Chapter 2
Fractal Calculus Fundamentals

2.1 Preliminaries

Many real physical processes possess "memory," which comes as follows: time connection between the process cause, $f(t)$, and the process effect, $g(t)$, is not immediate, and the condition of $g(t)$ is specified with the condition of $f(t)$ not at the same moment but delayed. This property is called *hereditary*.

Hereditary property, in particular, shows itself in such phenomena and processes as metal fatigue, magnetic and electrical hysteresis, motion of bodies through viscous medium, propagation of sound waves, diffusion, etc.

The key part in the hereditary theory belongs to the influence (memory) function, which generally reflects the complexity of the system and the process. One of the important properties of the memory function is self-similarity that is determined based on some general considerations. As it was shown in Chap. 1, self-similarity is the "generic indicator" of power functions.

Exponential functions are well known to be the most widely used functions in electrical engineering, theoretical physics, and mathematics. These functions (or their superpositions) are used to solve many problems that result in ordinary differential equations with integer-order derivatives.

There is a need to mathematically describe physical processes and phenomena that possess hereditary effect, which obey power and even logarithmic laws. Thus, applications of nonconventional mathematics have emerged such as fractal dimensions, fractional-order integrals and derivatives, and nonstandard distribution functions with infinite moments (Levy distributions).

Although fractional derivatives and integrals were introduced by the famous mathematicians Abel and Liouville as early as in the 30s of the nineteenth century, the significance and popularity of the "new" mathematical concepts started growing rapidly just in recent decades. This quick growth of interest was undoubtedly stimulated by the introduction of fractal geometry.

A.Kh. Gil'mutdinov et al., *Fractal Elements and their Applications*, Analog Circuits and Signal Processing, DOI 10.1007/978-3-319-45249-4_2

This chapter describes the fundamental concepts of fractional-order integrals and derivatives, and some of their properties and transforms in several fields such as electrical engineering, electrical chemistry, and impedancemetry that exhibits fractional-order dynamics.

2.2 Properties of Fractional-Order Integrals and Derivatives

2.2.1 Riemann-Liouville Fractional-Order Integral and Derivative

The definition of fractional integral follows from the generalization of the integer-order Cauchy formula. Let the integral operator is denoted by I; then the integration of $f(x)$ is defined as

$$ {}_0I_x^1 f(x) = \int_0^x f(t)dt. $$

Here, the subscripts specify the integration limits; the left one is the lower limit, while the right one is the upper limit. The superscript specifies the integration order.

The expression to calculate the integer-order n-fold multiple integral is defined by

$$ {}_0I_x^n f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1}dt\,. \tag{2.1} $$

One may generalize (2.1) to a non-integer integral of order α, which is denoted as the Riemann-Liouville (RL) definition of fractional-order integral, as follows:

$$ {}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt, \tag{2.2} $$

where $\Gamma(\alpha)$ is the well-known gamma function given by

$$ \Gamma(x) = \int_0^\infty t^{x-t}\mathrm{e}^{-t}dt,\ x\in\Re. \tag{2.3} $$

The simplest way to understand gamma function is to generalize the factorial of all real numbers. It may be shown, using integration by parts, that for integer values of $x = n$

$$\Gamma(n) = (n-1)\cdot(n-2)\cdot\ldots 2\cdot 1 = (n-1)!, \text{ and } \Gamma(n+1) = n\Gamma(n) = n!$$

Now, let D denotes a differential operator; then the relation between the fractional-order differential and integral operators is represented as $D^{\alpha}f(x) = I^{-\alpha}f(x)$. Consequently, the fractional-order derivative of order α can be deduced from (2.2) to yield

$${}_0D_x^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha+1}}dt. \tag{2.4}$$

It should be noted that the value of the lower limit of integration, which is zero here, could be arbitrary. In general, the integration (differentiation) limits are specified with subscripts. For example, a Riemann-Liouville derivative with non-zero lower limit will be defined as

$${}_bD_x^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_h^x \frac{f(t)}{(x-t)^{\alpha+1}}dt. \tag{2.5}$$

When the lower limit equals zero, for simplicity, one may replace ${}_0D_x^a f(x)$ by $D_x^a f(x)$. The RL derivatives and integrals are widely used in fractional calculus. It can be extended to the most general case when $n-1 < \beta \leq n$; for any integer number $n \geq 1$. The fractional-order derivative of order $n-1 < \beta \leq n$ can be expressed as

$${}_aD_x^{\beta}f(x) = \frac{d^n}{dx^n}\left[{}_aD_x^{-(n-\beta)}f(x)\right] = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\left[\int_a^x \frac{f(t)}{(x-t)^{\beta-n+1}}dt\right]. \tag{2.6}$$

2.2.2 Grunwald-Letnikov Fractional-Order Derivative and Integral

Unlike the Riemann-Liouville approach, which is based on the concept of multiple integrals, the Grunwald-Letnikov definition (hereinafter referred to as GL definition) of fractional-order derivative follows from the classical definition of integer-order derivative.

Consider the following definition of the first-order derivative:

$$D^1 f(x) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}.$$

To acquire the second-order derivative, differentiate $D^1 f(x)$ once more to get

$$D^2 f(x) = \lim_{dx \to 0} \frac{D^1 f(x+dx) - D^1 f(x)}{dx}$$
$$= \lim_{dx_2 \to 0} \frac{\lim\limits_{dx_2 \to 0} \frac{f(x+dx_1+dx_2) - f(x+dx_1)}{dx_2} - \lim\limits_{dx_2 \to 0} \frac{f(x+dx_2) - f(x)}{dx_2}}{dx_1}.$$

Assume that the increments are of equal size, i.e., $dx_1 = dx_2 = dx$; then the expression of the second-order derivative can be simplified to

$$D^2 f(x) = \lim_{dx \to 0} \frac{f(x+2dx) - 2f(x+dx) + f(x)}{dx^2}$$

Similarly, the nth-order derivative of the function $f(x)$ can be obtained by running this procedure n times. Hence,

$$D^n f(x) = \lim_{dx \to 0} \frac{1}{h^n} \sum\nolimits_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh); \quad h \equiv dx, \tag{2.7}$$

where $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$ are the binomial coefficients.

This expression can be generalized for arbitrary real number, $\alpha \in \Re$, by replacing the standard factorials with gamma function. Furthermore, the upper limit of the summation, $(t-a)/h$, (not an integer number) tends to infinity as $h \to 0$ (where t and a are the upper and the lower limits of differentiation). Then the resulting GL fractional derivative of $f(x)$ will be described as

$${}_aD_x^\alpha f(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{m=0}^{\left[\frac{t-a}{h}\right]} \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha - m + 1)} f(x - mh); h \equiv dx, \tag{2.8}$$

where [.] is the flooring operator.

Just like in the case of the RL fractional integral transform into the RL fractional derivative, the GL fractional derivative can also be transformed into the GL fractional integral. The most natural way to do it is to determine the expression for binomial coefficients (2.8) at $n < 0$. It can be shown that

$$\binom{-n}{m} = \frac{-n(-n-1)(-n-2)(-n-3)\ldots(-n-m+1)}{m!} = (-1)^m \frac{(n+m-1)!}{(n-1)!m!}.$$

and

$$\binom{-\alpha}{m} = (-1)^m \frac{\Gamma(\alpha+m)!}{\Gamma(\alpha)m!}. \tag{2.9}$$

Hence, from (2.9), the GL fractional-order integral can be written as follows:

$$ {}_aI_x^{\alpha} f(x) = \lim_{h\to 0} h^{\alpha} \sum_{m=0}^{\left[\frac{t-a}{h}\right]} \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} f(x-mh);\ h\equiv dx. \tag{2.10}$$

In general (2.8) can (2.10) can be combined in a single definition as follows:

$$ {}_aD_x^{\pm\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\mp\alpha}} \sum\nolimits_{j=0}^{\left[\frac{t-a}{h}\right]} C_j^{\pm\alpha} f((x-j)h), \tag{2.11}$$

where

$$C_j^{\pm\alpha} \equiv (-1)^j \binom{\pm\alpha}{j} = \left(1 - \left(\frac{1\pm\alpha}{j}\right)\right) C_{j-1}^{\pm\alpha};\ C_0^{\pm\alpha} = 1;\ j = 1, \ldots, n. \tag{2.12}$$

Despite the difference between the RL and GL definitions of fractional-order integral and derivative, they are actually equivalent. In real practice, the RL definition is widely used to analytically calculate fractional-order integrals and derivatives of relatively simple functions (x^a, e^x, sin(x), ... etc.). The GL definition, on the other hand, can easily be used for numerical calculations, where its accuracy depends on the step size, h.

2.2.3 Properties of Fractional-Order Derivatives

Consider the following basic properties of fractional-order derivatives.

1. If $f(x)$ is an analytical function of x, then its fractional derivative ${}_aD_x^a f(x)$ is also an analytical function of x and α.
2. If $\alpha = n$ and n is an integer number, then ${}_aD_x^a f(x)$ operation converges to the same value of the conventional integer-order n differentiation.
3. If $\alpha = 0$, then ${}_aD_x^a f(x) = {}_0D_x^0 f(x) = f(x)$.
4. Fractional-order differentiation, just like integer-order differentiation, is a linear operation:

$$ {}_0D_x^{\alpha}[af(x) + bg(x)] = a\left[{}_0D_x^{\alpha} f(x)\right] + b\left[{}_0D_x^{\alpha} g(x)\right].$$

5. Fractional differentiation is commutative, i.e.,

$$ {}_0D_x^{\alpha}\left[{}_0D_x^{\beta}f(x)\right]={}_0D_x^{\beta}\left[{}_0D_x^{\alpha}f(x)\right]={}_0D_x^{\alpha+\beta}f(x). $$

6. Since $I^{\alpha}f(x) = D^{-\alpha}f(x)$, all previous five properties still apply for fractional-order integration.

2.3 Laplace Transform of Fractional-Order Operators

The Laplace transform is widely used to solve engineering problems, including problems in radio engineering. In this section, we introduce some fundamentals of the Laplace transform of integer-order calculus, which will then be generalized to fractional-order calculus.

2.3.1 Fundamentals of Laplace Transform

The Laplace transform of a function $f(t)$, denoted as $F(s)$, where $s = \sigma + j\omega$, is given by

$$ F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} \mathrm{e}^{-st} f(t)dt. \tag{2.13} $$

Notice that (2.13) converges if $f(t)$ is both identically equal to zero for $t < 0$ and a single piecewise continuous function with finite number of the first kind discontinuities for $t > 0$. As $t \rightarrow \infty$, $f(t)$ should be bounded; that is, it should not grow faster than the exponential function $M\mathrm{e}^{\sigma 0t}$ for some finite σ_0 and a positive number M. The functions in the Laplace domain are usually denoted with capital letters, while the original functions are denoted with lower case letters.

The original function $f(t)$ can be restored out from $F(s)$ by means of the inverse Laplace transform; that is,

$$ f(t) = \mathcal{L}^{-1}\{F(s)\} = \int_{\sigma-j\infty}^{\sigma+j\infty} \mathrm{e}^{st} F(s)ds, \quad \sigma = \mathrm{Re}(s) > \sigma_0. \tag{2.14} $$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform operator or the inverse $\mathcal{L}$ transform; σ_0 is located in the right half of the absolute convergence plane of the Laplace integral (2.13).

The convolution of two functions, $f(t)$ and $g(t)$, denoted by $f(t)*g(t)$, is given by

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{2.15}$$

The Laplace transform of (2.15) yields the product of the Laplace transforms of the two corresponding functions, i.e.,

$$\mathcal{L}[f(t)*g(t)] = F(s)G(s) \tag{2.16}$$

provided that $F(s)$ and $G(s)$ both exist.

Another useful property is the Laplace transform of the nth-order derivative of $f(t)$:

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0), \tag{2.17}$$

which can be obtained from (2.13) via integration by parts with the assumption that the corresponding integrals exist.

2.3.2 Laplace Transform of Fractional-Order Integrals

Consider the RL definition of fractional-order integral for $\alpha > 0$ given by (2.2). The Laplace transform of $f(t)$ can be obtained by taking the Laplace transform of the convolution of two functions, $g(t) = t^{\alpha-1}$, and $f(t)$ as follows:

$$\mathcal{L}\{{}_0D_t^{-\alpha} f(t)\} = \mathcal{L}\{{}_0I_t^{\alpha} f(t)\} = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau = t^{\alpha-1} * f(t). \tag{2.18}$$

Notice that the Laplace transform of $t^{\alpha-1}$ is equal to

$$G(s) = \mathcal{L}\left\{g(t) = t^{\alpha-1}\right\} = \Gamma(\alpha)s^{-\alpha}. \tag{2.19}$$

Using (2.18) and (2.19) implies

$$\mathcal{L}\{{}_0D_t^{-\alpha} f(t)\} = s^{-\alpha} F(s). \tag{2.20}$$

It should be noted that the Laplace transform of the GL fractional-order integral is just the same as that of the RL one.

2.3.3 Laplace Transform of Fractional-Order Derivatives

To determine the Laplace transform of the RL fractional-order derivative of $f(t)$, assume that

$$ {}_0D_t^{\alpha} f(t) = g^n(t), \tag{2.21}$$

or

$$ g(t) = {}_0D_t^{-(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau; \ (n-1 \le \alpha \le n). \tag{2.22}$$

Applying (2.17) onto (2.22) yields

$$ \mathcal{L}\{ {}_oD_t^{\alpha} f(t)\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0). \tag{2.22}$$

Moreover, from (2.19), the Laplace transform of $g(t)$ is then equal to

$$ G(s) = s^{-(n-\alpha)} F(s) \tag{2.23}$$

and using (2.2) yields

$$ g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}} {}_0D_t^{-(n-\alpha)} f(t) = {}_0D_t^{\alpha-k-1} f(t). \tag{2.24}$$

Substituting from (2.23) and (2.24) into (2.22) yields the following Laplace transform of the RL fractional-order derivative for $\alpha > 0$:

$$ \mathcal{L}\{ {}_0D_t^{\alpha} f(t)\} = s^n G(s) - \sum_{k=0}^{n-1} s^k \left[f^{(n-k-1)}(0) \right]_{t=0}, \tag{2.25}$$

where $n - 1 \le \alpha < n$.

2.4 Fourier Transform of Fractional-Order Operators

2.4.1 Fundamentals of Fourier Transform

The Fourier transform of a continuous absolutely integrable function, $h(t)$, for $t \in (-\infty, \infty)$ and defined as follows:

$$H(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} \mathrm{e}^{-j\omega t} h(t) dt. \tag{2.26}$$

The corresponding original function, $h(t)$, can be restored from its Fourier transform $H(\omega)$ by means of the inverse Fourier transform:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \mathrm{e}^{+j\omega t} d\omega. \tag{2.27}$$

As in the case of Laplace transform, the original functions will be denoted with lower case letters, while their transforms with capital letters.

The Fourier transform of the convolution of two functions, $h(t)$ and $g(t)$, defined over $t \in (-\infty, \infty)$ equals to the product of their Fourier transforms, i.e.,

$$\mathcal{F}\{h(t)^* g(t)\} = H(\omega) G(\omega) \tag{2.28}$$

provided that $H(\omega)$ and $G(\omega)$ exist (here $G(\omega)$ is the Fourier transform of $g(t)$ function). Property (2.28) will be used to determine the RL Fourier transform of both fractional-order integrals and derivatives.

Another useful property of the Fourier transform often used to solve applied problems is the Fourier transform of the derivatives of $h(t)$. Namely if $h(t), h'(t), \ldots, h^{(n-1)}(t)$ tend to zero as $t \to \pm\infty$, then the Fourier transform of the nth derivative of $h(t)$ is equal to

$$\mathcal{F}\{h^n(t)\} = (-j\omega)^n H(\omega). \tag{2.29}$$

This expression is true provided that the function $h(t)$ with all its derivatives up to and including the $(n-1)$th one tends to zero as $t \to \pm\infty$.

2.4.2 Fourier Transform of Fractional-Order Integrals

To find the Fourier transform of any function, first estimate the Fourier transform of the RL fractional-order integral. Let the lower limit be $a = -\infty$. Then,

$${}_{-\infty}D_t^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1} g(\tau) d\tau, \tag{2.30}$$

where $0 < \alpha < 1$.

Now consider the following subsidiary function:

$$h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

where, from (2.8), its Laplace transform can be calculated as

$$\mathcal{L}\{h(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-st}dt = s^{-\alpha}. \tag{2.31}$$

Suppose $s = j\omega$, where ω is real. According to Dirichlet theorem, the integral (2.31) converges if $0 < \alpha < 1$. Then, the Fourier transform of the following causal signal, $h_+(t)$:

$$h_+(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & (t > 0) \\ 0, & (t \le 0) \end{cases}$$

is given

$$\mathcal{F}\{h_+(t)\} \; = \; (j\omega)^{-\alpha}. \tag{2.32}$$

Hence, the Fourier transform of the RL fractional-order integral (2.30) can be found using (2.28) as the Fourier transform of the convolution of the functions $h_+(t)$ and $g(t)$ as follows:

$$h_+(t) * g(t) \; = \; {}_{-\infty}D_t^{-\alpha} f(t) \tag{2.33}$$

Consequently,

$$\mathcal{F}\{{}_{-\infty}D_t^{-\alpha}\, g(t)\} \; = \; (j\omega)^{-\alpha}\, G(\omega). \tag{2.34}$$

The expression (2.34) is also known as the Fourier transform of the GL fractional-order integral ${}_{-\infty}D_t^{-\alpha}\, g(t)$, which is the same as that of the RL definition of fractional-order integral.

2.4.3 Fourier Transform of Fractional-Order Derivatives

Let the lower limit in (2.4) be $a = -\infty$. Using integration by parts, (2.4) of the RL derivative yields

$$
{}_{-\infty}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} \frac{g^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau = {}_{-\infty}D_t^{\alpha-n} g^{(n)}(t). \tag{2.35}
$$

From (2.34) and (2.29) we obtain the following expression of the Fourier transform of the RL derivative of $\mathcal{F}\ \{{}_0D_t^{\alpha}g(t)\}$, i.e.,

$$
\begin{aligned}
\mathcal{F}\left\{{}_0D_t^{\alpha}g(t)\right\} &= (-j\omega)^{\alpha-n}\mathcal{F}\{g(t)\} \\
&= (-j\omega)^{\alpha-n}(-j\omega)^{n}G(\omega) = (-j\omega)^{\alpha}G(\omega).
\end{aligned} \tag{2.36}
$$

This expression is formally the same as the Laplace transform of the integer-order derivative by replacing s by $j\omega$.

2.5 Dynamics of Fractional-Order Transfer Functions

2.5.1 Fractional-Order Transfer Functions

An arbitrary dynamical system of fractional order can be described by the following fractional-order differential equation:

$$
\begin{aligned}
& a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \ldots + a_0 D^{\alpha_0} y(t) = \\
& = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \ldots + b_0 D^{\beta_0} u(t)
\end{aligned} \tag{2.37}
$$

One may assume, without loss of generality, that $0 \leq \alpha_0 < \alpha_1 < \alpha_2 \ldots < \alpha_n$, and $0 \leq \beta_0 < \beta_1 < \beta_2 \ldots < \beta_m$. It follows from (2.37) that the transfer function $Y(s)/U(s)$ is given by

$$
\frac{Y(s)}{U(s)} \equiv T(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}. \tag{2.38}
$$

Here $D^{\gamma} \equiv {}_0D_t^{\gamma}$; a_i $(i=0,\ldots, n)$, b_i $(i=0,\ldots, m)$ are constant coefficients, while α_i $(i=0,\ldots, n)$, β_i $(i=0,\ldots, m)$ are arbitrary real numbers.

2.5.2 Mittag-Leffler Function

The significance of Mittag-Leffler function lies in their importance of solving fractional-order differential equations, similar to the importance of the exponential functions for the case of integer-order differential equations. A one-parameter Mittag-Leffler function is defined as

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \ \alpha > 0, \tag{2.39}$$

where $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $x \in \mathbb{C}$.

When $\alpha = 1$

$$E_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \mathrm{e}^x. \tag{2.40}$$

This is why the Mittag-Leffler functions are also known as the generalized exponential function. The two-parameter Mittag-Leffler functions, which are a generalization of (2.40), take the following form:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \tag{2.41}$$

where α, $\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $x \in \mathbb{C}$.

2.5.3 Solving Fractional-Order Differential Equation (FoDEQ) Using Laplace Transform

The response of a linear system to an arbitrary input signal can be found by convolving the input signal with the system's unit impulse response. The unit impulse response describes the system behavior due to a unit pulse, which corresponds to the homogeneous solution of the differential equation that describes the system dynamics.

In engineering practice, it is preferable to determine the unit impulse responses for linear dynamical systems, or their responses due to an arbitrary input signal, in the frequency domain by using Laplace transform and its inverse.

In the latter case, the Laplace transform of the differential equation (2.37) is the system's transfer function $T(S)$ given by (2.38). If $U(s)$, $(\mathcal{L}\{u(t)\}$, is known, then the response of the linear system in the frequency domain equals to $Y(s) = T(\mathrm{s})U(s)$.

If the dynamic system is linear and of an integer order, then the transfer function and the system's response can be considered as rational transfer functions of complex frequencies. Thus, the most common practice in determining the system's response of linear systems in time domain is by writing it as a sum of simple exponential functions of the form

$$y(t) = \sum\nolimits_{i=0}^{n} k_i \mathrm{e}^{s_i t},$$

where k_i are the residues of $Y(s)$, and s_i, $i = 1, 2, \ldots, n$ are the roots of the characteristic polynomial $Y(s) = 0$.

The same approach can be used in the case of linear systems that obey fractional-order dynamics. In this regard, there is a problem to expand the fractional-order system transfer function into simple fractions and to figure out the inverse Laplace transform for each of them since the characteristic polynomial exhibits infinite number of roots.

To clarify this point, let us consider some examples of solving fractional-order differential equations using Laplace transform.

Example 1 Consider a system with the following transfer function:

$$H(s) = \frac{1}{s^\alpha} \tag{2.42}$$

Here α can be both fractional and integer number.

From (2.31), the inverse Laplace transform of this function is the unit pulse response of the system; that is,

$$h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0. \tag{2.43}$$

Example 2 Consider a linear fractional-order system with the following transfer function:

$$H(s) = \frac{1}{s^\alpha - a}. \tag{2.44}$$

This function can be represented as a sum of the first q components of a geometric series as follows:

$$H(s) = \frac{1}{s^\alpha - a} = \frac{1}{s - a^q}\sum_{k=1}^{q} a^{k-1} s^{(1-k\alpha)}, \tag{2.45}$$

where $q = 1/\alpha$.

Taking the inverse Laplace transform of (2.45) yields

$$h(t) = \frac{1}{a}\sum_{k=1}^{q} a^k E_{1-k\alpha}(t, a^q). \tag{2.46}$$

Here $E_{1-k\alpha}(t, a^q) = \mathrm{e}^{a^q t}\sum_{k=0}^{\infty}\binom{1-k\alpha}{m}\frac{t^{-1+k\alpha+m}}{\Gamma(k\alpha+m)}a^{qm}$ is the Mittag-Leffler function.

Hence, the inverse Laplace transform of (2.44) using simple fractions expansion makes it possible to derive the time domain system's response as a sum of weighted Mittag-Leffler functions (i.e., generalized exponents).

2.6 Fractional-Order Electrical and Electronic Systems

2.6.1 Semi-infinite Transmission Line

Let us consider a homogeneous electric transmission line of length l. Let R, C, G, and L be the line parameters. The sending end of the line ($x = 0$) is supplied with electromotive force $e_0(t)$. The termination end of the line ($x = l$) is connected to a load Z. That described above is shown in Fig. 2.1.

Let $v(x, t)$ be the voltage difference between a point x on the line and the common rail. Let also $i(x, t)$ be the value of the current at point x and at the time moment t. The voltage drop across the unit length is a sum of the inductance voltage drop and the ohmic voltage drop:

$$\frac{\partial v(x,t)}{dx} = -L\frac{\partial i(x,t)}{dt} - Ri(x,t). \tag{2.47}$$

The current change in this area is a sum of the insulation leakage current and the capacitive leakage current:

$$\frac{\partial i(x,t)}{dx} = -C\frac{\partial v(x,t)}{dt} - Gv(x,t). \tag{2.48}$$

Taking the time Laplace transform of the (2.47) and (2.48) yields the following systems of equations:

$$\frac{dV(x,s)}{dx} = -(sL + R)I(x,s), \tag{2.49a}$$

$$\frac{dI(x,s)}{dx} = -(sC + G)V(x,s). \tag{2.49b}$$

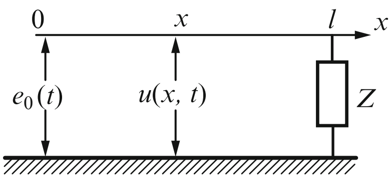

Fig. 2.1 Circuit diagram of a homogeneous electric transmission line

Let $Z_1(p) = sL + R$, and $Y_1(p) = pC + G$. The system of two first-order equations given by (2.49a and 2.49b) can be described as a single second-order equations for both the voltage and current as follows:

$$\frac{dV^2(x,s)}{dx^2} - \gamma^2 V(x,s) = 0; \tag{2.50a}$$

$$\frac{dI^2(x,s)}{dx^2} - \gamma^2 I(x,s) = 0, \tag{2.50b}$$

where $\gamma^2 = Z_1(s)Y_1(s)$.

The general solution of equation (2.50a) is given by

$$V(x,\ s)\ =\ A\mathrm{e}^{\gamma x}\ +\ B\mathrm{e}^{-\gamma x} \tag{2.51}$$

and the general solution of equation (2.49a) is equal to

$$I(x,s) = \frac{\gamma\left(B\mathrm{e}^{-\gamma x} - A\mathrm{e}^{\gamma x}\right.}{Z_1(s)}. \tag{2.52}$$

where the constant coefficients, A and B, are determined from the boundary conditions as follows:

$$V(0,\ s)\ =\ A\ +\ B; \tag{2.53a}$$

$$V(l,\ s)\ =\ A\mathrm{e}^{\gamma l}\ +\ B\mathrm{e}^{-\gamma l}. \tag{2.53b}$$

It is obvious that $V(0,\ s) = E_0(s)$ as long as the load impedance equals $Z(s) = V(l,\ s)/I(l,\ s)$.

The second boundary condition can be expressed as follows:

$$A\mathrm{e}^{\gamma x} + B\mathrm{e}^{-\gamma x} = Z(s)\frac{\gamma\left(B\mathrm{e}^{-\gamma x} - A\mathrm{e}^{\gamma x}\right.}{Z_1(s)}. \tag{2.54}$$

where the impedance $Z_1(p)/\gamma$ is called the *wave impedance* or the *characteristic impedance* of the system, i.e.,

$$Z_c(s) = \sqrt{Z_1(s)/Y_1(s)} \tag{2.55}$$

Therefore, in order to calculate the constant coefficients, A and B, one has to solve the following system of equations:

$$A + B = E_0(s); \tag{2.56}$$

$$\left(\frac{z(s)}{z_c(s)}+1\right)Ae^{\gamma l}=\left(\frac{z(s)}{z_c(s)}-1\right)Be^{-\gamma l}. \tag{2.57}$$

If the line load impedance equals the line characteristic impedance (i.e., the case of matched transmission line without reflected waves), then it follows from (2.53) that

$$A = 0; B = E_0(s); V(s) = E_0(s)e^{-\gamma x}; I(s) = V(s) / Z_c(s).$$

If $L=0$ and $G=0$, we obtain a semi-infinite RC line, the characteristic impedance of which equals

$$Z_c(s)=\sqrt{\frac{R}{C}}S^{-1/2}, \tag{2.58}$$

and that implies

$$I(s)=\sqrt{\frac{C}{R}}s^{-1/2}V(s). \tag{2.59}$$

Then the time-domain fractional-order differential equation of the semi-infinite RC line will be

$$i(t)=\sqrt{\frac{C}{R}}D_t^{0.5}v(t). \tag{2.60}$$

According to (2.60), the current in such line is directly obtained from the half-order derivative of the voltage.

2.6.2 Electrochemistry

The principal objective of electrochemical analysis is to determine the concentration $\rho(x, t)$ of electrochemically active elements (for example, ions of some substance in the solution) on the electrode surface (at $x=0$). Direct measurement of $\rho(x, t)$ is quite difficult. But it is much easier to experimentally measure the surface current density $j_x(0, t)$, which relates with concentration $\rho(x, t)$ as follows:

$$j_x(0,t)=\left|-K\frac{\partial\rho(x,t)}{\partial x}\right|_{x=0}. \tag{2.61}$$

Using (2.61) requires solving the diffusion equation at the right half-plane (inside the electrode):

$$\frac{\partial \rho(x,t)}{\partial t} = K \frac{\partial^2 \rho(x,t)}{\partial x^2}.$$

The process of solving this equation can be simplified if the "square root" of the operators in both sides is found. Substituting $\frac{\partial \rho(x,t)}{dx}\Big|_{x=0}$ from (2.61) into the above equation yields

$${}_0D_t^{0.5}\rho(0,t) = K^{0.5}\frac{\partial \rho(x,t)}{\partial x}\Big|_{x=0} = -K^{-0.5}j_x(0,t).$$

Hence, the concentration of chemically active elements on the electrode surface is calculated as

$$\rho(0,\ t) = -K^{-0.5}j_x(0,\ t).$$

The last expression is considered as the basis of creating modern chemical analysis and devices.

2.6.3 Rough Surface Impedance

The performance characteristics of electrochemical devices and their internal processes are largely specified with the properties of the surface of the metallic electrode that contacts the liquid or a solid electrolyte. The simplest model that affects the edges of the alternating current through the system is a serial connection of an external electrical capacitor and an electrolyte impedance resistor. In this case, the real part of the impedance of this model obviously does not depend on frequency, and the imaginary part is inversely proportional to frequency.

However, when the roughness of the electrode surface is quite significant, the system's behavior varies, at least within the limited frequency range. The frequency dependence impedance is also influenced by an additional power term of the form $A(j\omega)^{-\eta}$ where η is between zero and one $(0 < \eta < 1)$.

The input impedance of the "rough electrode-electrolyte" interface is modeled with a special element in the equivalent circuit of its input impedance. Since the phase frequency response of such an element is constant, it is called a *constant phase element* (CPE). The index η of such a CPE depends on the surface roughness; that is, the smoother the surface the closer η to 1 and vice versa.

Studies of rough electrode surfaces by means of electronic microscopes revealed the fact that their images had no natural scale of length (just like in the case of fractal surfaces) and these images were the same at different rates of magnification.

To model the "rough electrode-electrolyte" system and to estimate its impedance, we will use a Cantor kernel regular fractal (see Fig. 1.5). Here all black parts symbolize an electrolyte contacting the metallic electrode (white color). Every

$$Z(j\omega) = R + \cfrac{1}{j\omega C + \cfrac{2}{aR + \cfrac{1}{j\omega C + \cfrac{2}{a^2R + \ldots}}}}.$$

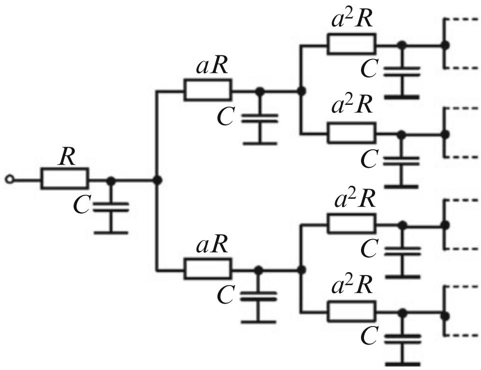

Fig. 2.2 Equivalent circuit of the "rough electrode-electrolyte" interface

stage of building the kernel for this system is as follows: the middle part of every segment is removed so that the length of each remaining segment equals $1/a$ ($a < 2$) of the initial segment length.

The current from the electrolyte towards the electrode encounters ohmic resistance of the electrolyte and the surface of electrical capacity at every area of the surface. The equivalent electrical circuit diagram of such system is shown in Fig. 2.2.

Every new stage of building of the Cantor manifold is reflected by the circuit branching. The impedance of every following branch increases a-fold as long as the thickness of the corresponding surface high point decreases a-fold. The number of capacity elements that model the surface capacities of the high points also increases at every following stage. All capacity elements have the same capacity value. The input impedance of the circuit depicted in Fig. 2.2 can be expressed with a continued fraction expansion (CFE) as follows:

The function $Z(\omega)$ can be written as the following scaling expression:

$$Z\left(\frac{\omega}{a}\right) = R + \frac{aZ(\omega)}{j\omega C \cdot Z(\omega) + 2}. \tag{2.62}$$

The expression of (2.62) is true when $Z(\omega) = A(j\omega)^{-\eta}$, where A is a constant coefficient, where $\eta = 1 - d$ and $d = \ln 2/\ln a$ represent the Cantor manifold fractal dimensions.

Hence, the circuit of Fig. 2.2 describes a constant phase element (CPE) as long as the Cantor manifold, d, satisfies $0 < d < 1$, and the parameter η is also limited between zero and one, $0 < \eta < 1$. Notice that $\eta = 3 - d_s$ for fractal dimension of the considered interface. When the surface is smooth, $d_s = 2$, which implies $\eta = 1$ and that coincides with the experimental results.

The power law dependence between impedance and frequency is specified with a reasonable set of resistive and capacitive current paths. Since capacitors block dc signals, a low-frequency signal propagates farther through the circuit before it "leaves" via the surface capacity, so the impedance is higher at lower frequencies. Real surfaces are self-similar only for scales of finite interval. This determines the frequency range within which the phase is constant.

The dependence between the effective complex value of the current and that of the voltage for rough surface (the generalized Ohm's law) is described as follows:

$$\dot{I}\,(j\omega) = \left[\dot{Z}\,(j\omega)\right]^{-1}\dot{U}\,(j\omega) = A^{-1}\cdot(j\omega)^{\eta}\dot{U}\,(j\omega).$$

The relation between the instantaneous current and voltage values will be

$$i(t) = A^{-1}{}_0D_t{}^{\eta}u(t).$$

The last expression shows that this dependence possesses fractional-order derivative in time domain.

Test questions

1. Explain why physical processes with "memory" are described in terms of fractional derivatives.
2. Give examples of physical and chemical processes that have hereditarity.
3. Formulate the concept of gamma-function.
4. Derive the Riemann-Liouville fractional integral expression via the integer-order multiple integral.
5. Derive the Grunwald-Letnikov fractional integral expression.
6. List the properties of fractional-order derivatives.
7. Develop the expression of the fractional-order integral (derivative) Laplace transform.
8. Demonstrate the relation between the fractional-order dynamic system differential equation and its transfer function.
9. Develop the Laplace transform method for solving the fractional-order differential equations.

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