Quadrature Based Approximations of Non-integer Order Integrator on Finite Integration Interval

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Abstract Implementation of non-integer order systems is the subject of an ongoing research. In this paper we consider the approximation of non-integer order integrator with the use of diffusive realization of pseudo differential operator. We propose a transformation of variables allowing easier approximation with use of quadratures. We then analyze the convergence and discuss the consequences of reduction in the integration interval.

Keywords Diffusive realization · Approximation · Non-integer order integrator · Quadratures

1 Introduction

Non-integer order systems take an increasing part in science and engineering. Their applications include, among the others, modeling, control and signal processing. In most of the problems the infinite memory of non-integer order systems introduces problems with realization. Because of that realizations cannot come directly from the definition but have to take form of integer order realizations.

In this paper, the problem of realization of non-integer order integrator, i.e., $1/s^{\alpha}$ is considered. A method based on diffusive realization of pseudo differential operators is investigated.

Currently realizations on non-integer order systems are focused in four areas. First area is based on approximating the s^{α} operator in the frequency domain. The most popular approaches are the Oustaloup's method [1, 2] and CFE (*Continuous fraction expansion*) method [3–5]. Both these methods are based on different premises

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but both allow obtaining relatively easy approximations. CFE method is generally considered as worse of the two in the aspect of frequency response representation [6]. Ostaloup's method gives a very good representation of frequency response at the cost of high numerical sensitivity. This sensitivity can be lessened with use of time domain realizations [7, 8].

Second method of approximation relies on discrete realizations based on truncation of series representations. In particular, the methods include truncation of Grünwald–Letnikov derivative definitions or power series expansions (PSE) in the z variable (discrete frequency) domain. These methods give good approximations only at high frequencies. Examples of the method can be found in [9]. Improvements on PSE methods with better band of correct approximation were investigated by Ferdi [10, 11].

Third method of approximation is based on approximation of non-integer order system impulse response with Laguerre functions. Early works in this area were unsuccessful in approximation of α order integrators [12, 13]. In [14] it was shown that under certain assumptions method is convergent in L^1 and L^2 norms when approximating asymptotically stable non-integer transfer functions of relative degree equal or greater than one. Most applications of the methods are in filtering and parametric optimization [15–18].

Fourth type of approximation that recently rises in popularity is the method using diffusive realization of pseudo differential operators [19] for approximation. First works in the area used finite dimensional approximation with trapezoidal integration [20]. Later works used simple diagonal matrix time domain realization with modified Oustaloup nodes from frequency domain method [21, 22]. This method is especially useful in analysis of infinite dimensional systems using operator theory. It can be used, for example, to prove such properties as stability of closed loop system [23].

In this paper, diffusive realization is used for construction of approximation using quadrature methods. In particular, frequency domain representation of non-integer order integral will be used, i.e., $1/s^\alpha$. For such system the input-output mapping will be approximated using two quadratures on finite integration intervals allowing creation of finite dimensional approximation in the form of sum of first order lags.

The rest of the paper is organized as follows. Next section presents a theorem that is a basis of investigation along with its proof. Next, the approximation method is presented and a variable transformation allowing effective use of quadratures. Then two types of quadratures are analyzed. Operation of both quadratures is investigated with the use of H^∞ norm as an indicator of approximation quality. Then results are discussed and future works directions are presented.

2 Diffusive Realization of Non-integer Order Integrator

As the basis for diffusive realization we consider the following theorem (see e.g. [21]) which determines equivalent form for integrator of order α.

Theorem 1 *For $\alpha \in (0, 1)$ and $s \in \mathbb{C}$ we have:*

$$\frac{1}{s^\alpha} = \int_0^\infty \frac{\sin \alpha\pi}{\pi} \frac{1}{x^\alpha} \frac{1}{s+x} x. \tag{1}$$

Proof Proof will be omitted, however it relies on computing inverse Laplace transform of $1/s^\alpha$ using contour integration using Bromwich contour (see Fig. 1).

Remark 1 Grabowski in [23] proved the same result in a different way. The given proof is based on direct calculation of right-hand side of (1) using integration contour as in Fig. 2. This proof, however, is not constructive.

The main advantage of the method used in Theorem 1 is that we do not need to integrate non-integer power of s. It allows to approximate the integral.

3 Approximation of Diffusive Realizations

The basic approach to approximation of integrals relies on using infinite sum of form

$$\int_0^\infty \frac{\sin \alpha\pi}{\pi} \frac{1}{x^\alpha} \frac{1}{s+x} \mathrm{x} \approx \sum_{i=0}^{n} \frac{b_i}{s+x_i} \tag{2}$$

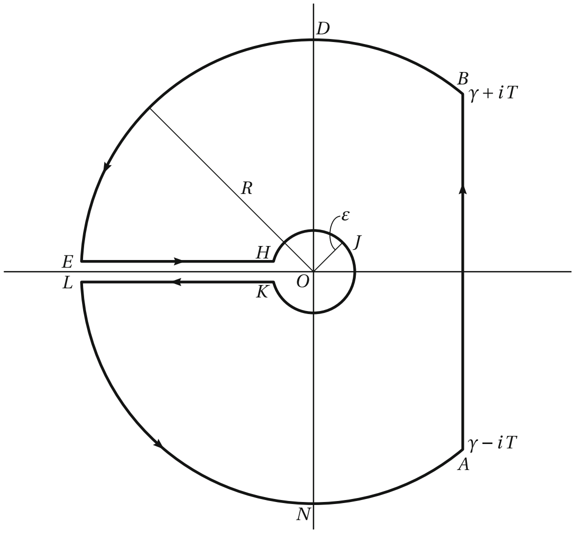

Fig. 1 Bromwich contour

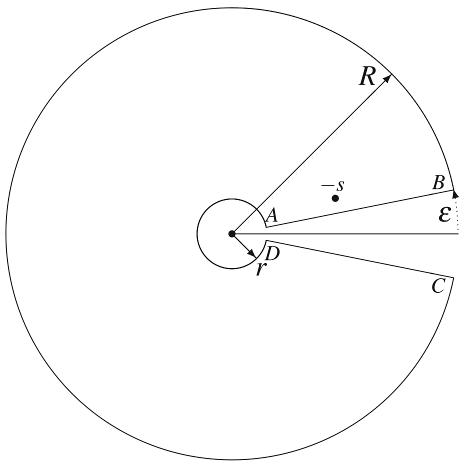

Fig. 2 Contour from [23]

where x_i, are denote nodes of quadrature while b_i are products of quadrature weights and integrand values. The choice of quadrature for approximation is not straightforward. In [20] the authors used trapezoidal rule. In [21], rectangle rule with non-uniform nodes was considered. Both approaches transform the integration interval from $[0,\ \infty]$ to $(0,\ \omega_{max}]$ as x_i are important only for high frequencies. It causes, however, errors in integrating, especially near ω_{max}.

Despite the fact that integral (2) is defined on infinite interval, multiple authors had considered limiting the interval only to the one including the interesting frequency band. In this paper we will focus on such bounded intervals, but using more advanced quadratures.

It can be inferred, that because approximant (2) is a sum of first order low pass filters the quality of frequency response representation will depend on time constants. Because we are interested in good representation of frequencies for different orders of magnitude a following transformation is proposed. Let us set

$$x = 10^{\theta}$$

Integrating (1) for $x \in [10^a, 10^b]$ reduces to

$$\int_a^b \frac{\sin \alpha\pi}{\pi} \log(10) 10^{\theta(1-\alpha)} \frac{1}{s + 10^{\theta}} \mathrm{d}\theta. \tag{3}$$

This formula, while at first glance complicated allows balancing high and low frequencies. Following section contains analysis of quadratures for finite intervals.

4 Quadratures on Infinite Intervals

In this section we will consider two types of quadratures on finite intervals, originating from interval $[-1, 1]$. We will however present very elegant matlab code examples given originally by Trefethen [24]. Provided programs are fully functional and self explanatory.

The first quadrature we consider is the Gauss quadrature, also known as Gauss–Legendre quadrature. It is a quadrature, nodes of which are roots of Legendre polynomials, while formulas for weights one can find for example in [25]. Main advantage of the Gauss–Legendre quadrature comes from the orthogonality of Legendre polynomials. In particular, let us consider a polynomial $p(x)$ of order $2N + 1$. Such polynomial can always be written as $p(x) = q(x) \cdot l_{N+1}(x) + r(x)$, where $q(x)$ is a polynomial of degree $N + 1$, $l_{N+1}(x)$ is a Legendre polynomial of degree $N + 1$ and $r(x)$ is a polynomial of degree N. When considering interpolation quadrature of degree $N + 1$ on nodes, we can easily see that $\int_{-1}^{1} q(x) \cdot l_N(x)\mathrm{d}x = 0$ because of orthogonality of Legendre polynomials and the integral of $r(x)$ is exact because of uniqueness of interpolation polynomials. These facts give that Gauss–Legendre quadratures on $N + 1$ nodes are exact for polynomials of order $2N + 1$. Following code uses eigenvalue decomposition to obtain both weights and nodes.

```
function I = gauss(f,n) % (n+1)-pt Gauss quadrature of f
beta = .5./sqrt(1-(2*(1:n)).^(-2)); % 3-term recurrence coeffs
T = diag(beta,1) + diag(beta,-1); % Jacobi matrix
[V,D] = eig(T); % eigenvalue decomposition
x = diag(D); [x,i] = sort(x); % nodes (= Legendre points)
w = 2*V(1,i).^2; % weights
I = w*feval(f,x); % the integral
```

The second quadrature is the Clenshaw–Curtis quadrature. It is an interpolation quadrature on Chebyshev nodes. In certain aspects it is also equivalent to discrete cosine transform, and can be computed using FFT. Again code given by Trefethen:

```
function I = clenshaw_curtis(f,n) % (n+1)-pt C-C quadrature of f
x = cos(pi*(0:n)'/n); % Chebyshev points
fx = feval(f,x)/(2*n); % f evaluated at these points
g = real(fft(fx([1:n+1 n:-1:2]))); % fast Fourier transform
a = [g(1); g(2:n)+g(2*n:-1:n+2); g(n+1)]; % Chebyshev coeffs
w = 0*a'; w(1:2:end) = 2./(1-(0:2:n).^2); % weight vector
I = w*a; % the integral
```

While theory gives us, that Gauss quadratures, thanks to orthogonality, are exact for polynomials of twice higher order than interpolation quadratures it is observed that in practice Clenshaw–Curtis quadrature gives very similar precision [24]. Moreover it has much smaller computational complexity. It should be also noted, that while based on interpolation both of these quadratures are immune to Runge's phenomenon, as both Legendre and Chebyshev nodes have asymptotic distribution of $1/(1 - x^2)$ for $[-1, 1]$ interval.

5 Approximation Analysis

In order to analyze the approximation performance we conducted error analysis using H^∞ norm. We considered the difference

$$e(s) = \frac{1}{s^\alpha} - \sum_{i=0}^{n} \frac{b_i}{s + x_i}$$

for frequency $[10^{-5},\ 10^5]$ rad/s. The order of quadrature was increased in order to observe the change of norm of $\|e\|_\infty$. For initial analysis we have considered the integration interval of $[-5, 5]$ for (3) corresponding to $[10^{-5}, 10^5]$ rad/s for original integral. Both quadratures were tested simultaneously. In illustrated results $\alpha = 1/2$ was considered.

In the Fig. 3 we depicted this analysis. As it can be observed after initial convergence error stops decreasing and fixes on a certain value. This error is caused by the limitation of infinite interval into a finite one. Indeed one can observe in the Fig. 4 that increasing the interval to $[-7, 7]$ results in dropping of error by an order of magnitude.

The fixed error is connected to the phase errors at low frequencies. As one can observe in the Fig. 5, 50th order approximate has a phase error, especially noticeable at low frequencies, where the gain of the integrator is the largest. Because of that any

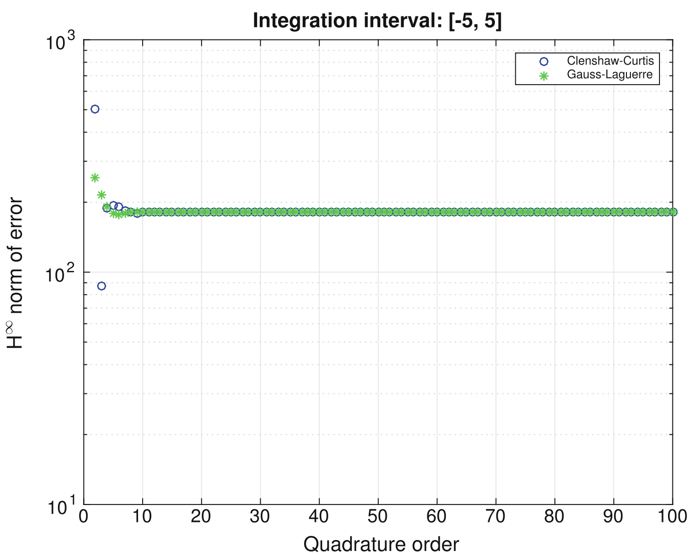

Fig. 3 Analysis of $\|e\|_\infty$ norm with increasing order of quadrature for the integration interval of $[-5, 5]$

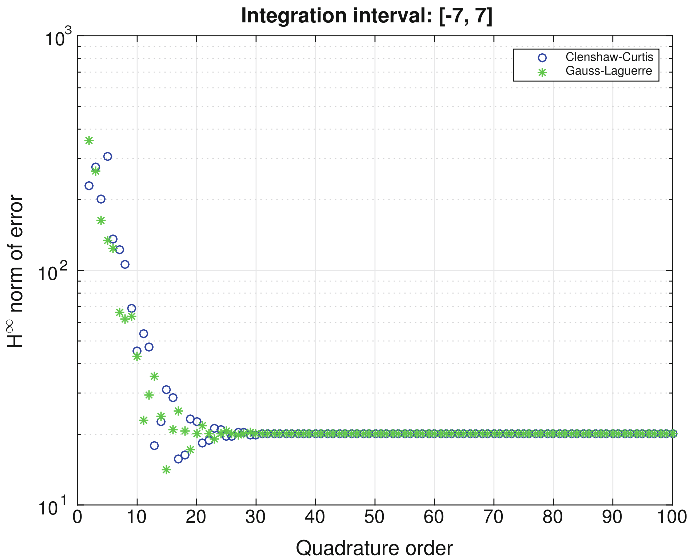

Fig. 4 Analysis of $\|e\|_\infty$ norm with increasing order of quadrature for the integration interval of $[-7, 7]$

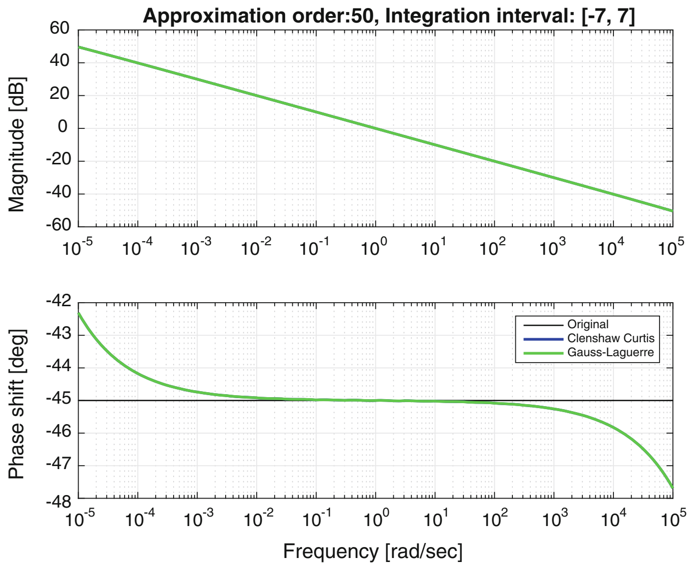

Fig. 5 Analysis of frequency response of 50th order approximation with the integration interval of $[-7, 7]$. Clenshaw–Curtis and Gauss–Legendre quadratures are overlapping

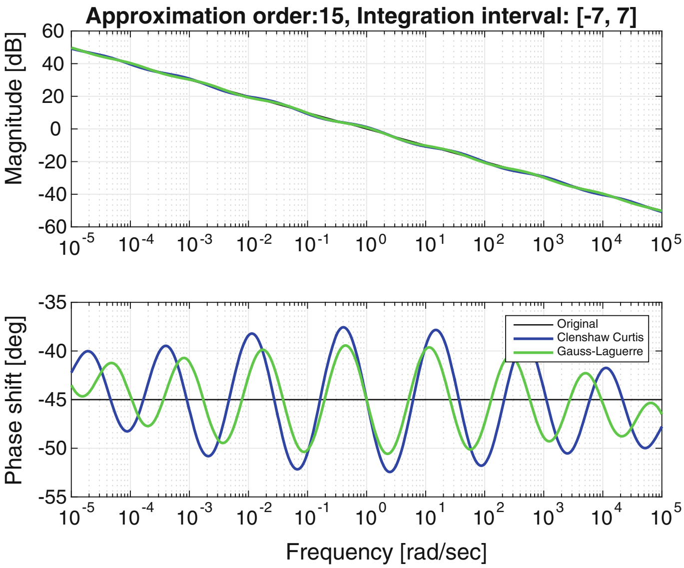

Fig. 6 Analysis of frequency response of 15th order approximation with the integration interval of $[-7, 7]$

error in phase is multiplied by approximately 300. What is interesting the magnitude is almost ideally represented.

As one can observe in the Fig. 4 there are values, where the error is smaller than the observed bias. For example such value is $N = 15$. In that case Gauss–Laguerre quadrature gives much better result. It is however caused by not obtaining convergence yet. It can be seen in Fig. 6, that the phase fluctuates strongly, and for such N and this type of quadrature is very close to the ideal value for low frequencies. Such approximation, however, is obviously not acceptable for use.

6 Conclusion

In this paper, we considered diffusive realization of integrator of order α and its approximation with use of quadratures on finite intervals. It can be observed, that initial convergence is relatively quick but bias caused by reduction of integration interval is fixed. This bias depends on integration interval, and as a-rule-of-a-thumb integration interval has to be greater than the band pass where the approximation is needed.

The performance indicator taken as the error of H^∞ norm is dominated by low-frequency terms, especially the phase errors. Nevertheless, the method should be further considered, especially in order to accelerate the convergence as there are certain methods which show better performance for lower orders (see, e.g. Oustaloup method). It is probably due to the slow decay of function in diffusive realization (1). However, high order of approximation should not be feared, as system is directly realized with independent linear differential equations of order one which are very easily discretized.

It should be also noted, that Oustaloup approximation, which has only geometric justification outperforms this approximation significantly for bounded intervals. This suggests, that this type of approximation has only the theoretical significance.

Further works should be concentrated on using quadratures on finite intervals along with asymptotic approximation on the 'tail' of the function.

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