Preface

Many contest problems concern series and sequences. Each year contest problems become more and more challenging and even an "A" math student without special preparation for such problems could feel frustrated and lost. Based on my own experience, the first time I had to consider a series, different from an arithmetic or geometric progression, it was at my city math olympiad when I was in 9^{th} grade. I remember that one of the problems looked like this.

> Evaluate the sum $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{1977}+\sqrt{1978}}$.

I knew that this sum exists because they would not ask to evaluate it otherwise. I noticed that there are 1977 terms to add, and of course it would not be math contest if there was not some interesting approach to find this sum without putting 1977 over the common denominator. The first thing I tried worked. I multiplied the numerator and denominator of one fraction by the quantity that differed from its denominator by only the sign. Then I applied the formula of the difference of two squares, which made the denominator of the fraction one.

For example, $\frac{1}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)} = \frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2} = \frac{\sqrt{3}-\sqrt{2}}{1} = \sqrt{3} - \sqrt{2}$.

Replacing each fraction as above by "rationalizing its denominator," I noticed that all radicals were canceled, except the first term and last terms, $\sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \sqrt{5} - \sqrt{4} + \ldots + \sqrt{1978} - \sqrt{1977} = \sqrt{1978} - 1$.

The answer was obtained!

After winning the City Olympiad, I was sent to the Regional Math Olympiad and was again surprised that two or three problems there were on topics that were not yet covered in our classes at school. One of the problems was on sequences, but again, it was different from the arithmetic and geometric progressions that we learned in algebra class. Here it is.

Find the formula for the n^{th} term of the sequence of numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, . . .

I remember that I looked at those numbers, noticed that each term starting from the third one is the sum of the two preceding terms. That allowed me to create the formula for the sequence: $a_{n+2} = a_n + a_{n+1},\ a_1 = a_2 = 1$.

Knowing nothing about the Fibonacci sequence and what this sequence actually described, I started thinking this way, "This sequence is not an arithmetic progression because the difference of any consecutive terms is not the same." I asked myself a question, "What if the terms of this sequence belong to a geometric progression?" Then they must satisfy the formula above. The idea appeared to be good and after manipulations, I found the answer. I solved this problem without any preliminary knowledge about the Fibonacci sequence and derived "my method" of dealing with the sequences given by recursion.

I show how I solved that problem from 10th grade in detail by demonstrating it as Problem 24, Chapter 1 of this book. Why do I write about these two examples from my own Olympiad experience and emphasize my lack of the knowledge about special sequences? There are several reasons for this but the first is to understand that nobody knows everything. We learn by organizing information and thoughts, not by simply storing them. I do not ask you to reinvent the wheel each time. However, I ask you to understand rather than simply memorize.

My method of teaching mathematics is constructed on four simple premises:

1. It is my opinion that creativity can be developed by considering some interesting approaches while also gaining routine background knowledge. For example, a difference of squares formula that I used to solve the first problem is not boring if considered in conjunction with an example of use such as one from my other book *Methods of Solving Nonstandard Problems* for the solution of $39999 \cdot 40001$ without a calculator.
2. Math education is now mostly oriented on teaching mathematics by "having fun." But "fun" should not be skin deep. I noticed that many math educators show their students the amazing Fibonacci sequence, generate it, and show its properties using videos or slides. Yes, students probably would recognize that the sequence given at the Math Olympiad in 1978 was Fibonacci. Many would be able to find some of its terms either by hand or by using a graphing calculator. I am not sure that many would derive the formula for its n^{th} term. A deeper understanding of mathematical concepts can be fun, and it is far more rewarding in the long term.
3. Concepts should not stand in isolation. For example, there is a connection between the golden ratio and the Fibonacci sequence which generates it. The golden ration of nature is the result of a simple recurrent relationship! Connections generate new insight and further enhance concepts.

4. The learning of mathematics should have a human purpose. Perhaps that purpose is merely to compete. That is good enough! Many modern contest problems have sequences and series either directly or as a part of a problem. Hence, as an instructor, it is a good idea to help those who want to participate in math contests to learn more about sequences and series by exploring the topics in order that one would create their own beautiful solutions to a problem.

If you are struggling with math, this book is for you. Most math books start from theoretical facts and give one or two examples and then a set of problems. In this book almost every statement is followed by problems. You are not just memorizing a theorem—you apply the knowledge immediately. Upon seeing a similar problem in the homework section you will be able to recognize and solve it. While each section of the book can be studied independently, the book is constructed to reinforce patterns developed at stages throughout the book. This helps you see how math topics are connected.

What Is This Book About?

This book is not a textbook. It is a learning and teaching tool that helps the reader to develop a creative learning experience. It gives many examples of series, partial, or infinite sums of which can be evaluated using methods taught in this book. Let us consider the problem to evaluate the series (Problem 50),

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{2016\cdot 2017}$.

I want to share my experience with my Calculus 2 class when learning series and sequences. Although some of my students answered correctly that there are 2016 terms and recognize the formula of the n^{th} term as $\frac{1}{n\cdot(n+1)}$, usually nobody in class can find this finite sum. I tell them a story how I evaluated this sum in 9th grade by noticing that each fraction can be written as a difference of two unit fractions

$$\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}, \quad \ldots. \quad \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

They quickly replace each term by the difference and evaluate the sum as $1-\frac{1}{2017}=\frac{2016}{2017}$. This would do nothing for most of my students and they would not remember this "trick" as nobody remembers telephone numbers anymore unless I asked them next class to evaluate the following infinite sum:

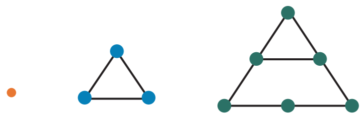
$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\ldots. \tag{P.1}$$

Those who recognized that this series is an infinite form of the finite series given before would evaluate its partial sum as $S_n=1-\frac{1}{n+1}$ and hence using the limit as n approaches infinity would state that the series is convergent to 1.

Further, we will discuss the so-called Leibniz harmonic triangle, related to Pascal's triangle, but that has only unit fractions recorded in the form of a triangle. The sum of the series represents the sum of all elements of the second diagonal of the infinite Leibniz triangle and my students learn that the sum is one using a different approach (Problem 67). Next, I ask my students to modify the Leibniz triangle so that it has only denominators of each fraction instead of fractions themselves (Figure 2.2). Let us construct a sequence of the denominators: $2, 6, 12, 20, 30, 42, \ldots, n(n+1), \ldots$. Students see that the same numbers belong to the second diagonal of the modified Leibniz triangle! I tell my students that these numbers are special and that each of them is a double so-called triangular number, known by the Ancient Greeks, 2000 BC.

Consider the sequence $1, 3, 6, 10, 15, 21, \ldots \frac{n(n+1)}{2}, \ldots$. Greeks visualized each such number placed in a triangle of side 1, 2, 3, 4, etc. The total number of the balls that could fit a triangle of side n would represent the n^{th} triangular number. We construct by hand several triangular numbers and learn their properties (Figure P.1).

Figure P.1 Triangular Numbers



Many properties are formulated and solved as problems in this book. For example,

> Can you explain why the formula for the n^{th} triangular number is $\frac{n(n+1)}{2}$?

This is where we recall how each was constructed and where my students see that it is the sum of all natural numbers between 1 and n. Thus,

$$\begin{aligned} 1 &= 1 \\ 3 &= 1 + 2 \\ 6 &= 1 + 2 + 3 \\ &\ldots \end{aligned}$$

It is useful to be reminded how the famous mathematician Gauss evaluated such a sum at the age of 10. The story is told frequently in algebra and calculus books, but here students can actually use the idea in deriving $T_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. For example, can we evaluate the sum of the first n triangular numbers, $1 + 3 + 6 + 10 + 15 + \ldots + \frac{n(n+1)}{2}$? Different methods of finding this and other sums are taught in this book. Younger students would probably enjoy a geometric approach, and calculus students would really benefit from applying sigma notation and well-known summation formulas (Problem 38).

What actually impresses all my present and former students is the connection between the sequence of natural numbers, sequence of triangular numbers, sequence of triangular numbers and tetrahedral numbers, etc. I demonstrate that the n^{th} partial sum of triangular numbers is the corresponding tetrahedral number (Problems 36 and 39). Denote a tetrahedral number by $Tr(n)$. Then $Tr(n) = \sum_1^n T_n = \sum_1^n \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$ (Figure P.2). Here we can briefly discuss that although formulas for the n^{th} terms of either triangular or tetrahedral numbers look like fractions, the numbers are always integers, because the product of two consecutive natural numbers is always a multiple of two and a product of three consecutive integers is always divisible by 6.

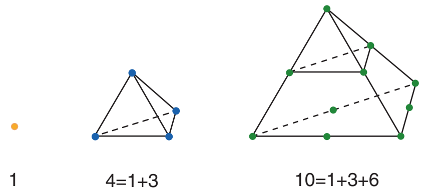


Figure P.2 Tetrahedral numbers

Let us return to the sum of the numbers in the second diagonal of the Leibniz-modified triangle (Figure 2.2),

$$2 + 6 + 12 + 20 + 30 + 42 + \ldots + n(n+1) = 2\sum_{k=1}^{n}\left(\frac{k(k+1)}{2}\right) = \sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

We can see that with n increasing, this sum will increase without bound and that the corresponding infinite sum of the reciprocals of each number $\left(\frac{1}{k(k+1)}\right)$ converges to unity. The series of Eq. P.1 is called a "telescopic series" and plays a very important role in the convergence of infinite series. Jacob Bernoulli used a slight modification of this series for the comparison test and found the upper boundary for the infinite sum of the Dirichlet series, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$. This problem is named Basel's problem and was solved by Leonhard Euler 40 years after being proposed (Chapters 2 and 3). Euler found that the series converges to $\frac{\pi^2}{6}$.

Finding the sum for infinite series for which it is impossible to evaluate the partial sum is often a challenging problem. Many mathematicians of all ages at some point of their life tried to find a number associated with certain infinite series.

The first step is to establish whether the series is convergent or not. This is why I describe the famous convergence theorems for numerical and functional series in Chapter 3. Chapter 3 might not look like competition material, but it does have many unique methods for finding partial and infinite sums. Let us consider one of the problems from Chapter 3.

Find the sum of an infinite series $\frac{1}{4}+\frac{1}{36}+\frac{1}{144}+\frac{1}{400}+\frac{1}{900}+\frac{1}{1764}+\ldots$.

This series can be rewritten in terms of the Dirichlet and telescopic series and converges to $\frac{\pi^2}{3}-3$.

I start from an exploration of the properties of well-known arithmetic and geometric sequences that are familiar to high school students. By giving my students many problems during the 25 years of my teaching experience, I noticed that they are very adept in pattern recognition. They might recognize that this is a Fibonacci sequence and determine succeeding terms by the two preceding terms. However, as I mentioned above, it is usually hard for them to analytically find the formula for the n^{th} term or even to add the numbers: $2+9+16+23+\ldots+352$.

Yes, they find that the terms differ by 7 and that the first term is 2. But many students panic because they do not know how many terms there are, in order to apply the Gauss counting approach. This is why students need to study arithmetic and geometric progressions. For example, the n^{th} term of the series, 352, can be written as $a_n = 2+(n-1)\cdot 7 = 7n-5 = 352 \Rightarrow n = 51$. Next, we can use Gauss's formula and evaluate the sum as

$$S_{51} = \left(\frac{2+352}{2}\right)\cdot 51 = 9027.$$

Many challenging problems of arithmetic, geometric, and other sequences can be found in the book. For example, knowledge of geometric series will allow you to solve interesting problems such as,

Find the sum of 2016 numbers $3+33+333+3333+\ldots+\underbrace{333\ldots3}_{2016}$.

Other methods will be used to evaluate a sum like,

Evaluate the sum, $S=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{2015}{2016!}$.

An important feature of this book is that most Statements, Lemmas, and Theorems have detailed proofs. I remember how one graduate student who was teaching geometry in a private high school rushed to report to me about finding "the formula for a prime number." He stated that it is $2^n - 1$. On my question "Why?," he replied that $2^2 - 1 = 3,\ 2^3 - 1 = 7$ are primes. When I asked what about $2^4 - 1 = 15$ that is not prime? He was confused and said "I did not go that far. . .." This story sounds like a joke, but it really happened and demonstrates that any statement must be proven. His "formula" was wrong and it was proven wrong by contradiction. Particular cases must be generalized and proven, for example, by mathematical induction, directly, or by contradiction.

This book is a collection of simple and complex problems on series and sequences that are selected to motivate the reader to start solving challenging problems. For example, the following problem requires similar ideas to Problem 50 and also generalizes the method and develops proof skills.

> The numbers $a_1,\ a_2,\ \ldots,\ a_n,\ a_{n+1}$ are terms of an arithmetic sequence.
> Prove that $\frac{1}{a_1 \cdot a_2} + \frac{1}{a_2 \cdot a_3} + \ldots + \frac{1}{a_n \cdot a_{n+1}} = \frac{n}{a_1 \cdot a_{n+1}}$.

After recognizing different sequences, one might like the following problem,

> Find the n^{th} term of a sequence 3, 13, 30, 54, 85, 123,. . ..

The given sequence is not an arithmetic sequence; however, the differences of two consecutive terms are 10, 17, 24, 31, 38, . . . and are in an arithmetic progression with common difference $d = 7$. This means that the given sequence of numbers 3, 13, 30, 54, . . . is the sequence of partial sums of this arithmetic progression and that its n^{th} term can be evaluated as $a_n = S_n(d = 7) = \frac{2 \cdot 3 + (n-1) \cdot 7}{2} \cdot n = \frac{(7n-1) \cdot n}{2}$.

The techniques used in this book are basic to understanding series and sequences. As early as 2000 BC, the Babylonians created tables of cubes and squares of the natural numbers and proved the summation of natural numbers, their squares, and cubes by a geometric approach. These formulas are in nearly every textbook still today and are used in finding other sums of finite series. While remembering these formulas by heart is a very good idea, it is better to be able to prove each formula by at least one of the methods demonstrated in this book. Memorization cannot replace understanding. Read the book with a pen and paper and be ready to derive a forgotten mathematical identity if it is needed.

Versions of problems solved by the Ancient Babylonians and Greeks often reappear in modern math contests. Their importance to modern mathematics is fundamental and unavoidable. For example, here is one of the problems of Chapter 1 of the book that was known to ancient mathematicians.

> Prove that a cube of a natural number n can be uniquely written as a sum of precisely n odd consecutive numbers.

We prove this statement and find formulas for the first and last odd numbers for any given n. For example, the cube of 7 is uniquely represented by the sum of seven odd numbers, $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55 = 343$, 1000 by 10 consecutive odd numbers, etc. Would you like to know how? The answer is in the book.

This book is not a textbook. Some knowledge of algebra and geometry such as what is introduced in secondary school is necessary to make full use of the material of Chapters 1 and 2. Knowledge of calculus is needed for better understanding of Chapter 3. However, a mastery of these subjects is not a prerequisite. You will use your knowledge of secondary school mathematics in order to better delve into the analysis of sequence and series and their properties as you develop problem-solving skills and your overall mathematical abilities.

The book is divided into four chapters: Introduction to Sequences and Series, Further Study of Sequences and Series, Series Convergence Theorems and Applications, Real-Life Applications of Arithmetic and Geometric Sequences. One hundred twenty homework problems with hints and detailed solutions are given at the end of the book. There are overlaps in knowledge and concepts between chapters. These overlaps are unavoidable since the threads of deduction we follow from the central ideas of the chapters are intertwined well within our scope of interest. For example, we will on occasion use the results of a particular lemma or theorem in a solution but wait to prove that lemma or theorem until it becomes essential to the thread at hand. If you know that property you can follow along right away and, if not, then you may find it in the following sections or in the suggested references.

Many figures are prepared with MAPLE, Excel, and Geometer's Sketchpad. Additionally, Chapters 1 and 4 have a number of screenshots produced by a popular graphing calculator by Texas Instruments. These graphs are shown especially for the benefit of students accustomed to using calculators in order to introduce them to analytical methods. Sometimes by comparing solutions obtained numerically and analytically, we can more readily see the advantages of analytical methods while referring to the numerically calculated graphs to give us confidence in our results. Following the new rules of the US Mathematics Olympiad, I suggest that you prepare all sketches by hand and urge you not to rely on a calculator or computer to solve the homework problems.

This book covers geometric, arithmetic and other sequences and their applications, sigma notation, and series. You will learn how to evaluate a limit in calculus analytically using arithmetic and geometric sequences and how to take an integral in just one step by recognizing a similarity with a sum like $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \ldots$.

Additionally, we will teach you how to find any term of a sequence given by a recursion formula and will introduce the so-called generating functions. You will have fun learning about figurate numbers and their properties and the application of mathematical induction to sequences and series. This book will also assist the reader in how to prove lemmas and theorems using different methods. Working on projects that Chapter 4 offers, you will see a connection of arithmetic and geometric series and sequences with real-life problems (radioactive decay, mortgage, loan, debts, etc.) and the wise use of technology for mathematics. This book will be very useful for beginners and for those who are looking challenge themselves. The book can be helpful for self-education, for people who want to do well in math classes, or for those preparing for competitions. It is also meant for math teachers and college professors who would like to use it as an extra resource in their classroom.

How Should This Book Be Used?

Here are my suggestions about how to use the book. Read the corresponding section and try to solve the problem without looking at my solution. If a problem is not easy, then sometimes it is important to find an auxiliary condition that is not a part of the problem, but that will help you to find a solution in a couple of additional steps. I will point out ideas we used in the auxiliary constructions so that you can develop your own experience and hopefully become an expert soon. If you find any question or section too difficult, skip it and go to another one. Later you may come back and try to master it. Different people respond differently to the same question. Return to difficult sections later and then solve all the problems. Read my solution when you have found your own solution or when you think you are just absolutely stuck. Think about related problems that you could solve using the same or similar approach and compare that to corresponding problems in the Homework section. Create your own problem and write it down along with your original solution. Now it is your powerful method. You will use it when it is needed.

I promise that this book will make you successful in problem solving. If you do not understand how a problem was solved or if you feel that you do not understand my approach, please remember that there are always other ways to do the same problem. Maybe your method is better than one proposed in this book. If a problem requires knowledge of trigonometry or number theory or another field of mathematics that you have not learned yet, then skip it and do other problems that you are able to understand and solve. This will give you a positive record of success in problem solving and will help you to attack the harder problem later. Do not ever give up!

I hope that upon finishing this book you will love math and its language as I do. Good luck and my best wishes to you!

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