Chapter 2
Related Work

Let us now formally specify a concurrent control system by an Interpreted Petri net, which is a powerful mathematical tool and naturally reflects concurrency in prototyped systems. Note, that the prototyping, analysis and decomposition methods presented in the book apply to the integrated concurrent control systems (cf. Chap. 1). Therefore, unless otherwise stated, we use the simple notations concurrent control system, concurrent controller in reference to this particular group.

2.1 Petri Nets and Interpreted Petri Nets

This section introduces basic definitions and notations regarding the Petri nets \[5, 7, 8, 10, 13, 15, 17, 19, 23–25, 27–31, 33–35, 39\].

**Definition 2.1** A Petri net is a 4-tuple:

\[ PN = (P, T, F, M_0), \]  

(2.1)

where

- \( P \) is a finite set of places,
- \( T \) is a finite set of transitions,
- \( F \subseteq (P \times T) \cup (T \times P) \) is a finite set of arcs,
- \( M_0 \) is an initial marking.

Set \( P \cup T \) is a set of nodes of a Petri net.
**Definition 2.2** (Marking) State of a Petri net is called a *marking*, which can be seen as a distribution of tokens in the net places. If a place contains one or more tokens, it is called a *marked place*. A marking can be changed by means of firing (execution) of a transition.

**Definition 2.3** (Transition firing) A transition \( t \) can be fired if every of its input places contains a token. Transition firing removes a token from every input place of \( t \) and adds a token to every output place of \( t \).

An exemplary Petri net \( PN_1 \) (taken from [14, 37]) is shown in Fig. 2.1. The net contains six places \( P = \{ p_1, \ldots, p_6 \} \) and three transitions \( T = \{ t_1, t_2, t_3 \} \).

**Definition 2.4** (Input and output places) Place \( p \) is an input place of transition \( t \), if \( (p, t) \in F \). Place \( p' \) is an output place of transition \( t \), if \( (t, p') \in F \). The set of input places of transition \( t \) is denoted by \( \bullet t \), while \( t \bullet \) denotes the set of output places of \( t \).

**Definition 2.5** (Input and output transitions) Transition \( t \) is an input transition of place \( p \), if \( (t, p) \in F \). Transition \( t' \) is an output transition of place \( p \), if \( (p, t') \in F \). The set of input transitions of place \( p \) is denoted by \( \bullet p \), while \( p \bullet \) denotes the set of output transitions of \( p \).

**Definition 2.6** (Reachable marking) A marking \( M' \) is reachable from marking \( M \), if \( M' \) can be obtained from \( M \) by a finite sequence of transition firings. When marking \( M \) is not specified explicitly, a reachable marking of a net is understood as a marking reachable from its initial marking \( M_0 \).

A state of a Petri net is obtained by a distribution markers (tokens) on the places. Figure 2.2 illustrates all the possible markings of \( PN_1 \). There are three reachable markings in the presented net, that can be reached by consecutive firing of transitions \( t_1, t_2 \) and \( t_3 \). Changes between particular states for this net can be simply denoted as: \( M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} M_0 \).

![Fig. 2.1](image_url) An exemplary Petri net \( PN_1 \).
Fig. 2.2 All reachable markings of $PN_1$

The state $M_0$ consists of three places $p_1$, $p_2$ and $p_6$, which means that those places are initially marked. After firing of the transition $t_1$, tokens from its input places ($p_1$ and $p_6$) are moved to the output places ($p_3$ and $p_4$) of $t_1$ and the net reaches marking $M_1$. Please note that token at place $p_2$ remains unchanged. State $M_1$ enables transition $t_2$, which execution leads to the third marking $M_2$. At this state three places are marked: $p_1$, $p_3$ and $p_5$. Finally, firing of $t_3$ returns tokens into the initial marking $M_0$.

Definition 2.7 (Enabled transitions in a given marking) For a Petri net, transition $t$ is enabled in marking $M$ if $\forall p \in \bullet t : p \in M$, that is, all its input places are simultaneously marked in $M$. Any transition enabled in $M$ can fire, changing the marking $M$ to $M'$. $M[t>]$ denotes that $t$ is enabled in $M$, while $M[>]$ indicates the set of all enabled transitions in $M$.

For $PN_1$, transition $t_1$ is enabled in $M_0$, which is denoted by $M_0[t_1>]$. Furthermore, firing of $t_1$ changes an initial marking $M_0$ to $M_1$. Such a situation is represented as $M_0[t_1>]M_1$. Similarly, $M_1[t_2>]M_2$. Finally, $M_2[t_3>]M_0$ moves tokens to the initial state.

Definition 2.8 (Liveness) A transition $t$ is live if for every reachable marking $M$, $t$ can be fired in $M$ or in a marking reachable from $M$. A Petri net is live if all transitions in the net are live.

Definition 2.9 (Safeness) A place of a net is safe if there is no reachable marking such that the place contains more than one token. A Petri net is safe if each place in the net is safe.

The characterization of liveness and safeness has been studied by the researchers all over the world. Fundamental theory regarding liveness and safeness of Petri nets was introduced in [6, 16]. Such analysis was extended in [15]. Finally, advanced algorithms allowing checking liveness and safeness of particular subclasses of Petri nets were proposed in [2–4, 21, 40].

Let us point out that analysis of liveness and safeness is out of the scope of this monograph. In our opinion, existing theorems and algorithms (especially those
presented in [2–4, 15, 16, 21, 40]) exhausted this subject enough. Therefore, we decided to focus on the concurrency and sequentiality aspects of analysis of Petri nets.

**Definition 2.10** (Reversibility) A Petri net is reversible if for each marking $M$, the initial marking $M_0$ is reachable from $M$. In other words, Petri net is reversible if it can always reach its initial state.

**Definition 2.11** (Well-formed net) A Petri net is well-formed if it is live, safe and reversible.

**Definition 2.12** (Conservativeness) A Petri net is conservative if all the reachable markings contain the same number of tokens.

**Definition 2.13** (Pure net) A Petri net is pure if it has no self-loops.

**Definition 2.14** (Path, strong connectedness) A path in a Petri net $PN$ is a sequence of nodes, connected by arcs. A net is strongly connected if for any pair $(n_i, n_j)$ of its nodes there is a path leading from $n_i$ to $n_j$.

**Theorem 2.1** [3] Let $PN$ be a live and safe Petri net. Then $PN$ is strongly connected.

Petri net $PN_1$ is live, safe and conservative, therefore it is well-formed. Moreover, the net is pure and conservative, since all the markings contain exactly three tokens. $PN_1$ is strongly connected, because for any pair of nodes there is a path that connects them.

**Definition 2.15** (Interpreted Petri net) An interpreted Petri net is a well-formed Petri net, defined as a 6-tuple:

$$PN = (P, T, F, M_0, X, Y), \quad (2.2)$$

where:

- $P$ is a finite set of places,
- $T$ is a finite set of transitions,
- $F \subseteq (P \times T) \cup (T \times P)$, is a finite set of arcs,
- $M_0$ is an initial marking,
- $X = \{x_1, x_2, \ldots, x_m\}$ is a binary vector of logic inputs,
- $Y = \{y_1, y_2, \ldots, y_n\}$ is a binary vector of logic outputs.

Interpreted Petri nets are very often used to specify the real-life controllers. The system communicates with the environment via input and output signals. The inputs are associated with transitions while its outputs are bounded to places. The transition is enabled if all the transition inputs are active (or condition tied to the transition is fulfilled). Therefore, input signals may preserve the net conflict-free. We shall show such a situation later in this section. Notice, that from the definition an interpreted
Petri net is well-formed, thus it is live, safe and reversible. It implies important additional properties. For example, based on the Theorem 2.1, each interpreted net is strongly connected.

Let us illustrate interpreted Petri nets by a real-life example. Figure 2.3 shows a modified system of the milling process, initially proposed in [38]. The main aim of the presented machine is to cut the square shapes from the wooden plank. The process is driven by a logic controller, specified by an interpreted Petri net $PN_2$, shown in Fig. 2.4.

There are 14 input signals denoted by $x_1, \ldots, x_{14}$, and 14 output signals marked as $y_1, \ldots, y_{14}$. Placement of a wooden plank on the tray (indicated by the sensor $x_{14}$) starts the whole process. First, the plank is moved (output signal $y_1$), until reached the right position (signalized by $x_1$). Simultaneously, a drill ($y_2$) is being set into the starting position ($x_2$). Next, the machine starts cutting the required shape from the wood (in the presented example—a square). The move of the drill is denoted by $y_4$ (immersion into the wood), $y_5$ (the drill moves to the right), $y_6$ (the drill moves to the down), $y_7$ (the drill moves to the left), $y_8$ (the drill moves to the top), $y_9$ (the drill goes up, to the initial position). Reaching the remaining positions is signalized by sensors $x_3, x_4, x_5, x_6, x_7$ and $x_8$, respectively. At the same time, while the shape is drilled, three concurrent actions are performed: a vacuum cleaner is turned on ($y_3$), and two assembly holes are drilled (signals $y_{10}, y_{11}$ and $y_{12}, y_{13}$. Finally, the drilled shape is moved to the platform ($y_{14}$) to the tray. When the plank is taken away ($x_{14}$), the system is ready for the further actions.

The net $PN_2$ consists of $|P| = 21$ places and $|T| = 17$ transitions. It is live, safe, reversible, and strongly connected. However, it is not conservative, since firing of $t_1$ moves and splits a single token from $p_1$ into $p_2$ and $p_4$.

Figure 2.5 shows another real-life example of a concurrent control system. The picture illustrates an advanced traffic lights system, driven by the FPGA device. There are three independent traffic lights that control particular lanes for cars (turning left, going straight, turning right). Additionally, two traffic lights for pedestrians are considered. In our consideration a simplified version of the controller (initially shown in [36]) will be presented. Assume, that all the car lanes operate in the same manner.
**Fig. 2.4** An interpreted Petri net $PN_2$ that controls the milling machine.
and they are treated as a single traffic light. Similarly, both traffic lights for pedestrians are coupled. Such a system can be easily described by an interpreted Petri net ($PN_3$), as it is presented in Fig. 2.6. Initially, the red lights for cars (place $p_4$, output $R_C$) and for pedestrians (place $p_6$, output $R_P$) are active. If there is no request from pedestrians to cross the street (latched input signal $req$ is inactive), the system enters the state, when the green light for cars (place $p_2$, output $G_C$) is shown (note, that the red light for pedestrians prevents collision). Such a situation takes place until a pedestrian wishing to cross the street pushes the button (signal $req$), which results in firing the transition $t_2$. Next, the yellow light for cars (place $p_3$, output $Y_C$) is flashed and the system goes back to the initial state. Since signal $req$ is active, the transition $t_4$ is enabled and executed. The green light for pedestrians is shown to cross the street (it is assumed, that signal $req$ is zeroed). The system returns to the initial state and the whole procedure is repeated.

![Fig. 2.5](image1.png) Traffic lights system

![Fig. 2.6](image2.png) An interpreted net $PN_3$ that controls the simplified version of traffic lights
Petri net $PN_3$ consists of $|P| = 6$ places and $|T| = 5$ transitions. It is live, safe, and reversible. The net is not conservative, because firing of $t_1$ consumes two tokens (from places $p_1$ and $p_4$), while only one token is moved to $p_2$.

Note that place $p_1$ has two output transitions: $p_1 \bullet = \{t_1, t_4\}$. Such a situation is called conflict in a Petri net. In case of interpreted Petri nets, it can be easily resolved by the input signals. In the particular example, signal $req$ indicates, which transition is enabled. If there are no conflicts in the net, it is called conflict-free Petri net. For example, $PN_1$ is conflict-free, because each place has exactly one output transition.

It is assumed that interpreted Petri nets shown in this book are free of conflicts. It means that either the net is conflict-free, or such a conflict is resolved by the input signals. Furthermore, any Petri net that is live, safe and conflict-free can be classified as an interpreted Petri net. Thus, $PN_1 = (P, T, F, M_0, \emptyset, \emptyset)$ is an interpreted Petri net, but the sets of its input and output signals are empty: $X = Y = \emptyset$. Of course one may say that a controller specified by such a net does not have any sense, since there is no communication with the environment. Obviously it is true. However, such nets can be successfully applied for theoretical purposes or at those prototyping stages (mainly analysis, cf. Chap. 5), where input and output signals are not considered.

**Definition 2.16** (Concurrency in interpreted Petri nets) Two places are concurrent if they are marked simultaneously at some reachable marking.

**Definition 2.17** (Sequentiality in interpreted Petri nets) Two places are sequential if they cannot be marked simultaneously, i.e., there is no marking that contains both places.

It is assumed, that (in case of interpreted Petri nets) concurrency and sequentiality are complementary, i.e., particular place is either concurrent or sequential to the other place.

**Definition 2.18** (Reachability set, concurrency set) Reachability set or concurrency set of a concurrent control system is the set of its reachable states (markings).

Reachability (concurrency) set, beside obvious concurrency analysis, can be also used to verify the correctness of the system behavior [11]. Popular representation of such a set is reachability graph.

**Definition 2.19** (Reachability graph) Reachability graph of a concurrent control system is the directed graph of its reachable states, and the direct transitions between them.

Figure 2.2 (shown at the beginning of current chapter) presents an exemplary reachability graph for Petri net $PN_1$ (from Fig. 2.1).

Petri nets are classified according to their structure. In our consideration we use the following classification of a Petri net (taken directly from [25]):

1. State Machine (SM),
2. Marked Graph (MG),
3. Free-Choice net (FC-net),
4. Extended Free-Choice net (EFC-net),
5. Simple Net (SN), also known as Asymmetric Choice net (AC-net).

Let us define each of the presented subclasses.

**Definition 2.20** (State machine, SM-net) A state machine is a Petri net for which every transition has exactly one input place and exactly one output place, i.e., \( \forall t \in T : |\bullet t| = |t \bullet| = 1 \).

**Definition 2.21** (Marked graph, MG-net) A marked graph is a Petri net for which every place has exactly one input transition and exactly one output transition, i.e., \( \forall p \in P : |\bullet p| = |p \bullet| = 1 \).

**Definition 2.22** (Free-choice, FC-net) A free-choice net is a Petri net for which every outgoing arc from a place is unique or is a unique incoming arc to a transition, i.e., \( \forall p \in P : |p \bullet| \leq 1 \) or \( \bullet(p \bullet) = \{p\} \).

**Definition 2.23** (Extended free-choice net, EFC-net) An Extended Free-Choice net is a Petri net for which every two places having a common output transition, have all their output transitions in common, i.e., \( \forall p_i, p_j \in P : p_i \bullet \cap p_j \bullet \neq \emptyset \Rightarrow p_i \bullet = p_j \bullet \).

**Definition 2.24** (Simple net, SN) A Simple net is a Petri net for which every two places having a common output transition, one of them has all the output transitions of the other (and possibly more), i.e., \( \forall p_i, p_j \in P : p_i \bullet \cap p_j \bullet \neq \emptyset \Rightarrow (p_i \bullet \subseteq p_j \bullet) \) or \( (p_i \bullet \supseteq p_j \bullet) \).

All the nets, that do not belong to any of above subclasses are just classified as Petri nets (PNs). For the interpreted Petri nets, we shall use the same abbreviations for their subclasses. For example an interpreted Petri net that is classified as an EFC-net, will be shortly classified as an interpreted EFC-net.

Particular subclasses are structured as follow: SM and MG are simplest possible structures of a Petri net. FC is a generalization of SMs and MGs. It means that each net that belongs to SM is also classified as an FC-net. Similarly, each MG is an FC-net, as well. Furthermore, EFC is a generalization of FC, while SN is a generalization of EFC. Finally, PN is a generalization of SN. Figure 2.7 illustrates the hierarchy of subclasses of Petri nets.

![Figure 2.7 Subclasses of Petri nets](image_url)
Recall net $PN_1$, shown in Fig. 2.1. Every place of $PN_1$ has exactly one input transition and exactly one output transition (there are no multiple arcs outgoing from places). It means that such a net belongs to MG. Moreover, it is automatically classified as FC-net, EFC-net, SN, and PN, as well. Similarly, net $PN_2$ is also classified as MG-net, but $PN_3$ belongs to SN.

Classification of a Petri net may be useful during analysis of the net [25]. For example, liveness, safeness of some subclasses (SM, MG, FC, EFC) can be checked polynomially [2–4, 15, 16, 21].

**Definition 2.25 (State machine component)** A state machine component (SMC, SM-component, S-component) of a Petri net $PN=(P, T, F, M_0)$ is a Petri net $S=(P', T', F', M_0)$ such that:

1. $S$ is an SM-net,
2. $S$ is strongly connected,
3. $P' \subseteq P$,
4. $T' = \bullet P' \cup P' \bullet$,
5. $F' = F \cap (P' \times T') \cup (T' \times P')$,
6. $S$ contains exactly one token in $M_0$.

Figure 2.8 shows all the state machine components that can be obtained in the net $PN_1$. There are four SMCs, consisting of the following places: $S_1 = \{p_1, p_4\}$, $S_2 = \{p_3, p_6\}$, $S_3 = \{p_2, p_5\}$, $S_4 = \{p_4, p_5, p_6\}$.

**Definition 2.26 (SM-decomposition)** State machine decomposition (SM-decomposition, S-decomposition) of a Petri net $PN = (P, T, F, M_0)$ is a set $S$ of elements (often called components or modules) $S = \{S_1, \ldots, S_n\}$ such that each place $p_i \in P$ is a place of exactly one component $S_j \in S$. Each component $S_j$ is an SM-net. If the particular place exists in more than one component, it is replaced by a place not belonging to $P$, called a non-operational place (NOP).
Definition 2.27 (Non-operational place, NOP) Let $PN$ be a Petri net and let $S$ be a set of SMCs achieved during SM-decomposition of $PN$. If place $p \in P$ exists in more than one $S \in S$, it is replaced by a non-operational place (NOP) in all $S \in S$, but one. The set $P' = \{p_1, \ldots, p_j\}$ of places of the same component $S \in S$ can be replaced by a single NOP if there exists a path in $P'$ leading from $p_i$ to $p_j$. Then, all transitions and arcs between $p_i$ and $p_j$ are removed, as well. A NOP is initially marked if any place substituted by it is initially marked.

Let us explain decomposition and NOPs by examples. Petri net $PN_1$ can be decomposed into three components: $S=\{S_1, S_2, S_3\}$, where $S_1 = \{p_1, p_4\}$, $S_2 = \{p_3, p_6\}$, and $S_3 = \{p_2, p_5\}$. This is the only one possibility to decompose $PN_1$. There are no places that exist in more than one component, thus there is no need to apply non-operational places.

Consider Petri net $PN_4$ shown in Fig. 2.9a. There are six places in the net $P = \{p_1, \ldots, p_6\}$ and four transitions $T = \{t_1, \ldots, t_4\}$. The net is safe, live, and reversible. There are three SMCs in the $PN_4$: $S_1 = \{p_1, p_3, p_4\}$, $S_2 = \{p_2, p_3, p_4, p_6\}$, $S_3 = \{p_5, p_6\}$.

Figure 2.9b illustrates one of the possible decompositions of $PN_4$. In the presented example, two NOPs are applied, therefore the final set of decomposed components consists of the following places: $S_1 = \{p_1, p_3, p_4\}$, $S_2 = \{p_2, NOP_1, p_6\}$, $S_3 = \{p_5, NOP_2\}$.

The first non-operational place ($NOP_1$) is used in module $S_2$. It substitutes places $p_3$ and $p_4$ that already exist in the first component $S_1$. All the transitions and arcs

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**Fig. 2.9** Petri net $PN_4$ (a) and decomposed $PN_4$ (b)
belonging to the path connecting such places are removed as well. Furthermore, \( NOP_2 \) replaces \( p_6 \) in the third component \( S_3 \), since \( p_6 \) already exists in \( S_2 \).

Let us point out that there are more ways to decompose \( PN_4 \), depending on the application of non-operational places. For example, the alternative decomposition of the net may consists of the following components: \( S_1 = \{ p_1, NOP_1 \} \), \( S_2 = \{ p_2, p_3, p_4, NOP_2 \} \), \( S_3 = \{ p_5, p_6 \} \), while \( NOP_1 \) substitutes places \( p_3, p_4 \), and \( NOP_2 \) replaces \( p_6 \).

More information about decomposition of Petri nets as well as decomposition algorithms can be found in Chap. 6.

**Definition 2.28** (SM-cover) State machine cover (SM-cover, S-cover) of a Petri net \( PN = (P, T, F, M_0) \) is a set \( C \) of state machine components \( C = \{ S_1, \ldots, S_n \} \) such that each place \( p_i \in P \) is a place of at least one component \( S_j \in C \).

SM-cover is very often confused with SM-decomposition. The main difference is that the particular place may exist in more than one component that belongs to the SM-cover. In opposite, the set of decomposed modules contains each place of the net exactly once. There are known conversion methods between SM-decomposition and SM-cover. Such a transformation can be done relatively easy, unless the conditions of the existence of SM-decomposition (or SM-cover) are not satisfied. More details can be found in [20].

Let us now introduce theorems regarding SM-decomposition and SM-cover. A very important relation between SM-cover and well-formed EFC-nets was shown in [9] (initially proposed for FC-nets in [15]):

**Theorem 2.2** [9] Well-formed EFC-nets are covered by SM-components.

Since every interpreted EFC-net is well-formed, we immediately have

**Theorem 2.3** Interpreted EFC-nets are covered by SM-components.

*Proof* Follows directly from Definition 2.15 and Theorem 2.2.

Furthermore, the existence of SM-decomposition of a net depends on its safeness, which was proved in [20]:

**Theorem 2.4** [20] For a Petri net exists an SM-decomposition, if and only if \( PN \) is safe.

Since every interpreted Petri net is safe by definition, we obtain a very important statement:

**Theorem 2.5** For an interpreted Petri net there always exists an SM-decomposition.

*Proof* Follows directly from Definition 2.15 and Theorem 2.4.
2.2 Computational Complexity of Algorithms

This section briefly introduces some preliminaries regarding computational complexity of algorithms. The presented notation we shall use during analysis of the computational complexity of algorithms shown in Chaps. 3–6.

The computational complexity presented in this book refers to the time complexity. Therefore, unless otherwise stated, we shall estimate the amount of time that is required in order to solve the problem. Note that only basic assumptions are presented. The detailed descriptions regarding computational complexity of algorithms can be found in [1, 12, 18, 22, 26, 32].

- **Time computational complexity of an algorithm** refers to the maximum number of interactions (steps) that the algorithm uses on a given size of a given input. Formally, time complexity of algorithm \( A \) is the function \( f : \mathbb{N} \rightarrow \mathbb{N} \), where \( f(n) \) is the maximum number of interactions (steps) that \( A \) uses on any input of length \( n \) [32].
- **Big-O notation** is used to estimate the upper bound for the number of interactions (steps) executed by an algorithm. Formally, for \( f(n) = O(g(n)) \) we say that \( g(n) \) is an upper bound for \( f(n) \), where \( f(n) \) is the maximum number of interactions of an algorithm [32]. Note, that Big-O refers to the worst-case complexity of the algorithm.
- **Polynomial bound of an algorithm** means that the upper bound of the algorithm can be represented in the form \( n^c \) for \( c > 0 \) [32]. In other words, algorithm for which the number of interactions is estimated as \( f(n) = O(n^c) \) for \( c > 0 \) is bounded by a polynomial in the size of inputs \( n \).
- **Exponential bound of an algorithm** means the bounds in the form \( c^n \) for \( c > 1 \). In other words, algorithm for which the number of interactions is estimated as \( f(n) = O(c^n) \) for \( c > 1 \) is bounded by an exponential in the size of inputs \( n \).
- **Total polynomial time, polynomial complexity** or just **polynomial time** means that the total run-time of an algorithm to generate all solutions (outputs) is bounded by a polynomial in the size of the input (cf. [18]).
- **Exponential complexity** or just **exponential time** means that the total run-time of an algorithm to generate all solutions (outputs) is bounded by an exponential in the size of the input (cf. [18]).
- **Polynomial delay** means that an algorithm generates results in such a way, that the time between subsequent outputs is bounded by a polynomial in the size of the input (cf. [12, 18]). We will say, that subsequent outputs are generated (computed, calculated) in polynomial time.
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