}_2^2(t) + \dot{y}_2^2(t)\right) + \frac{1}{2}M\dot{X}^2(t) - m_1 l_1 g \cos(\theta_1(t)) - m_2 l_2 g \cos(\theta_1(t)). \tag{2.108}$$

The equations of motion are given by the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \frac{\partial L}{\partial X} = u(t). \tag{2.109}$$

The control force $u(t)$ acts on the cart but not on the pendulums. Assuming small angles, $0 \leq |\theta_1(t)| \ll 1$ and $0 \leq |\theta_2(t)| \ll 1$, the equations of motion are linearized around the stationary point. Rewriting the second order differential equations as a controlled dynamical system with $P(t) = M\dot{X}(t)$, $p_1(t) = \dot{\theta}_1$, and $p_2(t) = \dot{\theta}_2(t)$ yields

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \boldsymbol{B}u(t), \tag{2.110}$$

with

$$\boldsymbol{x}(t) = (\theta_1(t), \theta_2(t), X(t), p_1(t), p_2(t), P(t))^T, \tag{2.111}$$

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{M} \\ \frac{g(m_1+M)}{l_1 M} & \frac{g m_2}{l_1 M} & 0 & 0 & 0 & 0 \\ \frac{g m_1}{l_2 M} & \frac{g(m_2+M)}{l_2 M} & 0 & 0 & 0 & 0 \\ -g m_1 & -g m_2 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.112}$$

$$\boldsymbol{B} = \left(0, 0, 0, -\frac{1}{l_1 M}, -\frac{1}{l_2 M}, 1\right)^T. \tag{2.113}$$

Kalman's controllability matrix is

$$
\boldsymbol{\mathcal{K}} = \left(\boldsymbol{B}|\boldsymbol{\mathcal{A}B}|\boldsymbol{\mathcal{A}}^2\boldsymbol{B}|\cdots|\boldsymbol{\mathcal{A}}^5\boldsymbol{B}\right)
= \begin{pmatrix}
0 & -\frac{1}{l_1 M} & 0 & -\frac{\alpha_3 g}{l_1^2 l_2 M^2} & 0 & -\frac{\beta_1 g^2}{l_1^3 l_2^2 M^3} \\
0 & -\frac{1}{l_2 M} & 0 & -\frac{\alpha_2 g}{l_1 l_2^2 M^2} & 0 & -\frac{\beta_2 g^2}{l_1^2 l_2^3 M^3} \\
0 & \frac{1}{M} & 0 & \frac{\alpha_1 g}{l_1 l_2 M^2} & 0 & \frac{\beta_3 g^2}{l_1^2 l_2^2 M^3} \\
-\frac{1}{l_1 M} & 0 & -\frac{\alpha_3 g}{l_1^2 l_2 M^2} & 0 & -\frac{\beta_1 g^2}{l_1^3 l_2^2 M^3} & 0 \\
-\frac{1}{l_2 M} & 0 & -\frac{\alpha_2 g}{l_1 l_2^2 M^2} & 0 & -\frac{\beta_2 g^2}{l_1^2 l_2^3 M^3} & 0 \\
1 & 0 & \frac{\alpha_1 g}{l_1 l_2 M} & 0 & \frac{\beta_3 g^2}{l_1^2 l_2^2 M^2} & 0
\end{pmatrix} \tag{2.114}
$$

with

$$
\alpha_1 = l_2 m_1 + l_1 m_2, \quad \alpha_2 = l_1 (m_2 + M) + l_2 m_1, \tag{2.115}
$$
$$
\alpha_3 = l_2 (m_1 + M) + l_1 m_2, \tag{2.116}
$$

and

$$
\beta_1 = l_1^2 m_2 (m_2 + M) + l_2 l_1 m_2 (2m_1 + M) + l_2^2 (m_1 + M)^2, \tag{2.117}
$$
$$
\beta_2 = l_2^2 m_1 (m_1 + M) + l_1 l_2 m_1 (2m_2 + M) + l_1^2 (m_2 + M)^2, \tag{2.118}
$$
$$
\beta_3 = l_1^2 m_2 (m_2 + M) + l_2^2 m_1 (m_1 + M) + 2 l_2 l_1 m_1 m_2. \tag{2.119}
$$

The matrix $\boldsymbol{\mathcal{K}}$ has full row rank,

$$
\operatorname{rank}(\boldsymbol{\mathcal{K}}) = 6, \tag{2.120}
$$

as long as the pendulums are not identical. Consequently, the system is controllable. A small deviation from the equilibrium position can be counteracted by control. Pendulums with identical mass and lengths, $m_2 = m_1$ and $l_2 = l_1$, respectively, yield

$$
\alpha_2 = \alpha_3, \quad \beta_1 = \beta_2, \tag{2.121}
$$

and the first and the second as well as the fourth and the fifth row of the matrix $\boldsymbol{\mathcal{K}}$ become identical. Consequently, the rank of $\boldsymbol{\mathcal{K}}$ changes, and two identical pendulums cannot be controlled.

Both examples show one important consequence of controllability: arbitrary many, parallel connected identical systems cannot be controlled (Kailath 1980; Chen 1998). Expressed in a less rigorous language, controllability renders balancing two identical brooms with only a single hand mathematically impossible.

2.4.3 Controllability for Systems Satisfying the Linearizing Assumption

Kalman's approach to controllability does not allow a direct generalization to nonlinear systems. Furthermore, nothing is said about the trajectory along which this transfer is achieved. To some extent, these questions can be addressed in the framework of exactly realizable trajectories. Here, a controllability matrix is derived which applies not only to LTI systems but also to nonlinear systems satisfying the linearizing assumption from Sect. 2.3.

Consider the controlled system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\,\boldsymbol{u}(t) \tag{2.122}$$

together with the linearizing assumption

$$\boldsymbol{\mathcal{Q}}\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{x} + \boldsymbol{\mathcal{Q}}\boldsymbol{b}. \tag{2.123}$$

Equation (2.123) implies a linear constraint equation for an exactly realizable desired trajectory $\boldsymbol{x}_d(t)$,

$$\boldsymbol{\mathcal{Q}}\dot{\boldsymbol{x}}_d(t) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{x}_d(t) + \boldsymbol{\mathcal{Q}}\boldsymbol{b}. \tag{2.124}$$

or, inserting $\mathbf{1} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{Q}}$ between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{x}_d$,

$$\boldsymbol{\mathcal{Q}}\dot{\boldsymbol{x}}_d(t) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t) + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t) + \boldsymbol{\mathcal{Q}}\boldsymbol{b}. \tag{2.125}$$

From now on, the parts $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$ and $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$ are considered as independent state components. The part $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$ is prescribed by the experimenter while the part $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$ is governed by Eq. (2.124). Equation (2.125) is a linear dynamical system for the variable $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$ with inhomogeneity $\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t) + \boldsymbol{\mathcal{Q}}\boldsymbol{b}$. Achieving a transfer from the initial state $\boldsymbol{x}_0$ to the finite state $\boldsymbol{x}_1$ means the realizable trajectory $\boldsymbol{x}_d(t)$ has to satisfy

$$\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0, \tag{2.126}$$

$$\boldsymbol{x}_d(t_1) = \boldsymbol{x}_1. \tag{2.127}$$

Consequently, the prescribed part $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$ satisfies

$$\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t_0) = \boldsymbol{\mathcal{P}}\boldsymbol{x}_0, \quad \boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t_1) = \boldsymbol{\mathcal{P}}\boldsymbol{x}_1, \tag{2.128}$$

while the part $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$ satisfies

$$\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_0) = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_0, \quad \boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_1) = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_1. \tag{2.129}$$

Being a linear equation, the solution $\mathcal{Q}\boldsymbol{x}_d(t)$ to the constraint equation (2.125) can be expressed as a functional of $\mathcal{P}\boldsymbol{x}_d(t)$,

$$\mathcal{Q}\boldsymbol{x}_d(t) = \exp\left(\mathcal{QAQ}(t-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 + \int_{t_0}^{t} d\tau \exp\left(\mathcal{QAQ}(t-\tau)\right)\mathcal{Q}\left(\mathcal{AP}\boldsymbol{x}_d(\tau)+\boldsymbol{b}\right). \tag{2.130}$$

See also Appendix A.1 for a derivation of the general solution to a forced linear dynamical system. The solution Eq. (2.130) satisfies the initial condition given by Eq. (2.129). Now, all initial and terminal conditions except $\mathcal{Q}\boldsymbol{x}_d(t_1) = \mathcal{Q}\boldsymbol{x}_1$ are satisfied. Enforcing this remaining terminal condition onto the solution Eq. (2.130) yields

$$\begin{aligned}\mathcal{Q}\boldsymbol{x}_1 &= \mathcal{Q}\boldsymbol{x}_d(t_1)\\ &= \exp\left(\mathcal{QAQ}(t_1-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 + \int_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}(t_1-\tau)\right)\mathcal{Q}\left(\mathcal{AP}\boldsymbol{x}_d(\tau)+\boldsymbol{b}\right).\end{aligned} \tag{2.131}$$

This is actually a condition for the part $\mathcal{P}\boldsymbol{x}_d(t)$. Therefore, the transfer from $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$ is achieved as long as the part $\mathcal{P}\boldsymbol{x}_d$ satisfies Eqs. (2.128) and (2.131). In between t_0 and t_1, the part $\mathcal{P}\boldsymbol{x}_d(t)$ of the realizable trajectory can be freely chosen by the experimenter. A system is controllable if at least one exactly realizable trajectory $\boldsymbol{x}_d(t)$ can be found such that the constraints Eqs. (2.128) and (2.131) are satisfied.

Analogously to the derivation of the Kalman rank condition in Sect. 2.4.2, one can ask for the conditions on the state matrices $\mathcal{A}$ and projectors $\mathcal{P}$ and $\mathcal{Q}$ such that the constraint Eq. (2.131) can be satisfied. Equation (2.131) is rearranged as

$$\begin{aligned}&\exp\left(-\mathcal{QAQ}(t_1-t_0)\right)\mathcal{Q}\boldsymbol{x}_1 - \mathcal{Q}\boldsymbol{x}_0 - \int_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}(t_0-\tau)\right)\mathcal{Q}\boldsymbol{b}\\ &= \int_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}(t_0-\tau)\right)\mathcal{QAP}\boldsymbol{x}_d(\tau),\end{aligned} \tag{2.132}$$

and an argument equivalent to the derivation of the Kalman rank condition (2.102) is applied. Due to the Cayley-Hamilton theorem, any power of matrices with $i \geq n$ can be expanded in terms of lower order matrix powers as

$$(\mathcal{QAQ})^i = \sum_{k=0}^{n-1} d_{ik}(\mathcal{QAQ})^k. \tag{2.133}$$

The r.h.s. of Eq. (2.132) can be simplified as

$$\begin{aligned}
&\int\limits_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}\left(t_0-\tau\right)\right)\mathcal{QAP}\boldsymbol{x}_d\left(\tau\right) \\
&= \sum_{k=0}^{\infty}\left(\mathcal{QAQ}\right)^k \mathcal{QAP}\int\limits_{t_0}^{t_1} d\tau \frac{\left(t_0-\tau\right)^k}{k!}\mathcal{P}\boldsymbol{x}_d\left(\tau\right) \\
&= \sum_{k=0}^{n-1}\left(\mathcal{QAQ}\right)^k \mathcal{QAP}\int\limits_{t_0}^{t_1} d\tau \frac{\left(t_0-\tau\right)^k}{k!}\mathcal{P}\boldsymbol{x}_d\left(\tau\right) \\
&\quad + \sum_{i=n}^{\infty}\left(\mathcal{QAQ}\right)^i \mathcal{QAP}\int\limits_{t_0}^{t_1} d\tau \frac{\left(t_0-\tau\right)^i}{i!}\mathcal{P}\boldsymbol{x}_d\left(\tau\right) \\
&= \sum_{k=0}^{n-1}\left(\mathcal{QAQ}\right)^k \mathcal{QAP}\int\limits_{t_0}^{t_1} d\tau \left(\frac{\left(t_0-\tau\right)^k}{k!}+\sum_{i=n}^{\infty} d_{ik}\frac{\left(t_0-\tau\right)^i}{i!}\right)\mathcal{P}\boldsymbol{x}_d\left(\tau\right),
\end{aligned} \tag{2.134}$$

such that Eq. (2.132) becomes a truncated sum

$$\begin{aligned}
&\exp\left(-\mathcal{QAQ}\left(t_1-t_0\right)\right)\mathcal{Q}\boldsymbol{x}_1-\mathcal{Q}\boldsymbol{x}_0-\int\limits_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}\left(t_0-\tau\right)\right)\mathcal{Q}\boldsymbol{b} \\
&\quad = \sum_{k=0}^{n-1}\left(\mathcal{QAQ}\right)^k \mathcal{QAP}\boldsymbol{\alpha}_k\left(t_1,t_0\right).
\end{aligned} \tag{2.135}$$

Define the $n \times 1$ vectors

$$\boldsymbol{\alpha}_k\left(t_1,t_0\right) = \int\limits_{t_0}^{t_1} d\tau \left(\frac{\left(t_0-\tau\right)^k}{k!}+\sum_{i=n}^{\infty} d_{ik}\frac{\left(t_0-\tau\right)^i}{i!}\right)\mathcal{P}\boldsymbol{x}_d\left(\tau\right). \tag{2.136}$$

The right hand side of Eq. (2.135) can be written with the help of the $n^2 \times 1$ vector

$$\boldsymbol{\alpha}\left(t_1,t_0\right) = \begin{pmatrix} \boldsymbol{\alpha}_0\left(t_1,t_0\right) \\ \boldsymbol{\alpha}_1\left(t_1,t_0\right) \\ \vdots \\ \boldsymbol{\alpha}_{n-1}\left(t_1,t_0\right) \end{pmatrix} \tag{2.137}$$

as

$$\exp\left(-\mathcal{QAQ}\left(t_1-t_0\right)\right)\mathcal{Q}\boldsymbol{x}_1-\mathcal{Q}\boldsymbol{x}_0$$
$$-\int_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}\left(t_0-\tau\right)\right)\mathcal{Q}\boldsymbol{b}=\tilde{\mathcal{K}}\boldsymbol{\alpha}\left(t_1,t_0\right). \tag{2.138}$$

The $n\times n^2$ *controllability matrix* $\tilde{\mathcal{K}}$ is defined by

$$\tilde{\mathcal{K}}=\left(\mathcal{QAP}|\mathcal{QAQAP}|\cdots|\left(\mathcal{QAQ}\right)^{n-1}\mathcal{QAP}\right). \tag{2.139}$$

The left hand side of Eq. (2.138) can be any point in $\mathcal{Q}\mathbb{R}^n=\mathbb{R}^{n-p}$. The mapping is surjective, i.e., every element on the left hand side has a corresponding element on the right hand side, if $\tilde{\mathcal{K}}$ has full rank $n-p$. Therefore, the nonlinear affine control system Eq. (2.122) satisfying the linearizing assumption Eq. (2.123) is controllable if

$$\operatorname{rank}\left(\tilde{\mathcal{K}}\right)=n-p. \tag{2.140}$$

Example 2.10: Single input diagonal LTI system
Consider the LTI system from Example 2.8. The two parts of $\mathcal{A}$ necessary for the computation of the controllability matrix $\tilde{\mathcal{K}}$ are

$$\mathcal{QAP}=\frac{1}{4}\begin{pmatrix}\lambda_1-\lambda_2 & \lambda_1-\lambda_2\\ \lambda_2-\lambda_1 & \lambda_2-\lambda_1\end{pmatrix},\quad \mathcal{QAQ}=\frac{1}{4}\begin{pmatrix}\lambda_1+\lambda_2 & \lambda_1-\lambda_2\\ \lambda_2-\lambda_1 & \lambda_1+\lambda_2\end{pmatrix}. \tag{2.141}$$

The controllability matrix is

$$\begin{aligned}\tilde{\mathcal{K}}&=\left(\mathcal{QAP}|\mathcal{QAQAP}\right)\\ &=\frac{1}{4}\begin{pmatrix}(\lambda_1-\lambda_2) & (\lambda_1-\lambda_2) & \frac{1}{2}\left(\lambda_1^2-\lambda_2^2\right) & \frac{1}{2}\left(\lambda_1^2-\lambda_2^2\right)\\ (\lambda_2-\lambda_1) & (\lambda_2-\lambda_1) & \frac{1}{2}\left(\lambda_2^2-\lambda_1^2\right) & \frac{1}{2}\left(\lambda_2^2-\lambda_1^2\right)\end{pmatrix}.\end{aligned} \tag{2.142}$$

The upper row of $\tilde{\mathcal{K}}$ equals the lower row times -1, i.e., the rows are linearly dependent and so $\tilde{\mathcal{K}}$ has rank

$$\operatorname{rank}\left(\tilde{\mathcal{K}}\right)=1. \tag{2.143}$$

If $\lambda_1=\lambda_2$, all entries of $\tilde{\mathcal{K}}$ vanish and then $\tilde{\mathcal{K}}$ has zero rank. The system Eq. (2.104) is controllable as long as $\lambda_1\neq\lambda_2$.

Example 2.11: Controllability of the activator-controlled FHN model
Controllability in form of a rank condition can be discussed for all models of the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x(t) + a_2 y(t) \\ R(x(t), y(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ B(x(t), y(t)) \end{pmatrix} u(t). \quad (2.144)$$

A prominent example is the activator-controlled FHN model. The $\mathcal{Q}$ part of the nonlinearity $\boldsymbol{R}$ is actually a linear function of the state $\boldsymbol{x}$,

$$\mathcal{Q}\boldsymbol{R}(\boldsymbol{x}(t)) = \begin{pmatrix} a_1 x(t) + a_2 y(t) \\ 0 \end{pmatrix} + \begin{pmatrix} a_0 \\ 0 \end{pmatrix} = \mathcal{Q}\mathcal{A}\boldsymbol{x}(t) + \mathcal{Q}\boldsymbol{b}, \quad (2.145)$$

i.e., this model satisfies the linearizing assumption with the matrix $\mathcal{A}$ and vector $\boldsymbol{b}$ defined by

$$\mathcal{Q}\mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}\boldsymbol{b} = \begin{pmatrix} a_0 \\ 0 \end{pmatrix}. \quad (2.146)$$

The controllability matrix $\tilde{\mathcal{K}}$ is

$$\tilde{\mathcal{K}} = (\mathcal{Q}\mathcal{A}\mathcal{P} | \mathcal{Q}\mathcal{A}\mathcal{Q}\mathcal{A}\mathcal{P}) = \begin{pmatrix} 0 & a_2 & 0 & a_1 a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.147)$$

and, for $a_2 \neq 0$, $\tilde{\mathcal{K}}$ has rank

$$\operatorname{rank}\left(\tilde{\mathcal{K}}\right) = 1 = n - p. \quad (2.148)$$

The activator-controlled FHN model is controllable as long as $a_2 \neq 0$, i.e., as long as the equation for the inhibitor x also depends on the activator y. The control directly affects the activator y. If $a_2 = 0$ in Eq. (2.144), the inhibitor evolves decoupled from the activator, and therefore cannot be affected by control.

Example 2.12: Controllability of the controlled SIR model
With the help of the projectors $\mathcal{P}$ and $\mathcal{Q}$ computed in Example 2.7, the controllability matrix is obtained as

$$\tilde{\mathcal{K}} = (\mathcal{QAP}|\mathcal{QAQAP}|\mathcal{QAQAQAP})$$
$$= \begin{pmatrix} \frac{\gamma}{4} & -\frac{\gamma}{4} & 0 & -\frac{\gamma^2}{8} & \frac{\gamma^2}{8} & 0 & \frac{\gamma^3}{16} & -\frac{\gamma^3}{16} & 0 \\ \frac{\gamma}{4} & -\frac{\gamma}{4} & 0 & -\frac{\gamma^2}{8} & \frac{\gamma^2}{8} & 0 & \frac{\gamma^3}{16} & -\frac{\gamma^3}{16} & 0 \\ -\frac{\gamma}{2} & \frac{\gamma}{2} & 0 & \frac{\gamma^2}{4} & -\frac{\gamma^2}{4} & 0 & -\frac{\gamma^3}{8} & \frac{\gamma^3}{8} & 0 \end{pmatrix}. \quad (2.149)$$

As long as $\gamma \neq 0$, the rank of $\tilde{\mathcal{K}}$ is

$$\text{rank}\left(\tilde{\mathcal{K}}\right) = 1 < n - p = 2. \quad (2.150)$$

Thus, the rank of $\tilde{\mathcal{K}}$ is smaller than $n - p$, and consequently the SIR model is *not* controllable. It is impossible to find a control to reach every final state $\boldsymbol{x}_1$ from every other initial state $\boldsymbol{x}_0$. Intuitively, the reason is simple to understand. The controlled SIR model satisfies a conservation law, see Example 1.3. Independent of the actual time dependence of the control signal $u(t)$, the total number N of individuals is conserved,

$$S(t) + I(t) + R(t) = N. \quad (2.151)$$

The value of N is prescribed by the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. For all times, the dynamics of the controlled SIR model is restricted to a two-dimensional surface embedded in the three-dimensional state space. Hence, the system's state vector can only reach points lying on this surface, and no control can force the system to leave it.

2.4.4 Discussion

A controllability matrix $\tilde{\mathcal{K}}$, Eq. (2.139), is derived in the framework of exactly realizable trajectories. If $\tilde{\mathcal{K}}$ satisfies the rank condition $\text{rank}\left(\tilde{\mathcal{K}}\right) = n - p$, the system is controllable. At least one control signal exists which achieves a transfer from an arbitrary initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ to an arbitrary final state $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$ within the finite time interval $t_1 - t_0$.

The controllability matrix $\tilde{\mathcal{K}}$ can be computed for all LTI system. We expect that the rank condition for controllability, Eq. (2.140) is fully equivalent to Kalman's rank condition, Eq. (2.102). If the system is controllable in terms of $\tilde{\mathcal{K}}$, it is also controllable in terms of $\mathcal{K}$, and vice versa. The advantage of controllability in terms of $\tilde{\mathcal{K}}$ is its applicability to a certain class of nonlinear systems. For affine dynamical systems satisfying the linearizing assumption (2.82), the rank condition for $\tilde{\mathcal{K}}$ remains a valid check for controllability. This class encompasses a number of simple nonlinear models which are of interest to physicists. In particular, it is proven that all mechanical control systems in one spatial dimension are controllable, see Example 2.11. Other systems satisfying the linearizing assumption are the controlled SIR model, Example 2.12, and the activator-controlled FHN model, see Example 2.11. Controllability as proposed here cannot be applied to the inhibitor-controlled FHN model because the corresponding constraint equation is nonlinear. Checking its controllability requires a notion of nonlinear controllability for general nonlinear systems. Nonlinear controllability cannot be defined in form of a simple rank condition for a controllability matrix but demands more difficult concepts.

Exactly realizable trajectories allow a characterization of the entirety of state trajectories along which a state transfer can be achieved. Any desired trajectory which satisfies the constraint equation

$$\mathcal{Q}\left(\dot{\boldsymbol{x}}_d(t) - \mathcal{A}\boldsymbol{x}_d(t) - \boldsymbol{b}\right) = \boldsymbol{0}, \tag{2.152}$$

and the initial and terminal conditions

$$\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{x}_d(t_1) = \boldsymbol{x}_1 \tag{2.153}$$

does the job. For example, a second order differential equation for $\boldsymbol{x}_d(t)$ can accommodate both initial and terminal conditions Eq. (2.153). A successful transfer from $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$ is achieved if $\boldsymbol{x}_d(t)$ additionally satisfies the constraint equation (2.152). The control signal is given by

$$\boldsymbol{u}(t) = \mathcal{B}^{+}\left(\dot{\boldsymbol{x}}_d(t) - \mathcal{A}\boldsymbol{x}_d(t) - \boldsymbol{b}\right). \tag{2.154}$$

Equation (2.154) can be used to obtain an expression for the control which depends only on the part $\mathcal{P}\boldsymbol{x}_d(t)$. According to Eq. (2.130), the solution for $\mathcal{Q}\boldsymbol{x}_d(t)$ can be expressed in terms of a functional of $\mathcal{P}\boldsymbol{x}_d(t)$,

$$\begin{aligned}\mathcal{Q}\boldsymbol{x}_d[\mathcal{P}\boldsymbol{x}_d(t)] = {}& \exp\left(\mathcal{QAQ}(t-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 \\ & + \int_{t_0}^{t} d\tau \exp\left(\mathcal{QAQ}(t-\tau)\right)\mathcal{Q}\left(\mathcal{AP}\boldsymbol{x}_d(\tau) + \boldsymbol{b}\right).\end{aligned} \tag{2.155}$$

Consequently, Eq. (2.154) becomes a functional of $\mathcal{P}\boldsymbol{x}_d(t)$ as well,

$$\begin{aligned}\boldsymbol{u}[\mathcal{P}\boldsymbol{x}_d(t)] = {}& \mathcal{B}^{+}\left(\mathcal{P}\dot{\boldsymbol{x}}_d(t) - \mathcal{PAP}\boldsymbol{x}_d(t) - \mathcal{PA}\exp\left(\mathcal{QAQ}(t-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 - \mathcal{P}\boldsymbol{b}\right) \\ & - \mathcal{B}^{+}\mathcal{A}\int_{t_0}^{t} d\tau \exp\left(\mathcal{QAQ}(t-\tau)\right)\mathcal{Q}\left(\mathcal{AP}\boldsymbol{x}_d(\tau) + \boldsymbol{b}\right).\end{aligned} \tag{2.156}$$

Thus, any reference to $\mathcal{Q}\boldsymbol{x}_d(t)$ except for the initial condition $\mathcal{Q}\boldsymbol{x}_0$ is eliminated from the expression for the control signal. The control signal is entirely expressed in terms of the part $\mathcal{P}\boldsymbol{x}_d(t)$ prescribed by the experimenter.

Using the complementary projectors $\mathcal{P}$ and $\mathcal{Q}$, the state matrix $\mathcal{A}$ can be split up in four parts as

$$\mathcal{A} = \mathcal{P}\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}\mathcal{Q} + \mathcal{Q}\mathcal{A}\mathcal{P} + \mathcal{Q}\mathcal{A}\mathcal{Q}. \tag{2.157}$$

Note that the controllability matrix $\tilde{\mathcal{K}}$, Eq. (2.139), does only depend on the parts $\mathcal{Q}\mathcal{A}\mathcal{P}$ and $\mathcal{Q}\mathcal{A}\mathcal{Q}$, but not on $\mathcal{P}\mathcal{A}\mathcal{P}$ and $\mathcal{P}\mathcal{A}\mathcal{Q}$. This fact extends the validity of the controllability matrix $\tilde{\mathcal{K}}$ to nonlinear systems satisfying the linearizing assumption. Furthermore, only the parts $\mathcal{Q}\mathcal{A}\mathcal{P}$ and $\mathcal{Q}\mathcal{A}\mathcal{Q}$ must be known to decide if a system is controllable. Thus, it can be possible to decide about controllability of a system without knowing all details of its dynamics. This insight might be useful for experimental systems with incomplete or approximated model equations.

2.5 Output Controllability

Consider the dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\,\boldsymbol{u}(t), \tag{2.158}$$

together with the output

$$\boldsymbol{z}(t) = \boldsymbol{h}(\boldsymbol{x}(t)). \tag{2.159}$$

Here, $\boldsymbol{z}(t) = (z_1(t), \ldots, z_m(t))^T \in \mathbb{R}^m$ with $m \leq n$ components is called the *output vector* and the *output function* $\boldsymbol{h}$ maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

A system is called output controllable if it is possible to achieve a transfer from an initial output state

$$\boldsymbol{z}(t_0) = \boldsymbol{z}_0 \tag{2.160}$$

at time $t = t_0$ to a terminal output state

$$\boldsymbol{z}(t_1) = \boldsymbol{z}_1 \tag{2.161}$$

at the terminal time $t = t_1$. In contrast to output controllability, the notion of controllability discussed in Sect. 2.4 is concerned with the controllability of the state $\boldsymbol{x}(t)$ and is often referred to as state or full state controllability. Note that a state controllable system is not necessarily output controllable. Similarly, an output controllable system is not necessarily state controllable. For $m = n$ and an output function equal to the identity function, $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x}$, output controllability is equivalent to state controllability.

2.5.1 Kalman Rank Condition for the Output Controllability of LTI Systems

The notion of state controllability developed in form of a Kalman rank condition for a controllability matrix can be adapted to output controllability (Kalman 1959, 1960; Chen 1998). Consider the LTI system with n-dimensional state vector $\boldsymbol{x}(t)$ and p-dimensional control signal $\boldsymbol{u}(t)$,

$$\dot{\boldsymbol{x}}(t) = \mathcal{A}\boldsymbol{x}(t) + \mathcal{B}\boldsymbol{u}(t). \tag{2.162}$$

The output is assumed to be a linear relation of the form

$$\boldsymbol{z}(t) = \mathcal{C}\boldsymbol{x}(t) \tag{2.163}$$

with $m \times n$ *output matrix* $\mathcal{C}$. The $m \times np$ *output controllability matrix* is defined as

$$\mathcal{K}_{\mathcal{C}} = \left(\mathcal{C}\mathcal{B}|\mathcal{C}\mathcal{A}\mathcal{B}|\mathcal{C}\mathcal{A}^2\mathcal{B}|\cdots|\mathcal{C}\mathcal{A}^{n-1}\mathcal{B}\right). \tag{2.164}$$

The LTI system Eq. (2.162) with output (2.163) is output controllable if $\mathcal{K}_{\mathcal{C}}$ satisfies the rank condition

$$\operatorname{rank}(\mathcal{K}_{\mathcal{C}}) = m. \tag{2.165}$$

A proof of Eq. (2.165) proceeds along the same lines as the proof for the Kalman rank condition for state controllability in Sect. 2.4.2. Using Eq. (2.94), the solution for $\boldsymbol{z}(t)$ is

$$\boldsymbol{z}(t) = \mathcal{C}\boldsymbol{x}(t) = \mathcal{C}e^{\mathcal{A}(t-t_0)}\boldsymbol{x}_0 + \mathcal{C}\int_{t_0}^{t} d\tau e^{\mathcal{A}(t-\tau)}\mathcal{B}\boldsymbol{u}(\tau). \tag{2.166}$$

Evaluating Eq. (2.166) at the terminal time $t = t_1$ and enforcing the terminal output condition Eq. (2.161) yields a condition for the control signal $\boldsymbol{u}$,

$$\boldsymbol{z}_1 = \boldsymbol{z}(t_1) = \mathcal{C}e^{\mathcal{A}(t_1-t_0)}\boldsymbol{x}_0 + \mathcal{C}\int_{t_0}^{t_1} d\tau e^{\mathcal{A}(t_1-\tau)}\mathcal{B}\boldsymbol{u}(\tau). \tag{2.167}$$

Exploiting the Cayley–Hamilton theorem and proceeding analogously to Eq. (2.97) yields

$$z_1 - \mathcal{C}e^{\mathcal{A}(t_1-t_0)}\boldsymbol{x}_0 = \mathcal{C}\sum_{k=0}^{n-1}\mathcal{A}^k\mathcal{B}\tilde{\boldsymbol{\beta}}_k(t_1,t_0) = \mathcal{K}_{\mathcal{C}}\tilde{\boldsymbol{\beta}}(t_1,t_0) \tag{2.168}$$

with $p \times 1$ vectors $\tilde{\boldsymbol{\beta}}_k$ defined as

$$\tilde{\boldsymbol{\beta}}_k(t_1,t_0) = \int_{t_0}^{t_1} d\tau \frac{(t_1-\tau)^k}{k!}\boldsymbol{u}(\tau) + \sum_{i=n}^{\infty} c_{ik}\int_{t_0}^{t_1} d\tau \frac{(t_1-\tau)^i}{i!}\boldsymbol{u}(\tau) \tag{2.169}$$

and the $np \times 1$ vector

$$\tilde{\boldsymbol{\beta}}(t_1,t_0) = \begin{pmatrix} \tilde{\boldsymbol{\beta}}_0(t_1,t_0) \\ \vdots \\ \tilde{\boldsymbol{\beta}}_{n-1}(t_1,t_0) \end{pmatrix}. \tag{2.170}$$

The linear map from $\tilde{\boldsymbol{\beta}}(t_1,t_0)$ to z_1 must be *surjective*. This is the case if and only if the matrix $\mathcal{K}_{\mathcal{C}}$ defined in Eq. (2.164) has full row rank, i.e.,

$$\operatorname{rank}(\mathcal{K}_{\mathcal{C}}) = m. \tag{2.171}$$

2.5.2 Output Controllability for Systems Satisfying the Linearizing Assumption

Using the framework of exactly realizable trajectories, we can generalize the condition for output controllability in form of a matrix rank condition to nonlinear affine control systems satisfying the linearizing assumption from Sect. 2.3. Consider the affine control system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \mathcal{B}(\boldsymbol{x}(t))\boldsymbol{u}(t) \tag{2.172}$$

with linear output

$$z(t) = \mathcal{C}\boldsymbol{x}(t). \tag{2.173}$$

The constraint equation for exactly realizable desired trajectories $\boldsymbol{x}_d(t)$ is linear,

$$\mathcal{Q}\dot{\boldsymbol{x}}_d(t) = \mathcal{Q}\mathcal{A}\boldsymbol{x}_d(t) + \mathcal{Q}\boldsymbol{b}, \tag{2.174}$$

and has the solution

$$\mathcal{Q}\boldsymbol{x}_d(t) = \exp\left(\mathcal{QAQ}(t-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 + \int_{t_0}^{t} d\tau \exp\left(\mathcal{QAQ}(t-\tau)\right)\mathcal{Q}\left(\mathcal{AP}\boldsymbol{x}_d(\tau)+\boldsymbol{b}\right). \tag{2.175}$$

Enforcing the desired output value at the terminal time $t = t_1$ yields

$$\begin{aligned} z_1 &= z_d(t_1) = \mathcal{C}\boldsymbol{x}_d(t_1) = \mathcal{CP}\boldsymbol{x}_d(t_1) + \mathcal{CQ}\boldsymbol{x}_d(t_1) \\ &= \mathcal{CP}\boldsymbol{x}_d(t_1) + \mathcal{CQ}\exp\left(\mathcal{QAQ}(t_1-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 \\ &\quad + \mathcal{CQ}\int_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}(t_1-\tau)\right)\mathcal{Q}\left(\mathcal{AP}\boldsymbol{x}_d(\tau)+\boldsymbol{b}\right). \end{aligned} \tag{2.176}$$

That is a condition for the part $\mathcal{P}\boldsymbol{x}_d(\tau)$. Exploiting the Cayley–Hamilton theorem and proceeding as in Eq. (2.134) yields

$$\begin{aligned} &z_1 - \mathcal{CQ}\exp\left(\mathcal{QAQ}(t_1-t_0)\right)\mathcal{Q}\boldsymbol{x}_0 - \mathcal{CQ}\int_{t_0}^{t_1} d\tau \exp\left(\mathcal{QAQ}(t_1-\tau)\right)\mathcal{Q}\boldsymbol{b} \\ &= \mathcal{CP}\boldsymbol{x}_d(t_1) + \mathcal{CQ}\sum_{k=0}^{n-1}\left(\mathcal{QAQ}\right)^k \tilde{\boldsymbol{\alpha}}_k(t_1,t_0). \end{aligned} \tag{2.177}$$

In Eq. (2.177) we defined the $n \times 1$ vectors

$$\tilde{\boldsymbol{\alpha}}_k(t_1,t_0) = \int_{t_0}^{t_1} d\tau \left(\frac{(t_1-\tau)^k}{k!} + \sum_{i=n}^{\infty} d_{ik}\frac{(t_1-\tau)^i}{i!}\right)\mathcal{QAP}\boldsymbol{x}_d(\tau). \tag{2.178}$$

The right hand side of Eq. (2.177) can be written with the help of the $n(n+1)\times 1$ vector

$$\tilde{\boldsymbol{\alpha}}(t_1,t_0) = \begin{pmatrix} \mathcal{P}\boldsymbol{x}_d(t_1) \\ \boldsymbol{\alpha}_0(t_1,t_0) \\ \vdots \\ \boldsymbol{\alpha}_{n-1}(t_1,t_0) \end{pmatrix} \tag{2.179}$$

and the $m \times n(n+1)$ *output controllability matrix*

$$\tilde{\mathcal{K}}_{\mathcal{C}} = \left(\mathcal{CP}|\mathcal{CQAP}|\cdots|\mathcal{CQ}\left(\mathcal{QAQ}\right)^{n-1}\mathcal{QAP}\right) \tag{2.180}$$

as

$$z_1 - \mathcal{C}\mathcal{Q}\exp\left(\mathcal{Q}\mathcal{A}\mathcal{Q}\left(t_1 - t_0\right)\right)\mathcal{Q}\boldsymbol{x}_0 - \mathcal{C}\mathcal{Q}\int_{t_0}^{t_1} d\tau \exp\left(\mathcal{Q}\mathcal{A}\mathcal{Q}\left(t_1 - \tau\right)\right)\mathcal{Q}\boldsymbol{b}$$
$$= \tilde{\mathcal{K}}_{\mathcal{C}}\tilde{\boldsymbol{\alpha}}\left(t_1, t_0\right). \tag{2.181}$$

The linear map from $\tilde{\boldsymbol{\alpha}}\left(t_1, t_0\right)$ to z_1 is surjective if $\tilde{\mathcal{K}}_{\mathcal{C}}$ has full row rank, i.e., if

$$\operatorname{rank}\left(\tilde{\mathcal{K}}_{\mathcal{C}}\right) = m. \tag{2.182}$$

Thus, a nonlinear affine control system satisfying the linearizing assumption is output controllable with linear output Eq. (2.173) if the matrix $\mathcal{K}_{\mathcal{C}}$ satisfies the *output controllability rank condition* Eq. (2.182).

With $m = n$ and $\mathcal{C} = \mathbf{1}$, the notion of output controllability reduces to the notion of full state controllability. Indeed, note that for $\mathcal{C} = \mathbf{1}$, $\tilde{\mathcal{K}}_{\mathcal{C}}$ can be written in terms of the controllability matrix for realizable trajectories $\tilde{\mathcal{K}}$ given by Eq. (2.139) as

$$\tilde{\mathcal{K}}_{\mathcal{C}} = \left(\mathcal{P}|\tilde{\mathcal{K}}\right). \tag{2.183}$$

Because $\tilde{\mathcal{K}}$ has no components in the direction of $\mathcal{P}$, i.e., $\tilde{\mathcal{K}} = \mathcal{P}\tilde{\mathcal{K}} + \mathcal{Q}\tilde{\mathcal{K}} = \mathcal{Q}\tilde{\mathcal{K}}$, the matrix $\tilde{\mathcal{K}}_{\mathcal{C}}$ as given by Eq. (2.183) has rank

$$\operatorname{rank}\left(\tilde{\mathcal{K}}_{\mathcal{C}}\right) = p + \operatorname{rank}\left(\tilde{\mathcal{K}}\right) = n. \tag{2.184}$$

This proves that the rank condition for output controllability, Eq. (2.182), indeed reduces, for $\mathcal{C} = \mathbf{1}$, to the rank condition for full state controllability as given by Eq. (2.140).

Output controllability is discussed by means of two examples.

Example 2.13: Output controllability of the activator-controlled FHN model

The model

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x(t) + a_2 y(t) \\ R\left(x(t), y(t)\right) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \tag{2.185}$$

satisfies the linearizing assumption such that the constraint equation is linear with state matrix

$$\mathcal{Q}\mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \tag{2.186}$$

see Examples 2.5 and 2.11 for more details. We check for the controllability of a general desired output with 1×2 output matrix $\mathcal{C} = \left(c_1,\, c_2\right)$,

$$z_d(t) = \mathcal{C}\boldsymbol{x}_d(t) = c_1 x_d(t) + c_2 y_d(t)\,. \tag{2.187}$$

The 1×6 output controllability matrix $\tilde{\mathcal{K}}_{\mathcal{C}}$ becomes

$$\begin{aligned}\tilde{\mathcal{K}}_{\mathcal{C}} &= \left(\mathcal{C}\mathcal{P}|\mathcal{C}\mathcal{Q}\mathcal{A}\mathcal{P}|\mathcal{C}\mathcal{Q}\mathcal{A}\mathcal{Q}\mathcal{A}\mathcal{P}\right)\\ &= \left(0\ c_2\ 0\ a_2 c_1\ 0\ a_1 a_2 c_1\right).\end{aligned} \tag{2.188}$$

The rank of $\tilde{\mathcal{K}}_{\mathcal{C}}$ is at most one. Example 2.11 showed that the system Eq. (2.185) is not controllable if $a_2 = 0$. In this case, $\tilde{\mathcal{K}}_{\mathcal{C}}$ simplifies to

$$\tilde{\mathcal{K}}_{\mathcal{C}} = \left(0\ c_2\ 0\ 0\ 0\ 0\right). \tag{2.189}$$

Thus, $\tilde{\mathcal{K}}_{\mathcal{C}}$ still has rank one as long as $c_2 \neq 0$. In conclusion, a model which is not controllable can nevertheless have a controllable output. Although for $a_2 = 0$ in Eq. (2.185), the inhibitor $x(t)$ evolves uncoupled from the activator dynamics, activator and inhibitor are still coupled in the output $z_d(t)$, Eq. (2.187). In that way the activator $y_d(t)$ can counteract the inhibitor $x_d(t)$ to control the desired output. If additionally $c_2 = 0$, this is not possible, and the output is not controllable. Indeed, for $c_2 = 0$, the output controllability matrix Eq. (2.189) reduces to the zero matrix with vanishing rank.

Example 2.14: Output controllability of the SIR model
The controlled state equation for the SIR model was developed in Example 1.3 and is repeated here for convenience,

$$\begin{aligned}&\dot{S}(t) = -\left(\beta + u(t)\right)\frac{S(t)\,I(t)}{N}, \quad \dot{I}(t) = \left(\beta + u(t)\right)\frac{S(t)\,I(t)}{N} - \gamma I(t)\,,\\ &\dot{R}(t) = \gamma I(t)\,.\end{aligned} \tag{2.190}$$

The controllability of the SIR model was discussed in Example 2.12. We check for the controllability of a general single component desired output with 1×3 output matrix $\mathcal{C} = \left(c_1,\, c_2,\, c_3\right)$,

$$z_d(t) = \mathcal{C}\boldsymbol{x}_d(t) = c_1 S_d(t) + c_2 I_d(t) + c_3 R_d(t)\,. \tag{2.191}$$

The output controllability matrix $\tilde{\mathcal{K}}_{\mathcal{C}}$ becomes

$$\begin{aligned}\tilde{\mathcal{K}}_{\mathcal{C}} &= \left(\mathcal{CP}|\mathcal{CQAP}|\mathcal{CQAQAP}|\mathcal{CQAQAQAP}\right)\\ &= \begin{pmatrix}\kappa_1 & -\kappa_1 & 0 & \gamma\kappa_2 & -\gamma\kappa_2 & 0 & -\frac{1}{2}\gamma^2\kappa_2 & \frac{\gamma^2}{2}\kappa_2 & 0 & \frac{\gamma^3}{4}\kappa_2 & -\frac{1}{4}\gamma^3\kappa_2 & 0\end{pmatrix},\end{aligned} \tag{2.192}$$

with $\kappa_1 = \frac{1}{2}(c_1 - c_2)$ and $\kappa_2 = \frac{1}{4}(c_1 + c_2 - 2c_3)$. The rank of $\tilde{\mathcal{K}}_{\mathcal{C}}$ is at most one. Depending on the values of the output parameters c_1, c_2, c_3, and the system parameter γ, the rank of $\tilde{\mathcal{K}}_{\mathcal{C}}$ changes. Two cases are discussed in detail.

First, if $c_1 = c_2 = 0$ and $c_3 \neq 0$, then $\kappa_1 = 0$ and the output is $z_d(t) = c_3 R_d(t)$ and prescribes the number of recovered people over time. As can be seen from Eq. (2.190), $R(t)$ is decoupled from the controlled part of the equations if $\gamma = 0$. Indeed, in this case $\tilde{\mathcal{K}}_{\mathcal{N}}$ reduces to the zero matrix with vanishing rank. In conclusion, a desired output equal to the number $R_d(t)$ of recovered people cannot be controlled if $\gamma = 0$.

Second, for $c_1 = c_2 = c_3 = c$ the desired output becomes

$$z_d(t) = c\left(S_d(t) + I_d(t) + R_d(t)\right) = cN = \text{const.}, \tag{2.193}$$

with N being the total number of individuals. This conservation law can easily be derived from the system dynamics Eq. (2.190) and remains true for the controlled system. We expect that this output is not controllable because the value of N is fixed by the initial conditions and cannot be changed by control. Indeed, if $c_1 = c_2 = c_3$ then $\kappa_1 = 0$ and $\kappa_2 = 0$. The output controllability matrix $\tilde{\mathcal{K}}_{\mathcal{C}}$ becomes the zero matrix with vanishing rank, and the output Eq. (2.193) is not controllable.

2.6 Output Realizability

2.6.1 General Procedure

For a desired trajectory $\boldsymbol{x}_d(t)$ to be exactly realizable, it must satisfy the constraint equation

$$\mathcal{Q}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) = \boldsymbol{0}. \tag{2.194}$$

This equation fixes $n - p$ components of the n components of $\boldsymbol{x}_d(t)$. Our convention was to choose these $n - p$ independent components as $\boldsymbol{y}_d(t) = \mathcal{Q}(\boldsymbol{x}_d(t))\boldsymbol{x}_d(t)$, while the p independent components $\boldsymbol{z}_d(t) = \mathcal{P}(\boldsymbol{x}_d(t))\boldsymbol{x}_d(t)$ of the desired state trajectory are prescribed by the experimenter. Equation (2.194) becomes a non-autonomous differential equation for $\boldsymbol{y}_d(t)$,

$$\dot{\boldsymbol{y}}_d(t) = \boldsymbol{\mathcal{Q}}\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right)\boldsymbol{R}\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right) + \dot{\boldsymbol{\mathcal{Q}}}\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right)\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right). \tag{2.195}$$

Here, $\dot{\boldsymbol{\mathcal{Q}}}$ denotes the short hand notation

$$\dot{\boldsymbol{\mathcal{Q}}}(\boldsymbol{x}(t)) = \frac{d}{dt}\boldsymbol{\mathcal{Q}}(\boldsymbol{x}(t)) = \left(\dot{\boldsymbol{x}}^T(t)\nabla\right)\boldsymbol{\mathcal{Q}}(\boldsymbol{x}(t)). \tag{2.196}$$

The explicit time dependence rendering Eq. (2.195) a non-autonomous differential equation comes from the term $\boldsymbol{z}_d(t)$. The initial condition for Eq. (2.195) is

$$\boldsymbol{y}_d(t_0) = \boldsymbol{\mathcal{Q}}(\boldsymbol{x}(t_0))\,\boldsymbol{x}(t_0), \tag{2.197}$$

while $\boldsymbol{z}_d(t_0)$ has to satisfy

$$\boldsymbol{z}_d(t_0) = \boldsymbol{\mathcal{P}}(\boldsymbol{x}(t_0))\,\boldsymbol{x}(t_0). \tag{2.198}$$

Because of $\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t)) = \boldsymbol{0}$, the corresponding control signal $\boldsymbol{u}(t)$ is given as

$$\begin{aligned}\boldsymbol{u}(t) &= \boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right)\\ &= \boldsymbol{\mathcal{B}}^+\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right)\left(\dot{\boldsymbol{z}}_d(t) - \boldsymbol{R}\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right)\right)\\ &\quad - \boldsymbol{\mathcal{B}}^+\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right)\dot{\boldsymbol{\mathcal{P}}}\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right)\left(\boldsymbol{y}_d(t) + \boldsymbol{z}_d(t)\right).\end{aligned} \tag{2.199}$$

Solving Eq. (2.195) for $\boldsymbol{y}_d(t)$ in terms of $\boldsymbol{z}_d(t)$, the term $\boldsymbol{y}_d(t)$ is eliminated from Eq. (2.199), resulting in a control signal expressed in terms of $\boldsymbol{z}_d(t)$ only. The dependence of the control signal $\boldsymbol{u}(t)$ on $\boldsymbol{z}_d(t)$ is in form of a functional,

$$\boldsymbol{u}(t) = \boldsymbol{u}\left[\boldsymbol{z}_d(t)\right]. \tag{2.200}$$

However, the choice $\boldsymbol{z}_d(t) = \boldsymbol{\mathcal{P}}(\boldsymbol{x}_d(t))\,\boldsymbol{x}_d(t)$ is not the only possible desired output. A general approach prescribes an arbitrary m-component output

$$\boldsymbol{z}_d(t) = \boldsymbol{h}(\boldsymbol{x}_d(t)). \tag{2.201}$$

The function $\boldsymbol{h}$ maps the state space $\mathbb{R}^n$ to a space $\mathbb{R}^m$. Using the constraint equation (2.194), one can attempt to eliminate $n - m$ components of $\boldsymbol{x}_d(t)$ in the control signal and obtain a control signal depending on $\boldsymbol{z}_d(t)$ only. If it is possible to do so, also the controlled state trajectory $\boldsymbol{x}(t)$ can be expressed in terms of the desired output $\boldsymbol{z}_d(t)$ only. The output $\boldsymbol{z}_d(t)$ is an *exactly realizable desired output*. Clearly, not all desired outputs can be realized, and the question arises under which conditions it is possible to exactly realize a desired output $\boldsymbol{z}_d(t)$. For example, if the dimension m of the output signals is larger than the dimension p of the control signals, $m > p$, it should be impossible to express the control signal in terms of the output. Here, we are not able to give a definite answer to this question. We discuss some general aspects of the problem in Sect. 2.6.2, and treat some explicit examples in Sect. 2.6.3.

A remark in order to minimize the confusion: $\boldsymbol{z}_d$ as given by $\boldsymbol{z}_d = \boldsymbol{\mathcal{P}}(\boldsymbol{x}_d)\boldsymbol{x}_d$ is an n-component vector, but has only p independent components. Starting with Eq. (2.201), the output $\boldsymbol{z}_d(t)$ is regarded as a p-component vector with p independent components, as it is customary for outputs.

2.6.2 Output Trajectory Realizability Leads to Differential-Algebraic Systems

Consider a desired output trajectory $\boldsymbol{z}_d(t)$ depending linearly on the desired state trajectory $\boldsymbol{x}_d(t)$,

$$\boldsymbol{z}_d(t) = \boldsymbol{\mathcal{C}}\boldsymbol{x}_d(t). \quad (2.202)$$

The desired output $\boldsymbol{z}_d(t)$ has $m \leq n$ independent components and $\boldsymbol{\mathcal{C}}$ is assumed to be a constant $m \times n$ *output matrix* with full rank,

$$\operatorname{rank}(\boldsymbol{\mathcal{C}}) = m. \quad (2.203)$$

Equation (2.202) is viewed as an underdetermined system of linear equations for the desired state $\boldsymbol{x}_d(t)$. See the Appendix A.2 for an introduction in solving underdetermined systems of equations.

For the linear output given by Eq. (2.202), we can define two complementary projectors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ by

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{C}}^+\boldsymbol{\mathcal{C}}, \quad \boldsymbol{\mathcal{N}} = \mathbf{1} - \boldsymbol{\mathcal{M}}. \quad (2.204)$$

Here, the Moore–Penrose pseudo inverse $\boldsymbol{\mathcal{C}}^+$ of $\boldsymbol{\mathcal{C}}$ is given by

$$\boldsymbol{\mathcal{C}}^+ = \boldsymbol{\mathcal{C}}^T\left(\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{C}}^T\right)^{-1}. \quad (2.205)$$

The projectors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ are symmetric $n \times n$ matrices. The inverse of the $m \times m$ matrix $\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{C}}^T$ exists because $\boldsymbol{\mathcal{C}}$ has full rank by assumption. The ranks of the projectors are

$$\operatorname{rank}(\boldsymbol{\mathcal{M}}) = m, \quad \operatorname{rank}(\boldsymbol{\mathcal{N}}) = n - m. \quad (2.206)$$

Multiplying $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ with $\boldsymbol{\mathcal{C}}$ from the left and right yields

$$\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{C}}^T = \boldsymbol{\mathcal{C}}^T, \quad \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{C}}, \quad \boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{C}}^T = \mathbf{0}, \quad \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{N}} = \mathbf{0}. \quad (2.207)$$

Multiplying the state-output relation (2.202) by $\boldsymbol{\mathcal{C}}^+$ from the left gives

$$\mathcal{M}\boldsymbol{x}_d(t) = \mathcal{C}^+ \boldsymbol{z}_d(t). \tag{2.208}$$

Using the last equation, the desired state $\boldsymbol{x}_d(t)$ can be separated in two parts as

$$\boldsymbol{x}_d(t) = \mathcal{M}\boldsymbol{x}_d(t) + \mathcal{N}\boldsymbol{x}_d(t) = \mathcal{C}^+ \boldsymbol{z}_d(t) + \mathcal{N}\boldsymbol{x}_d(t). \tag{2.209}$$

Thus, the part $\mathcal{M}\boldsymbol{x}_d(t)$ can be expressed in terms of the output $\boldsymbol{z}_d(t)$ while the part $\mathcal{N}\boldsymbol{x}_d(t)$ is left undetermined.

In the following, we enforce the first part of the linearizing assumption, namely, we assume constant projectors

$$\mathcal{P}(\boldsymbol{x}) = \mathcal{P} = \text{const.}, \quad \mathcal{Q}(\boldsymbol{x}) = \mathbf{1} - \mathcal{P} = \text{const.}, \tag{2.210}$$

in the constraint equation (2.194). The constraint equation becomes

$$\mathcal{Q}\dot{\boldsymbol{x}}_d(t) = \mathcal{Q}\boldsymbol{R}(\boldsymbol{x}_d(t)). \tag{2.211}$$

Using the projectors $\mathcal{M}$ and $\mathcal{N}$ introduced in Eq. (2.209), the constraint equation can be written as the *output constraint equation*

$$\mathcal{Q}\mathcal{N}\dot{\boldsymbol{x}}_d(t) = \mathcal{Q}\boldsymbol{R}\left(\mathcal{C}^+ \boldsymbol{z}_d(t) + \mathcal{N}\boldsymbol{x}_d(t)\right) - \mathcal{Q}\mathcal{C}^+ \dot{\boldsymbol{z}}_d(t). \tag{2.212}$$

This is a system of equations for the part $\mathcal{N}\boldsymbol{x}_d(t)$. However, note that the rank of the matrix product $\mathcal{Q}\mathcal{N}$ is

$$\begin{aligned} r &= \text{rank}(\mathcal{Q}\mathcal{N}) \leq \min(\text{rank}(\mathcal{Q}), \text{rank}(\mathcal{N})) \\ &= \min(n-p, n-m). \end{aligned} \tag{2.213}$$

In the most extreme case, $\mathcal{Q}\mathcal{N} = \mathbf{0}$ and so $r = 0$, and Eq. (2.212) reduces to a purely algebraic equation for $\mathcal{N}\boldsymbol{x}_d(t)$,

$$\mathbf{0} = \mathcal{Q}\boldsymbol{R}\left(\mathcal{C}^+ \boldsymbol{z}_d(t) + \mathcal{N}\boldsymbol{x}_d(t)\right) - \mathcal{Q}\mathcal{C}^+ \dot{\boldsymbol{z}}_d(t). \tag{2.214}$$

In general, Eq. (2.212) is a system differential-algebraic equations for the part $\mathcal{N}\boldsymbol{x}_d(t)$, and the order of the differential equation depends on the rank of $\mathcal{Q}\mathcal{N}$. For $m = p$, the system consists of r independent differential equations and $n - p - r$ algebraic equations. See the books Campbell (1980, 1982) and Kunkel and Mehrmann (2006) for more information about differential-algebraic equations.

Changing the order of differential equations implies consequences for its initial conditions. For example, evaluating Eq. (2.214) at the initial time $t = t_0$,

$$\mathbf{0} = \mathcal{Q}\boldsymbol{R}\left(\mathcal{C}^+ \boldsymbol{z}_d(t_0) + \mathcal{N}\boldsymbol{x}_d(t_0)\right) - \mathcal{Q}\mathcal{C}^+ \dot{\boldsymbol{z}}_d(t_0), \tag{2.215}$$

uncovers an additional relation between $\boldsymbol{x}_d(t_0)$ and $\boldsymbol{z}_d(t_0)$ which also involves the time derivative $\dot{\boldsymbol{z}}_d(t_0)$. If in an experiment the initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ of the system can be prepared, Eq. (2.215) yields the value for the part $\boldsymbol{\mathcal{N}}\boldsymbol{x}_d(t_0)$, while the part $\boldsymbol{\mathcal{M}}\boldsymbol{x}_d(t_0)$ is given by

$$\boldsymbol{\mathcal{M}}\boldsymbol{x}_d(t_0) = \boldsymbol{\mathcal{C}}^+\boldsymbol{z}_d(t_0). \tag{2.216}$$

On the other hand, if the initial state of the system cannot be prepared, Eq. (2.215) enforces an explicit relation between $\boldsymbol{\mathcal{N}}\boldsymbol{x}_d(t_0)$ and $\dot{\boldsymbol{z}}_d(t_0)$. In general, for $m = p$ and $r = \text{rank}(\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{N}}) < n - p$, $n - p - r$ additional conditions have to be satisfied by the initial time derivatives of the desired output trajectory $\boldsymbol{z}_d(t)$. We discuss output realizability with help of several examples.

2.6.3 Realizing a Desired Output: Examples

Example 2.15: Realizing a desired output for the activator-controlled FHN model

Consider the model

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x(t) + a_2 y(t) \\ R(x(t), y(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \tag{2.217}$$

with nonlinearity

$$R(x, y) = R(y) - x. \tag{2.218}$$

The function $R(y) = y - \frac{1}{3}y^3$ corresponds to the standard FHN nonlinearity. The constraint equation is linear

$$\dot{x}_d(t) = a_0 + a_1 x_d(t) + a_2 y_d(t). \tag{2.219}$$

In Example 2.5, the conventional choice of prescribing the activator variable $y_d(t)$ was applied. The constraint equation (2.219) was regarded as a differential equation for $x_d(t)$. Consequently, by eliminating $x_d(t)$ in the control $u(t) = \dot{y}_d(t) - R(x_d(t), y_d(t))$, $u(t)$ was expressed entirely in terms of $y_d(t)$. In contrast, here the output $z_d(t)$ is chosen as a linear combination of activator and inhibitor,

$$z_d(t) = h(x_d(t), y_d(t)) = c_1 x_d(t) + c_2 y_d(t). \tag{2.220}$$

Rearranging Eq. (2.220) gives

$$y_d(t) = \frac{1}{c_2}\left(z_d(t) - c_1 x_d(t)\right). \tag{2.221}$$

Using the last relation in the constraint equation (2.219) yields a linear ODE for x_d with inhomogeneity z_d,

$$\dot{x}_d(t) = \left(a_1 - \frac{c_1 a_2}{c_2}\right) x_d(t) + a_0 + \frac{a_2}{c_2} z_d(t). \tag{2.222}$$

Its solution is, with $\kappa = a_1 - \frac{a_2 c_1}{c_2}$,

$$\begin{aligned} x_d(t) = x_d(t_0)\, e^{\kappa(t-t_0)} + e^{\kappa t}\frac{a_2}{c_2}\int\limits_{t_0}^{t} \exp(-\kappa\tau)\, z_d(\tau)\, d\tau \\ + \frac{a_0}{\kappa}\left(e^{\kappa(t-t_0)} - 1\right), \end{aligned} \tag{2.223}$$

Using relation Eq. (2.221) for y_d together with the Eq. (2.223), x_d and y_d can be eliminated in terms of $z_d(t)$ from the control signal $u(t)$. The result is an expression in terms of the desired output z_d only (not shown).

One remark about the initial condition for the desired state $z_d(t)$. For any desired trajectory $\boldsymbol{x}_d(t)$ to be exactly realizable, its initial condition $\boldsymbol{x}_d(t_0)$ must agree with the initial condition $\boldsymbol{x}(t_0)$ of the controlled state $\boldsymbol{x}(t)$. This naturally restricts the initial value of the desired output to satisfy $z_d(t_0) = c_1 x_d(t_0) + c_2 y_d(t_0)$.

In conclusion, the control as well as the desired state trajectory $\boldsymbol{x}_d(t)$ is expressed solely in terms of the desired output $z_d(t)$. A numerical simulation of the controlled model shown in Fig. 2.5 demonstrates the successful realization of the desired output $z_d(t) = 4\sin(2t)$. Initially at time $t_0 = 0$, the state is set to $x_0 = y_0 = 0$, which complies with the initial value of the desired output $z_d(0) = 0$. A comparison of the desired output $z_d(t)$ with the output $z(t)$ obtained by numerical simulations of the controlled system demonstrates perfect agreement (see Fig. 2.5 top left), and a plot of $z(t) - z_d(t)$ reveals differences within numerical precision (see Fig. 2.5 top right). The controlled state trajectories $x(t)$ and $y(t)$ are shown in Fig. 2.5 bottom left, and the control signal is shown in Fig. 2.5 bottom right.

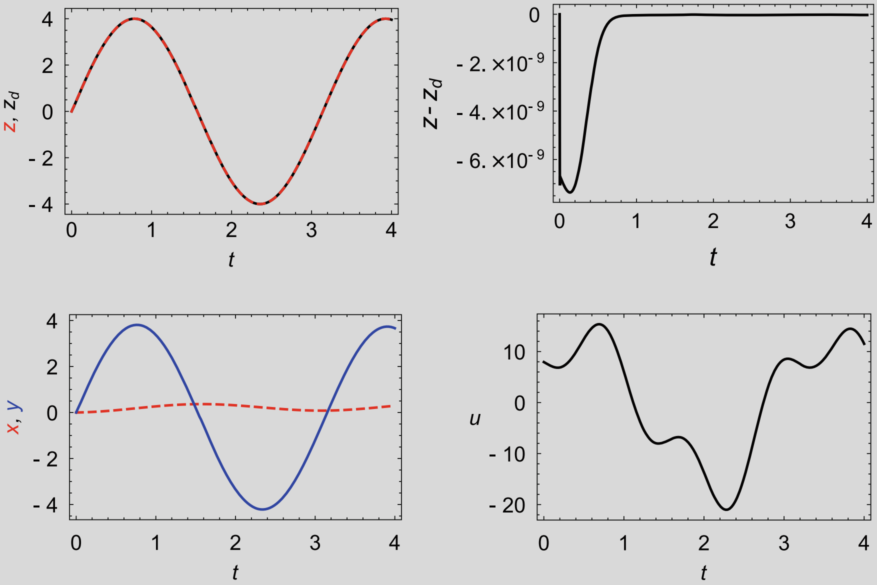

Fig. 2.5 Realizing a desired output in the activator-controlled FHN model. The numerical result z (*red dashed line*) for the output lies on top of the desired output trajectory z_d (*black line*), see *top left figure*. The difference $z - z_d$ is within the range of numerical precision (*top right*). The *bottom left figure* shows the corresponding state trajectories x (*red dashed line*) and y (*blue line*) and the control u (*bottom right*)

Example 2.16: Controlling the number of infected individuals in the SIR model

The controlled state equation for the SIR model was developed in Example 1.3, and its output controllability was discussed in Example 2.14. An uncontrolled time evolution is assumed for all times $t < t_0$, upon which the control is switched on. Starting at time $t = t_0$, the number of infected people over time is prescribed. The desired output is

$$z_d(t) = I_d(t). \tag{2.224}$$

The constraint equation consists of two independent equations

$$\begin{pmatrix} \frac{1}{2}\left(\gamma z_d(t) + \dot{z}_d(t) + \dot{S}_d(t)\right) \\ -\gamma z_d(t) + \dot{R}_d(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.225}$$

The constraint equation is considered as two differential equations for $S_d(t)$ and $R_d(t)$. Their solutions are readily obtained as

$$S_d(t) = -\gamma \int_{t_0}^{t} d\tau z_d(\tau) - z_d(t) + S_d(t_0) + z_d(t_0), \tag{2.226}$$

$$R_d(t) = R_d(t_0) + \gamma \int_{t_0}^{t} d\tau z_d(\tau). \tag{2.227}$$

Equations (2.226) and (2.227) express $S_d(t)$ and $R_d(t)$ solely in terms of the desired output $z_d(t)$ and the initial conditions. For any desired trajectory $\boldsymbol{x}_d(t)$ to be exactly realizable, its initial condition $\boldsymbol{x}_d(t_0)$ must comply with the initial condition $\boldsymbol{x}(t_0)$. For the initial conditions of R_d and S_d follows

$$R_d(t_0) = R(t_0), \quad S_d(t_0) = S(t_0), \tag{2.228}$$

while from $z_d(t) = I_d(t)$ follows

$$z_d(t_0) = I(t_0), \tag{2.229}$$

with $I(t_0)$ being the number of infected people at time $t = t_0$ when control measures are started.

The solution for the control signal realizing a desired trajectory $\boldsymbol{x}_d(t)$ is

$$\begin{aligned} u(t) &= \left(\boldsymbol{B}^T(\boldsymbol{x}_d(t))\boldsymbol{B}(\boldsymbol{x}_d(t))\right)^{-1}\boldsymbol{B}^T(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) \\ &= N\frac{\gamma I_d(t) + \dot{I}_d(t) - \dot{S}_d(t)}{2I_d(t)S_d(t)} - \beta, \end{aligned} \tag{2.230}$$

and using the solutions for $S_d(t)$ and $R_d(t)$ in terms of $z_d(t)$, the control signal becomes

$$u(t) = N\frac{\gamma z_d(t) + \dot{z}_d(t)}{z_d(t)\left(S(t_0) + I(t_0) - z_d(t) - \gamma\int_{t_0}^{t} d\tau z_d(\tau)\right)} - \beta. \tag{2.231}$$

The desired number of infected individuals $z_d(t)$ shall follow a parabolic time evolution,

$$z_d(t) = b_2 t^2 + b_1 t + b_0. \tag{2.232}$$

Three conditions are necessary to determine the three constants b_0, b_1, and b_2. The first condition follows from Eq. (2.229). Second, the number of infected individuals shall vanish at time $t = t_1$,

$$z_d(t_1) = 0, \tag{2.233}$$

such that $t_1 - t_0$ is the duration of the epidemic. To obtain a third relation, we demand that initially, the control signal vanishes. Evaluating Eq. (2.231) at $t = t_0$ yields

$$u(t_0) = N\frac{\gamma I(t_0) + \dot{z}_d(t_0)}{I(t_0) S(t_0)} - \beta = 0. \tag{2.234}$$

This relation can be used to obtain a relation for $\dot{z}_d(t_0)$ as

$$\dot{z}_d(t_0) = \frac{\beta}{N} I(t_0) S(t_0) - \gamma I(t_0). \tag{2.235}$$

Equation (2.234) guarantees a smooth transition of the time-dependent transmission rate $\beta(t) = \beta + u(t)$ across $t = t_0$.

Figure 2.6 shows a numerical solution. Up to time $t = t_0$, the system evolves uncontrolled, upon which all initial state values $S(t_0)$, $I(t_0)$, and $R(t_0)$ are measured. Starting at time $t_0 = 10$, the control signal $u(t)$, Eq. (2.231), acts on the system. To prevent an unphysical negative transmission rate $\beta(t) = \beta + u(t)$, the control $u(t)$ is clipped,

$$\hat{u}(t) = \begin{cases} u(t), & u(t) > -\beta, \\ -\beta, & u(t) \leq -\beta. \end{cases} \tag{2.236}$$

As can be seen in Fig. 2.6 bottom right, $\beta + u(t)$ reaches zero at an approximate time $\tilde{t}_1 \approx 56$, upon which the system evolves again uncontrolled. At this time, the epidemic has reached a reproductive number (see Example 1.3)

$$R_0 = \frac{\beta + u(\tilde{t}_1)}{\gamma} = 0 < 1, \tag{2.237}$$

and further spreading of the epidemic is prevented. Comparison of the controlled output $z(t) = I(t)$ with its desired counterpart $z_d(t) = I_d(t)$ shows perfect agreement for times $t_0 < t < \tilde{t}_1$ when control measures are operative, see bottom left of Fig. 2.6. Comparing the left and right top figures of Fig. 2.6 reveals a less dramatic epidemic in case of control (top right) than in case without control (top left), with a lower maximum number of infected individuals $I(t)$ (red) and a smaller final number of recovered individuals $R(t)$ (black).

Note that $R(t)$ is equivalent to the cumulative number of peoples affected by the epidemic.

While no exact analytical solution is known for the uncontrolled SIR model, we easily managed to find an exact analytical solution for the control as well as for the controlled state over time. This simple analytical approach provides statements as "If the number of infected individuals Δt days from now shall not exceed $I_{\Delta t}$, the transmission rate has to be lowered by $\Delta\beta$ within the next Δt_1 days" without much computational effort. It is a way to predict the effectiveness versus cost of control measures. Of course, application of this result to real world systems requires a model for the cost of quarantine measures or vaccination programs and their impact on the transmission rate $\beta(t)$.

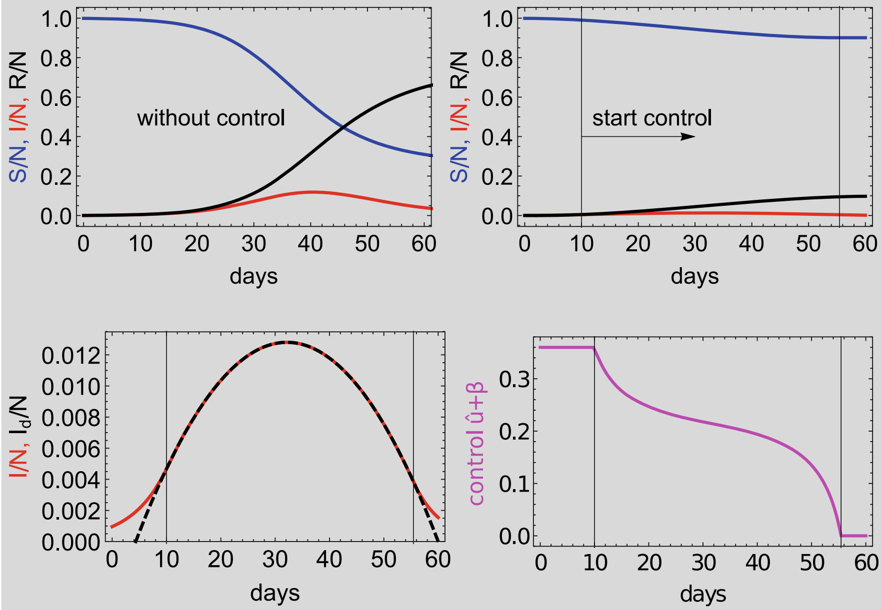

Fig. 2.6 Control of an epidemic in the SIR model. Without control (*top left*), much more individuals become infected (*red*) than with control measures starting at $t = 10$ (*top right*). A comparison of the desired output $z_d(t) = I_d(t)$ (*black dashed line*) of infected individuals with the actual output trajectory $z(t) = I(t)$ (*red solid line*) of the controlled dynamical system reveals perfect agreement for times $t_0 < t < \tilde{t}_1$ when control measures are operative (*bottom left*). The *bottom right figure* shows the control signal which is clipped such that the time-dependent transmission rate $\beta(t) = \beta + \hat{u}(t) > 0$ is always positive

Example 2.17: Activator as output for the inhibitor-controlled FHN model
Consider the model with coupling vector $\boldsymbol{B} = \left(1,\ 0\right)^T$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x(t) + a_2 y(t) \\ R(x(t), y(t)) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \tag{2.238}$$

and with nonlinearity

$$R(x, y) = R(y) - x. \tag{2.239}$$

The function $R(y) = y - \frac{1}{3}y^3$ corresponds to the standard FHN nonlinearity. Example 2.6 applied the conventional choice and prescribed the inhibitor variable $x_d(t)$ as the desired output, while $y_d(t)$ was determined as the solution to the corresponding constraint equation. In contrast, here the desired output is given by the activator $y_d(t)$

$$z_d(t) = y_d(t). \tag{2.240}$$

The control signal in terms of the desired trajectory $\boldsymbol{x}_d(t) = \left(x_d(t), y_d(t)\right)^T$ is

$$u(t) = \dot{x}_d(t) - a_0 - a_1 x_d(t) - a_2 y_d(t). \tag{2.241}$$

The constraint equation for $\boldsymbol{x}_d(t)$ becomes a nonlinear differential equation for $z_d(t) = y_d(t)$,

$$\dot{z}_d(t) = R(z_d(t)) - x_d(t). \tag{2.242}$$

To realize the desired output $y_d(t)$, any reference to the inhibitor $x_d(t)$ has to be eliminated from the control signal Eq. (2.241). To achieve that, the constraint equation (2.242) must be solved for $x_d(t)$ in terms of the desired output $z_d(t)$. This is a very simple task because Eq. (2.242) is a linear algebraic equation for $x_d(t)$. The solution is

$$x_d(t) = R(z_d(t)) - \dot{z}_d(t). \tag{2.243}$$

Using the last relation, $x_d(t)$ can be eliminated from the control signal Eq. (2.241) to get

$$\begin{aligned} u(t) &= \dot{x}_d(t) - a_0 - a_1 x_d(t) - a_2 z_d(t) \\ &= R'(z_d(t))\,\dot{z}_d(t) - \ddot{z}_d(t) - a_0 - a_1 z_d(t) - a_2\left(R(z_d(t)) - \dot{z}_d(t)\right). \end{aligned} \tag{2.244}$$

In conclusion, the control signal $u(t)$ as well as the desired state $\boldsymbol{x}_d(t)$ is expressed solely in terms of the desired output $z_d(t)$. Although the system does not satisfy the linearizing assumption because the constraint equation is a nonlinear differential equation, only a linear algebraic equation had to be solved. Thus, linear structures underlying nonlinear control systems may exist independently of the linearizing assumption. Interestingly, the approach of open loop control proposed here yields a similar result for the control as feedback linearization, see e.g. Khalil (2001). This hints at deep connections between our approach and feedback linearization. The framework of exactly realizable trajectories might open up a way to generalize feedback linearization to open loop control systems.

A remark about the initial conditions. For exactly realizable trajectories the initial state of the desired trajectory must be equal to the initial system state, $\boldsymbol{x}_d(t_0) = \boldsymbol{x}(t_0)$. Due to Eq. (2.243), the initial value for x_d is fully determined by the initial value of the desired output $z_d(t_0)$ and its time derivative $\dot{z}_d(t_0)$. For a fixed desired output trajectory $z_d(t)$, the system must be prepared in the initial state

$$x(t_0) = R(z_d(t_0)) - \dot{z}_d(t_0), \tag{2.245}$$

$$y(t_0) = z_d(t_0). \tag{2.246}$$

On the other hand, if the system cannot be prepared in a certain initial state, Eq. (2.243) imposes an additional condition on the desired output trajectory $z_d(t)$. In fact, not only is the initial value $z_d(t_0)$ prescribed by Eq. (2.246), but also the initial value of the time derivative $\dot{z}_d(t_0)$ is fixed by Eq. (2.245).

Figure 2.7 shows the result of a numerical simulation of the controlled FHN model with the prescribed activator

$$y_d(t) = 4\sin(2t) \tag{2.247}$$

as the desired output trajectory. At the initial time $t = t_0 = 0$, the system is prepared in a state such that Eqs. (2.245) and (2.246) are satisfied,

$$\left(x_0, y_0\right)^T = \left(-8, 0\right)^T. \tag{2.248}$$

Numerically solving the controlled system and comparing the controlled state trajectories $\boldsymbol{x}(t)$ with the corresponding desired reference trajectories reveals a perfect agreement within numerical precision, see the bottom row of Fig. 2.7.

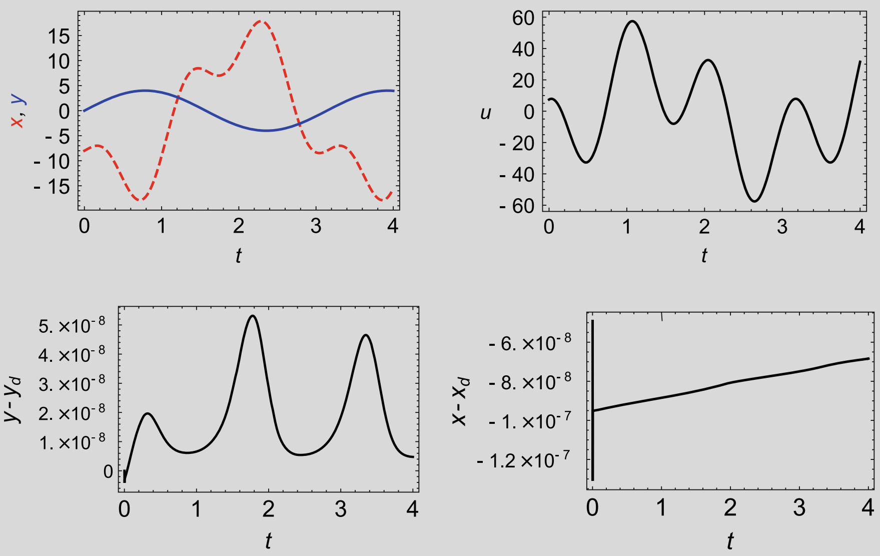

Fig. 2.7 Inhibitor-controlled FHN model with activator y_d as the desired output trajectory. *Top left* The desired activator (*blue solid line*) is prescribed as in Eq. (2.247), while the desired inhibitor x_d (*red dashed line*) behaves as given by Eq. (2.243). The control u is shown in the *top right panel*. Comparing the difference between desired activator (*bottom left*) and inhibitor (*bottom right*) with the corresponding controlled time evolution obtained from a numerical solution demonstrates agreement within numerical precision

Example 2.18: Modified Oregonator model
The modified Oregonator model is a model for the light sensitive Belousov–Zhabotinsky reaction (Krug et al. 1990; Field et al. 1972; Field and Noyes 1974). In experiments, the intensity of illuminated light is used to control the system. The Belousov–Zhabotinsky reaction has been used as an experimental play ground for ideas related to the control of complex systems, see e.g. Mikhailov and Showalter (2006) for examples. The system equations for the activator y and inhibitor x read as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} y(t) - x(t) \\ \frac{1}{\tilde{\epsilon}}\left(y(t)(1 - y(t)) + fx(t)\frac{q - y(t)}{q + y(t)} \right) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\tilde{\epsilon}}\frac{q - y(t)}{q + y(t)} \end{pmatrix} u(t). \qquad (2.249)$$

The control signal $u(t)$ is proportional to the applied light intensity. In experiments, the inhibitor is visible and can be recorded with a camera. The measured gray scale depends linearly on the inhibitor and is used as the output z,

$$z(t) = h(x(t)) = I_0 + cx(t). \tag{2.250}$$

For a desired trajectory to be exactly realizable, it has to satisfy the linear constraint equation,

$$\dot{x}_d(t) = y_d(t) - x_d(t). \tag{2.251}$$

Equation (2.251) is solved for $y_d(t)$ to obtain

$$y_d(t) = \dot{x}_d(t) + x_d(t) = \frac{1}{c}\dot{z}_d(t) + \frac{1}{c}(z_d(t) - I_0). \tag{2.252}$$

The inhibitor $x_d(t)$ was substituted with the desired output $z_d(t)$ given by Eq. (2.250). The control signal $u(t)$ can be expressed entirely in terms of the desired output $z_d(t)$ as

$$\begin{aligned} u(t) &= \frac{q + y_d(t)}{q - y_d(t)}\left(\tilde{\epsilon}\dot{y}_d(t) + y_d(t)(y_d(t) - 1)\right) - fx_d(t) \\ &= \frac{\tilde{\epsilon}}{c}\frac{cq + \dot{z}_d(t) + z_d(t) - I_0}{cq - \dot{z}_d(t) - z_d(t) + I_0}\left(\ddot{z}_d(t) + \dot{z}_d(t)\right) \\ &\quad + \frac{1}{c^2}\frac{cq + \dot{z}_d(t) + z_d(t) - I_0}{cq - \dot{z}_d(t) - z_d(t) + I_0}\left(\dot{z}_d(t) + z_d(t) - I_0\right)\left(\dot{z}_d(t) + z_d(t) - I_0 - c\right) \\ &\quad - \frac{f}{c}(z_d(t) - I_0). \end{aligned} \tag{2.253}$$

Since only the output $z(t)$ can be observed in experiments, the initial state $\boldsymbol{x}_0(t) = (x_0, y_0)^T$ of the system must be determined from $z(t)$. Solving Eq. (2.250) for $x(t)$ and using also Eq. (2.252) yields

$$x_0 = x(t_0) = \frac{1}{c}(z(t_0) - I_0), \tag{2.254}$$

$$y_0 = y(t_0) = \frac{1}{c}\dot{z}(t_0) + \frac{1}{c}(z(t_0) - I_0). \tag{2.255}$$

Thus, observation of the full initial state requires knowledge of the output $z(t_0)$ as well as its time derivative $\dot{z}(t_0)$. A generalization of this fact leads to the notion of observability, see e.g. Chen (1998) for the definition of observability in the context of linear systems. On the other hand, assuming it is impossible to prepare the system in a desired initial state, the desired output trajectory $z_d(t)$ has to satisfy specific initial conditions to comply with the initial system state $\left(x_0, y_0\right)^T$. In fact, these conditions are identical in form to Eqs. (2.254) and (2.255),

$$x_0 = \frac{1}{c}\left(z_d(t_0) - I_0\right), \tag{2.256}$$

$$y_0 = \frac{1}{c}\dot{z}_d(t_0) + \frac{1}{c}\left(z_d(t_0) - I_0\right). \tag{2.257}$$

In conclusion, for a successful realization of the desired output z_d, not only the initial value of z_d but also its time derivative must be prescribed. This result hints at a connection between output realizability and observability. A similar connection between observability and controllability is known as the principle of duality since the initial work of Kalman (1959), see also Chen (1998).

2.7 Conclusions

2.7.1 Summary

A common approach to control, especially in the context of LTI systems, is concerned with states as the objects to be controlled. Suppose a controlled system, often called a plant in this context, has a certain point $\boldsymbol{x}_1$ in state space, sometimes called the operating point, at which the system works efficiently. The control task is then to bring the system to the operating point $\boldsymbol{x}_1$, and keep it there. This naturally leads to a definition of controllability as the possibility to achieve a state-to-state transfer from an initial state $\boldsymbol{x}_0$ to the operating point $\boldsymbol{x}_1$ within finite time (Kalman 1959; Chen 1998).

In contrast to that, here an approach to control is developed which centers on the state trajectory $\boldsymbol{x}(t)$ as the object of interest. Of course, both approaches to control are closely related. A single operating point in state space at which the system is to be kept is nothing more than a degenerate state trajectory. Equivalently, any state trajectory can be approximated by a succession of working points.

We distinguish between the controlled state trajectory $\boldsymbol{x}(t)$ and the desired trajectory $\boldsymbol{x}_d(t)$. The former is the trajectory which the time-dependent state $\boldsymbol{x}$ traces out in state space under the action of a control signal. The latter is a fictitious reference trajectory for the state over time. It is prescribed in analytical or numerical form by the experimenter. Depending on the choice of the desired trajectory $\boldsymbol{x}_d(t)$, the controlled state $\boldsymbol{x}(t)$ may or may not follow $\boldsymbol{x}_d(t)$.

For affine control systems, the class of exactly realizable desired trajectories is defined in Sect. 2.2. For this subset of desired trajectories, a control signal exists which enforces the controlled state to follow the desired trajectory exactly,

$$\boldsymbol{x}(t) = \boldsymbol{x}_d(t), \tag{2.258}$$

for all times $t \geq t_0$. Exactly realizable desired trajectories satisfy the constraint equation

$$\mathbf{0} = \boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right), \tag{2.259}$$

with projector

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mathcal{B}}(\boldsymbol{x})\boldsymbol{\mathcal{B}}^{+}(\boldsymbol{x}) \tag{2.260}$$

with rank $n-p$. The vector of control signals $\boldsymbol{u}(t)$ is expressed in terms of the desired trajectory as

$$\boldsymbol{u}(t) = \boldsymbol{\mathcal{B}}^{+}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right). \tag{2.261}$$

The matrix $\boldsymbol{\mathcal{B}}^{+}(\boldsymbol{x})$ is the Moore–Penrose pseudo inverse of the coupling matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$. Equation (2.261) establishes a one-to-one relationship between the p-dimensional control signal $\boldsymbol{u}(t)$ and p out of n components of the desired trajectory $\boldsymbol{x}_d(t)$. The constraint equation (2.259) fixes those $n-p$ components of the desired trajectory $\boldsymbol{x}_d(t)$ without a one-to-one relationship to the control signal. The projectors $\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x})\boldsymbol{\mathcal{B}}^{+}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mathcal{P}}(\boldsymbol{x})$ allow a coordinate-free separation of the state $\boldsymbol{x}$ as well as the controlled state equation in two parts. The part of the state equation proportional to $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ determines the control signal. This approach allows the elimination of the control signal Eq. (2.261) from the system. The remaining part of the state equation, which is proportional to $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$, is the constraint equation (2.259). For the control of exactly realizable trajectories, only the constraint equation must be solved.

Note that the control signal Eq. (2.261) does not depend on the state of the system and is therefore an open loop control. As such, it may suffer from instability. An exactly realizable desired trajectory might or might not be stable against perturbations of the initial conditions or external perturbations as e.g. noise.

On the basis of the control signal Eq. (2.261) and constraint equation (2.259), a hierarchy of desired trajectories $\boldsymbol{x}_d(t)$ comprising 3 classes is established:

(A) desired trajectories $\boldsymbol{x}_d(t)$ which are solutions to the uncontrolled system,
(B) desired trajectories $\boldsymbol{x}_d(t)$ which are exactly realizable,
(C) arbitrary desired trajectories $\boldsymbol{x}_d(t)$.

Desired trajectories of class (A) satisfy the uncontrolled state equation

$$\dot{\boldsymbol{x}}_d(t) = \boldsymbol{R}(\boldsymbol{x}_d(t)). \tag{2.262}$$

This constitutes the most specific class of desired trajectories. Because of Eq. (2.262), the constraint equation (2.259) is trivially satisfied and the control signal as given by Eq. (2.261) vanishes,

$$\boldsymbol{u}(t) = \boldsymbol{0}. \tag{2.263}$$

Equation (2.263) implies a non-invasive control signal, i.e., the control signal vanishes upon achieving the control target. Because of Eq. (2.263), the open loop control approach proposed here cannot be employed for desired trajectories of class (A). Instead, these desired trajectories require feedback control. Class (A) encompasses several important control tasks, as e.g. the stabilization of unstable stationary states (Sontag 2011). A prominent example extensively studied by the physics community is the control of chaotic systems by small perturbations (Ott et al. 1990; Shinbrot et al. 1993). One of the fundamental aspects of chaos is that many different possible motions are simultaneously present in the system. In particular, an infinite number of unstable periodic orbits co-exist with the chaotic motion. All orbits are solutions to the uncontrolled system dynamics Eq. (2.262). Using non-invasive feedback control, a particular orbit may be stabilized. See also Schöll and Schuster (2007) and Schimansky-Geier et al. (2007) and references therein for more information and examples.

Desired trajectories of class (B) satisfy the constraint equation (2.259) and yield a non-vanishing control signal $\boldsymbol{u}(t) \neq \boldsymbol{0}$. The approach developed in this chapter applies to this class. Several other techniques developed in mathematical control theory, as e.g. feedback linearization and differential flatness, also work with this class of desired trajectories (Khalil 2001; Sira-Ramírez and Agrawal 2004). Class (B) contains the desired trajectories from class (A) as a special case. For desired trajectories of class (A) and class (B), the solution of the controlled state trajectory is simply given by $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$.

Finally, class (C) is the most general class of desired trajectories and contains class (A) and (B) as special cases. In general, these desired trajectories do not satisfy the constraint equation,

$$\boldsymbol{0} \neq \boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t)) \left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right), \tag{2.264}$$

such that, in general, the approach developed in this chapter cannot be applied to desired trajectories of class (C). No general expression for the control signal in terms of the desired trajectory $\boldsymbol{x}_d(t)$ is available. In general, the solution for the controlled state trajectory $\boldsymbol{x}(t)$ is not simply given by $\boldsymbol{x}_d(t)$, $\boldsymbol{x}(t) \neq \boldsymbol{x}_d(t)$. Thus, a solution to control problems defined by class (C) does not only consist in finding an expression for the control signal, but also involves finding a solution for the controlled state trajectory $\boldsymbol{x}(t)$ as well. One possible method to solve such control problems is optimal control.

The linearizing assumption of Sect. 2.3 defines a class of nonlinear control systems which essentially behave like linear control system. Models satisfying the linearizing assumption allow exact analytical solutions in closed form even if no analytical solutions for the uncontrolled system exists, see e.g. the SIR model in Example 2.16. The linearizing assumption uncovers a hidden linear structure underlying nonlinear control systems. Similarly, feedback linearization defines a huge class of nonlinear control systems possessing an underlying linear structure. The class of feedback linearizable systems contains the systems satisfying the linearizing assumption as a trivial case. However, the linearizing assumption defined here goes much further than feedback linearization. In fact, while general nonlinear control systems require a fairly abstract treatment for the definition of controllability (Slotine and Li 1991; Isidori 1995), we were able to apply the relatively simple notion of controllability in terms of a rank condition to systems satisfying the linearizing assumption, see Sect. 2.4. This is a direct extension of the properties of linear control systems to a class of nonlinear control systems. Furthermore, as will be shown in the next two chapters, the class defined by the linearizing assumption exhibits a linear structure even in case of optimal control for arbitrary, not necessarily exactly realizable desired trajectories. This enables the determination of exact, closed form expressions for optimal trajectory tracking in Chap. 4.

The approach to control proposed here shares many similarities to theories developed in mathematical control theory. We already mentioned inverse dynamics in the context of mechanical systems in Example 2.4. For more information about inverse dynamics, we refer the reader to the literature about robot control (Lewis et al. 1993; de Wit et al. 2012; Angeles 2013). In the following, we analyze the similarities and differences of our approach with differential flatness.

2.7.2 Differential Flatness

Similar to the concept of exactly realizable trajectories proposed in this chapter, differential flatness provides an open loop method for the control of dynamical systems. We first give a short introduction to differential flatness to be able to compare the similarities and differences to our approach. For more information about differential flatness as well as many examples, we refer the reader to Fliess et al. (1995), Van Nieuwstadt and Murray (1997), Sira-Ramírez and Agrawal (2004) and Levine (2009). The presentation follows (Sira-Ramírez and Agrawal 2004).

Differential flatness relies on the notion of differential functions. A function ϕ is a differential function of $\boldsymbol{x}(t)$ if it depends on $\boldsymbol{x}(t)$ and its time derivatives up to order β,

$$\phi(t) = \phi\left(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \ldots, \boldsymbol{x}^{(\beta)}(t)\right). \tag{2.265}$$

The symbol

$$\boldsymbol{x}^{(\beta)}(t) = \frac{d^\beta}{dt^\beta}\boldsymbol{x}(t) \tag{2.266}$$

denotes the time derivative of order β. An affine control system with n-component state vector $\boldsymbol{x}$ and p-component control signal $\boldsymbol{u}$ satisfies

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\,\boldsymbol{u}(t)\,. \tag{2.267}$$

Applying differentiation with respect to time to Eq. (2.267), the differential function $\tilde{\boldsymbol{\phi}}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ can be expressed as a function of $\boldsymbol{x}$ and $\boldsymbol{u}$

$$\boldsymbol{\phi}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \tilde{\boldsymbol{\phi}}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)) = \tilde{\boldsymbol{\phi}}(\boldsymbol{x}(t), \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\,\boldsymbol{u}(t))\,. \tag{2.268}$$

Similarly, the differential function $\tilde{\boldsymbol{\phi}}\left(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \dots, \boldsymbol{x}^{(\beta)}(t)\right)$ can be expressed as a differential function $\boldsymbol{\phi}\left(\boldsymbol{x}(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dots, \boldsymbol{u}^{(\beta-1)}(t)\right)$.

The system Eq. (2.267) is called differentially flat if there exists a p-component fictional output $\boldsymbol{z}(t) = \left(z_1(t), \dots, z_p(t)\right)^T$ such that (Sira-Ramírez and Agrawal 2004)

1. the output $\boldsymbol{z}(t)$ is representable as a differential function of the state $\boldsymbol{x}(t)$ and the vector of control signals $\boldsymbol{u}(t)$ as

$$\boldsymbol{z}(t) = \boldsymbol{\phi}\left(\boldsymbol{x}(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \ddot{\boldsymbol{u}}(t), \dots, \boldsymbol{u}^{(\beta-1)}(t)\right), \tag{2.269}$$

2. the state $\boldsymbol{x}(t)$ and the vector of control signals $\boldsymbol{u}(t)$ are representable as a differential function of the output $\boldsymbol{z}(t)$ as (with finite integer α)

$$\boldsymbol{x}(t) = \boldsymbol{\chi}\left(\boldsymbol{z}(t), \dot{\boldsymbol{z}}(t), \ddot{\boldsymbol{z}}(t), \dots, \boldsymbol{z}^{(\alpha)}(t)\right), \tag{2.270}$$

$$\boldsymbol{u}(t) = \boldsymbol{\psi}\left(\boldsymbol{z}(t), \dot{\boldsymbol{z}}(t), \ddot{\boldsymbol{z}}(t), \dots, \boldsymbol{z}^{(\alpha+1)}(t)\right), \tag{2.271}$$

3. the components of the output $\boldsymbol{z}(t)$ are differentially independent, i.e., they satisfy no differential equation of the form

$$\boldsymbol{\Omega}\left(\boldsymbol{z}(t), \dot{\boldsymbol{z}}(t), \ddot{\boldsymbol{z}}(t), \dots, \boldsymbol{z}^{(\beta)}(t)\right) = \boldsymbol{0}. \tag{2.272}$$

For a differentially flat system, the full solution for the state over time $\boldsymbol{x}(t)$ as well as the control signal $\boldsymbol{u}(t)$ can be expressed in terms of the output over time $\boldsymbol{z}(t)$. Mathematically, the relation between control signal $\boldsymbol{u}(t)$ and output $\boldsymbol{z}(t)$ is a differential function. This has the great advantage that the determination of the control signal can be done in real time at time t by computing only a finite number of time derivatives of $\boldsymbol{z}(t)$. This would not be possible if $\boldsymbol{u}(t)$ also involves time integrals of the output $\boldsymbol{z}$ because these would require summation over all previous times as well.

Another advantage is that no differential equations need to be solved to obtain the control signal and state trajectory. Usually, all expressions are generated by simply differentiating the controlled state equations with respect to time. The output $\boldsymbol{z}(t)$ has the same number p of components as the number of independent input signals $\boldsymbol{u}(t)$ available to control the system. If the control signal determined by $\boldsymbol{\psi}$, Eq. (2.271), is applied to the system, then the system's output is $\boldsymbol{z}(t)$. Differentially flat systems are not necessarily affine in control but can be nonlinear in the control as well. However, only certain systems are differentially flat, and it is not known under which conditions a controlled dynamical system is differentially flat if the number of independent control signals is larger than one, $p > 1$.

Similar to differential flatness, this chapter proposes an open loop control method. A solution for control signals exactly realizing a desired output $\boldsymbol{z}_d(t)$ with p components is determined. In the discussion of output trajectory realizability in Sect. 2.6, the control signal is expressed solely in terms of the desired output $\boldsymbol{z}_d(t)$ and the initial conditions for the state. This implies that the controlled state trajectory, given as the solution to the controlled state equation, can also be expressed in terms of the desired output and the initial conditions for the state. These facts fully agree with the concept of differential flatness. In contrast to the approach here, the literature about differential flatness does usually not distinguish explicitly between desired trajectory $\boldsymbol{x}_d(t)$ and controlled state trajectory $\boldsymbol{x}(t)$, but implicitly assumes this identity from the very beginning.

The most striking difference between the approach here and differential flatness is the restriction to differential functions. In general, our approach yields a control signal in terms of a functional of the desired output,

$$\boldsymbol{u}(t) = \boldsymbol{u}[\boldsymbol{z}_d(t)]. \tag{2.273}$$

Note that a functional is a more general expression than a differential function. Using the Dirac delta function $\delta(t)$, any time derivative of order β can be expressed as a functional,

$$\begin{aligned} \boldsymbol{z}_d^{(\beta)}(t) &= \int_{-\infty}^{\infty} d\tau \delta(\tau - t)\, \boldsymbol{z}_d^{(\beta)}(\tau) = -\int_{-\infty}^{\infty} d\tau \delta'(\tau - t)\, \boldsymbol{z}_d^{(\beta-1)}(\tau) \\ &\vdots \\ &= (-1)^{\beta} \int_{-\infty}^{\infty} d\tau \delta^{(\beta)}(\tau - t)\, \boldsymbol{z}_d(\tau). \end{aligned} \tag{2.274}$$

Therefore, any differential function of $\boldsymbol{z}(t)$ can be expressed in terms of a function of functionals of $\boldsymbol{z}(t)$, while the reverse is not true. The restriction to differential functions might also explain why only certain systems are differentially flat. In contrast, the approach proposed here can be applied to any affine control system. As an advantage, differential flatness yields expressions for state and control which are computationally more efficient because they do not require the solution of differential equations or integrals, which is not the case here.

2.7.3 Outlook

The framework of exactly realizable trajectories is interpreted as an open loop control method. However, it may be possible to extend this approach to feedback control. As discussed in Sect. 2.6, a control $\boldsymbol{u}(t)$ realizing a p-component desired output $z_d(t) = \boldsymbol{h}(\boldsymbol{x}_d(t))$ is expressed entirely in terms of the desired output. The dependence of $\boldsymbol{u}(t)$ on $z_d(t)$ is typically in form of a functional,

$$\boldsymbol{u}(t) = \boldsymbol{u}[z_d(t)]. \tag{2.275}$$

A generalization to feedback control yields a control signal which does not only depend on the desired output $z_d(t)$ but also on the monitored state $\boldsymbol{x}(t)$ of the controlled system,

$$\boldsymbol{u}(t) = \boldsymbol{u}[z_d(t), \boldsymbol{x}(t)]. \tag{2.276}$$

In general, the control signal is allowed to depend on the history of $\boldsymbol{x}(t)$ such that the dependence of $\boldsymbol{u}(t)$ on $\boldsymbol{x}(t)$ is also in form of a functional. Such a generalization of the approach to control proposed here certainly changes the stability properties of the controlled system and may result in an improved stability of the controlled trajectory.

A fundamental problem affecting not only exactly realizable trajectories but also feedback linearization, differential flatness, and optimal control, is the requirement of exactly knowing the system dynamics. This must be contrasted with the fact that the majority of physical models are idealizations. Unknown external influences in control systems can be modeled as noise or structural perturbations, which might both depend on the system state itself. To ensure a successful control in experiments, the proposed control methods must not only be stable against perturbations of the initial conditions, but must be sufficiently stable against structural perturbations as well. Stability against structural perturbations is also known as robustness in the context of control theory (Freeman and Kokotovic 1996). Before applying the control method developed in this chapter to real world problems, a thorough investigation of the stability of the control problem at hand must be conducted. In case of instability, countermeasures as e.g. additional stabilizing feedback control must be applied (Khalil 2001).

Section 2.3 introduces the linearizing assumption. On the one hand, this assumption is restrictive, but on the other hand it has far reaching consequences and results in significant simplifications for nonlinear affine control systems. A possible generalization of the linearizing assumption might be as follows. First, relax condition

Eq. (2.80) and allow a state dependent projector $\mathcal{Q}(\boldsymbol{x})$ which, however, does only depend on the state components $\mathcal{P}\boldsymbol{x}$,

$$\mathcal{Q}(\boldsymbol{x}) = \mathcal{Q}(\mathcal{P}\boldsymbol{x} + \mathcal{Q}\boldsymbol{x}) = \mathcal{Q}(\mathcal{P}\boldsymbol{x}). \tag{2.277}$$

Second, also relax condition Eq. (2.82) and assume a nonlinearity $\boldsymbol{R}(\boldsymbol{x})$ with the following structure,

$$\mathcal{Q}(\boldsymbol{x})\boldsymbol{R}(\boldsymbol{x}) = \mathcal{Q}(\mathcal{P}\boldsymbol{x})\mathcal{A}(\mathcal{P}\boldsymbol{x})\mathcal{Q}(\mathcal{P}\boldsymbol{x})\boldsymbol{x} + \mathcal{Q}(\mathcal{P}\boldsymbol{x})\boldsymbol{b}(\mathcal{P}\boldsymbol{x}). \tag{2.278}$$

The matrix $\mathcal{A}(\mathcal{P}\boldsymbol{x})$, the projector $\mathcal{Q}(\mathcal{P}\boldsymbol{x})$ and the inhomogeneity $\boldsymbol{b}(\mathcal{P}\boldsymbol{x})$ may all depend on the state components $\mathcal{P}\boldsymbol{x}$.

Together with

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{Q}(\mathcal{P}\boldsymbol{x}_d)\boldsymbol{x}_d\right) = {} & \dot{\mathcal{Q}}(\mathcal{P}\boldsymbol{x}_d)\mathcal{P}(\mathcal{P}\boldsymbol{x}_d)\boldsymbol{x}_d \\ & + \dot{\mathcal{Q}}(\mathcal{P}\boldsymbol{x}_d)\mathcal{Q}(\mathcal{P}\boldsymbol{x}_d)\boldsymbol{x}_d + \mathcal{Q}(\mathcal{P}\boldsymbol{x}_d)\dot{\boldsymbol{x}}_d,\end{aligned} \tag{2.279}$$

the constraint equation becomes

$$\frac{d}{dt}\left(\mathcal{Q}\boldsymbol{x}_d(t)\right) = \left(\dot{\mathcal{Q}} + \mathcal{Q}\mathcal{A}\right)\mathcal{Q}\boldsymbol{x}_d(t) + \dot{\mathcal{Q}}\mathcal{P}\boldsymbol{x}_d(t) + \mathcal{Q}\boldsymbol{b}. \tag{2.280}$$

The arguments are suppressed and it is understood that $\dot{\mathcal{Q}}$, $\mathcal{Q}$, $\mathcal{P}$, $\mathcal{A}$, and $\boldsymbol{b}$ may depend on the part $\mathcal{P}\boldsymbol{x}_d(t)$. Equation (2.280) is a linear equation for $\mathcal{Q}\boldsymbol{x}_d(t)$ and can thus be solved with the help of its state transition matrix, see Appendix A.1. However, the matrix $\mathcal{A} = \mathcal{A}(\mathcal{P}\boldsymbol{x}_d(t))$ exhibits an explicit time dependence through its dependence on $\mathcal{P}\boldsymbol{x}_d(t)$. This necessitates modifications for the notion of controllability from Sect. 2.4, see also Chen (1998).

A central assumption of the formalism presented in this chapter is that the $n \times p$ coupling matrix $\mathcal{B}(\boldsymbol{x})$ has full rank p for all values of $\boldsymbol{x}$. This assumption leads to a Moore–Penrose pseudo inverse $\mathcal{B}^+(\boldsymbol{x})$ of $\mathcal{B}(\boldsymbol{x})$ given by

$$\mathcal{B}^+(\boldsymbol{x}) = \left(\mathcal{B}^T(\boldsymbol{x})\mathcal{B}(\boldsymbol{x})\right)^{-1}\mathcal{B}^T(\boldsymbol{x}). \tag{2.281}$$

If $\mathcal{B}(\boldsymbol{x})$ does not have full rank for some or all values of $\boldsymbol{x}$, the inverse of $\mathcal{B}^T(\boldsymbol{x})\mathcal{B}(\boldsymbol{x})$ does not exist. However, a unique Moore–Penrose pseudo inverse $\mathcal{B}^+(\boldsymbol{x})$ does exist for any matrix $\mathcal{B}(\boldsymbol{x})$, regardless of its rank. No closed form expressions exist for the general case, but $\mathcal{B}^+(\boldsymbol{x})$ can nevertheless be computed numerically by singular value decomposition, for example. Because $\mathcal{B}^+(\boldsymbol{x})$ exists in any case, the $n \times n$ projector defined by

$$\mathcal{P}(\boldsymbol{x}) = \mathcal{B}(\boldsymbol{x})\mathcal{B}^+(\boldsymbol{x}) \tag{2.282}$$

exists as well. Thus, using the general Moore–Penrose pseudo inverse $\mathcal{B}^{+}(\boldsymbol{x})$, the formalism developed in this chapter can be extended to cases with $\mathcal{B}(\boldsymbol{x})$ not having full rank for some values of $\boldsymbol{x}$.

A mathematically more rigorous treatment of the notion of exactly realizable trajectories is desirable. An important question is the following. Under which conditions does the constraint equation

$$\mathbf{0} = \mathcal{Q}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) \tag{2.283}$$

have a unique solution for $\mathcal{Q}(\boldsymbol{x}_d(t))\,\boldsymbol{x}_d(t)$? Note that Eq. (2.283) is a non-autonomous nonlinear system of differential equations for $\mathcal{Q}(\boldsymbol{x}_d(t))\,\boldsymbol{x}_d(t)$ with the explicit time dependence caused by the part $\mathcal{P}(\boldsymbol{x}_d(t))\,\boldsymbol{x}_d(t)$. Therefore, a related question is for conditions on the part $\mathcal{P}(\boldsymbol{x}_d(t))\,\boldsymbol{x}_d(t)$ prescribed by the experimenter. For example, is $\mathcal{P}(\boldsymbol{x}_d(t))\,\boldsymbol{x}_d(t)$ required to be a continuously differentiable function or is it allowed to have jumps? Although some general answers might be possible, such questions are simpler to answer for specific control systems.

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