Chapter 2
Quasi-Newtonian Fluids

Experimental observations show that the viscosity, as measured by viscosimeters, is constant for some fluids as water or honey, that are called Newtonian fluids (see Sect. 1.6, p. 21). Nevertheless, experiments show that the viscosity is no more constant for many others fluids, such as pastes or polymer solutions. These fluids are called non-Newtonian or complex fluids. Measurements showed that the viscosity varies upon the imposed rate of deformation. The simplest idea to describe complex fluids is to plot the viscosity measurements versus the imposed shear rate, and then, to fit the obtained curve with a simple template viscosity function, adjusting some few parameters. This is the main idea of quasi-Newtonian fluids models. Quasi-Newtonian fluids could be viewed as a first step inside the world of complex fluids: they could be considered as a simple approximation of more complex viscoplastic or viscoelastic fluids, that will be studied in Chaps. 3 and 4, respectively.

This chapter starts by reviewing some classical viscosity functions with few adjustable parameters. Then, the boundary value problem defining the non-Newtonian fluid is defined. The solution of both Poiseuille and Couette flows are then explicitly computed when using a simple viscosity function, the power law. When dealing with more general viscosity functions, geometries or flow condition, explicit computations are no more possible and we turn to the computation of an approximation of the solution. The problem is recast into the research of a saddle point of a Lagrangian and we obtain the variational formulation. Two algorithms are examined: the fixed point and the Newton methods. Algorithms are illustrated by some numerical computations for a Poiseuille flow in a pipe with a non-circular cross section.

2.1 Viscosity Function

Let us start with some few definitions.

Definition 2.1 (*tensor norm*)
For all $\boldsymbol{\tau} \in \mathbb{R}^{3\times 3}$, the following *tensor norm* is considered

$$|\boldsymbol{\tau}| = \left(\frac{\boldsymbol{\tau} : \boldsymbol{\tau}}{2}\right)^{\frac{1}{2}} = \left(\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{i,j}^2\right)^{\frac{1}{2}}$$

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Remark 2.1 (*choice of the tensor norm*)
An alternate definition of the tensor norm could be $|\boldsymbol{\tau}|_m = (\boldsymbol{\tau}:\boldsymbol{\tau})^{\frac{1}{2}}$. As $|\boldsymbol{\tau}|_m = \sqrt{2}|\boldsymbol{\tau}|$, these two norms are equivalent from a mathematical point of view and the only change is eventually the values of the material parameters involved in constitutive equations. Choosing $|.|$ instead of $|.|_m$ is motivated in this book by the following reason: for a simple shear flow, the diagonal components of $D(\boldsymbol{u})$ are zero while extra-diagonal components are counted twice in the $|.|_m$ norm, due to symmetry. The $1/2$ factor in the definition of the $|.|$ tensor norm deletes this twice counting effect. This simplification will be illustrated with the following Poiseuille and Couette examples, that are simple shear flows. Moreover, the choice of the $|.|$ tensor norm is conventional in rheology: values for material parameters of constitutive equations are estimated from experimental measurements on simple shear flows and using this choice for the tensor norm.

Definition 2.2 (*shear rate*)
The *shear rate*, denoted by $\dot{\gamma}$ is defined by $\dot{\gamma} = |2D(\boldsymbol{u})|$.

For simple shear flow, such as Poiseuille or Couette flows, $\dot{\gamma}$ coincide, up to the sign, with the time derivative of the shear deformation, often denoted as γ. For a general flow, $\dot{\gamma} = |2D(\boldsymbol{u})|$ is no more a time derivative of a quantity: this is just a conventional notation, used by extension. Recall (see Definition 1.9, p. 14) that the Cauchy stress tensor admits the decomposition $\boldsymbol{\sigma}_{\text{tot}} = \boldsymbol{\sigma} - p\boldsymbol{I}$ where $\boldsymbol{\sigma}$ is the stress deviator and p is the pressure.

Definition 2.3 (*quasi-Newtonian fluid*)
The fluid is said quasi-Newtonian when there exists a positive function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ called the *viscosity function*, such that the stress deviator $\boldsymbol{\sigma}$ expresses as

$$\boldsymbol{\sigma} = 2\eta(|2D(\boldsymbol{u})|^2)\ \text{dev}\,(D(\boldsymbol{u})) \tag{2.1}$$

Note that the constitutive equation is non-linear: this is a major difference with Newtonian fluids. The viscosity function is supposed to be known by some experimental measurements. This function is expressed here for convenience in terms of the square of the shear rate, instead of the shear rate directly. Without any loss of generality, this choice will simplify later some computations. The simplest viscosity function is defined by the power law.

Definition 2.4 (*power law viscosity*) The power law expresses the viscosity as

$$\eta(\xi) = K\xi^{\frac{-1+n}{2}}, \quad \forall \xi \in \mathbb{R}^+ \tag{2.2}$$

where $n > 0$ and $K \geqslant 0$ are given real constants, called respectively the *power index* and the *consistency*.

Note that when $n = 1$ this function is constant and the fluid is Newtonian. In the context of polymer solutions, the power-law was proposed in 1929 by Ostwald [229]

and this author called it the Waele-Ostwald power law. Independently, in the context of steel metallurgy, also in 1929, Norton [225] proposed a one-dimensional version of the power law and in 1954, Hoff [144] generalized it to the three-dimensional case, using a tensor norm., and this law is referred as the Norton-Hoff one. In the context of glaciers and ice sheet models for climate change prediction the power law is referred as the Glen's law (see e.g. [70, p. 85]). This viscosity function is useful because of its simplicity, but only approximately describes the behavior of a real non-Newtonian fluid. For example, if n were less than one, the power law predicts that the effective viscosity would decrease with increasing shear rate indefinitely, requiring a fluid with infinite viscosity at rest and zero viscosity as the shear rate approaches infinity, but a real fluid has both a minimum and a maximum effective viscosity that depend on the physical chemistry at the molecular level. Therefore, the power law is only a good description of fluid behavior across the range of shear rates to which the coefficients were fitted. There are a number of other models that better describe the entire flow behavior of shear-dependent fluids, but they do so at the expense of simplicity, so the power law is still used to describe fluid behavior, permit mathematical predictions, and correlate experimental data.

Definition 2.5 (*Carreau's law viscosity*)
The Carreau's law expresses the viscosity as

$$\eta(\xi) = \eta_\infty + (\eta_0 - \eta_\infty)(1 + \lambda\xi)^{\frac{-1+n}{2}}, \quad \forall \xi \in \mathbb{R}^+ \tag{2.3}$$

where $n > 0$ and $\eta_0 \geqslant 0$, $\eta_\infty \geqslant 0$ and $\lambda \geqslant 0$ are given real constants, satisfying $\eta_0 \geqslant \eta_\infty$ when $n \leqslant 1$ and $\eta_0 \leqslant \eta_\infty$ when $n \geqslant 1$.

The Carreau's law is represented on Fig. 2.1. The Carreau's law was first proposed in 1968 by Carreau et al. [42]. A popular extension to the Carreau's law is the Yasuda [318] law, often called the Yasuda-Carreau's law and a closely related model is the Cross law [69]. Such models are of common use for blood flows (see e.g. [1]) with biological applications.

Note that when $n = 1$ then both the power law and the Carreau's law reduces to a Newtonian fluid model where the viscosity is constant. When $n < 1$, the viscosity is decreasing with the shear rate and the fluid is said *shear thinning* or *pseudo-plastic*. An example is solutions of large, polymeric molecules in a solvent with smaller molecules. It is generally supposed that the large molecular chains tumble at random and affect large volumes of fluid under low shear, but that they gradually align themselves in the direction of increasing shear and produce less resistance. A common household example of a strongly shear-thinning fluid is styling gel. If one were to hold a sample of hair gel in one hand and a sample of corn syrup or glycerin in the other, they would find that the hair gel is much harder to pour off the fingers (a low shear application), but that it produces much less resistance when rubbed between the fingers (a high shear application). When $n > 1$, the viscosity is increasing with the shear rate and the fluid is said *shear thickening* or *dilatant*. Such fluids are rarely encountered, but one common example is an uncooked paste of

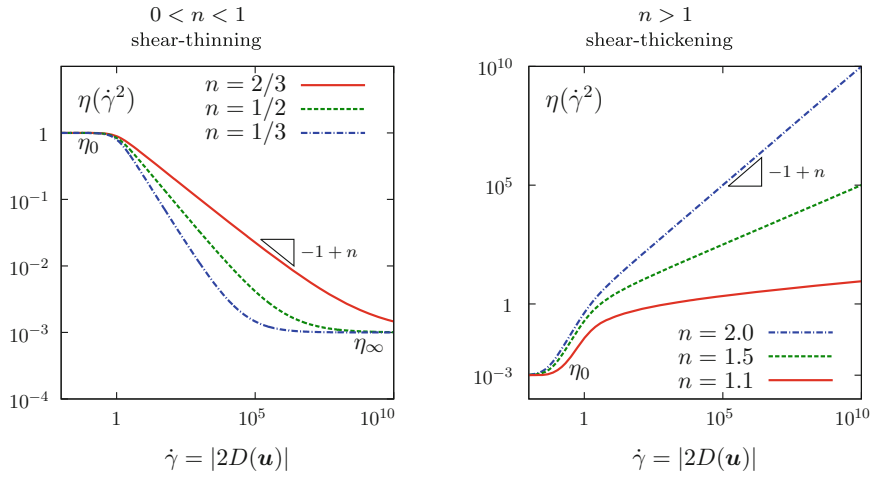

Fig. 2.1 Quasi-Newtonian flow: Carreau's law for the viscosity

cornstarch and water: under high shear rates the water is squeezed out from between the starch molecules, which are able to interact more strongly.

2.2 Problem Statement

As we consider here isothermal liquids, the density could be supposed as constant and the material as incompressible. The mass and momentum conservation equations (1.3) and (1.17) are completed with (1.22) for the Cauchy stress tensor and the constitutive equation (2.1) for quasi-Newtonian fluids. The set of equations is closed by appropriate initial and boundary conditions. The problem writes:

(P): find $\boldsymbol{u}$ and p, defined in $]0, T[\times\Omega$, such that

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla\boldsymbol{u}\right) - \mathbf{div}\left\{2\eta(|2D(\boldsymbol{u})|^2)\, D(\boldsymbol{u})\right\} + \nabla p = \rho\boldsymbol{g} \ \text{ in } \ ]0, T[\times\Omega \tag{2.4a}$$

$$-\operatorname{div}\boldsymbol{u} = 0 \ \text{ in } \ ]0, T[\times\Omega \tag{2.4b}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \ \text{ in } \ \Omega \tag{2.4c}$$

$$\boldsymbol{u} = \boldsymbol{u}_\Gamma \text{ on }]0, T[\times\partial\Omega \tag{2.4d}$$

2.3 Example: Poiseuille Flow

We consider a quasi-Newtonian fluid with a power law viscosity function. Let us reuse the notations of Sect. 1.5, p. 18 for the Poiseuille flow of a Newtonian fluid. Assume that the flow is laminar: the pressure is given by $p(x, y, z) = -fz + p_0$ where $f > 0$ and $p_0 \in \mathbb{R}$ are given. We first consider the case of a flow between two parallel plates. The velocity is parallel to the plates $\boldsymbol{u}(x) = (0, 0, u_z(x))$. Its gradient is given by

$$\nabla \boldsymbol{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_z' & 0 & 0 \end{pmatrix}$$

and the shear rate by $\dot{\gamma} = |2D(\boldsymbol{u})| = |u_z'|$. The stationary problem (2.4a)–(2.4d) reduces to

(P): find $u_z :]-L, L[\longrightarrow \mathbb{R}$ such that

$$\begin{cases} -\left(K|u_z'|^{n-1}u_z'\right)' = f \text{ in }]-L, L[\\ u_z(-L) = u_z(L) = 0 \end{cases}$$

This problem presents four physical parameters: n, K, L and f. In order to reduce the set of parameters, let us perform a dimensional analysis. As usual, the dimensionless quantities are denoted with tildes:

$$\tilde{x} = \frac{x}{L} \text{ and } \tilde{u}(\tilde{x}) = \frac{u_z(L\tilde{x})}{U}$$

where $U > 0$ will be chosen later. After this change of unknown, the problem becomes:

(P): find $\tilde{u} :]-1, 1[\longrightarrow \mathbb{R}$ such that

$$\begin{cases} -\dfrac{KU^n}{fL^{1+n}}\left(|\tilde{u}'|^{n-1}\tilde{u}'\right)' = 1 \text{ in }]-1, 1[\\ \tilde{u}(-1) = \tilde{u}(1) = 0 \end{cases}$$

Then, choose $U = (fL^{1+n}/K)^{1/n}$: the factor in the first equation disappears and there is only one physical parameter in this dimensionless problem, the power law index n. It means that, for a fixed n, the solution of the original problem is invariant up to a linear change of unknown. By using the symmetry of the geometry, the problem also reduces to the $]0, 1[$ interval. The dimensionless problem becomes:

(P): find $\tilde{u} :]0, 1[\longrightarrow \mathbb{R}$ such that

$$\begin{cases} -\left(|\tilde{u}'|^{n-1}\tilde{u}'\right)' = 1 \text{ in }]0, 1[\\ \tilde{u}'(0) = 0 \text{ and } \tilde{u}(1) = 0 \end{cases}$$

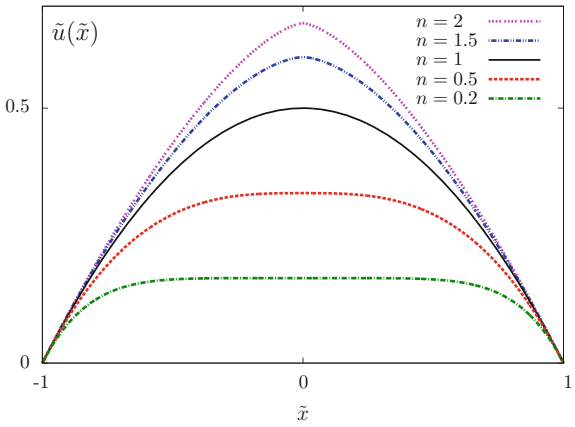

Fig. 2.2 Poiseuille flow with a power law viscosity

A first integration leads to: $-|\tilde{u}'|^{n-1}\tilde{u}' = \tilde{x} + c$ and $c = 0$ thanks to the symmetry condition $\tilde{u}'(0) = 0$. The previous relation leads to $\tilde{u}' < 0$ when $\tilde{x} \in]0, 1[$ and then $\tilde{u}' = -\tilde{x}^{1/n}$ when $\tilde{x} \in]0, 1[$. After a second integration and extending to $]-1, 1[$ by symmetry, we get $\tilde{u}(\tilde{x}) = \left(1 - |\tilde{x}|^{1+\frac{1}{n}}\right)/(1 + 1/n)$, for all $\tilde{x} \in]-1, 1[$. Finally, going back to dimensional quantities:

$$u_z(x) = \left(\frac{fL^{1+n}}{K}\right)^{\frac{1}{n}} \left(\frac{1 - \left(\frac{|x|}{L}\right)^{1+\frac{1}{n}}}{1 + \frac{1}{n}}\right), \quad \forall x \in]-L, L[.$$

Figure 2.2 plots the solution in dimensionless form for various values of the power law index n. When $n = 1$ we obtain the Newtonian solution already studied in Sect. 1.5, which is parabolic. Note that when $n < 1$, i.e. for shear thinning fluids, the solution is more flat: this is a very common experimental observation with many non-Newtonian fluids. Note also that when $n > 1$ the solution presents a peak at the center of the flow: the case $n > 1$ is less common. The case of a flow in a circular pipe requires similar computations: the solution is simply obtained by replacing f by $f/2$ and x by r in the previous expression.

2.4 Example: Couette Flow

We reuse the notations of Sect. 1.6, p. 21 for the Couette flow. For a laminar flow, the velocity writes $\boldsymbol{u} = (0, u_\theta(r), 0)$ in the cylindrical coordinate system (r, θ, z). By introducing the angular velocity $\omega(r)$ such that $u_\theta(r) = r\,\omega(r)$, the gradient of velocity writes

$$\nabla \boldsymbol{u} = \begin{pmatrix} 0 & 0 & 0 \\ r\omega' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The shear rate is $\dot{\gamma} = r|w'|$. There are two main experimental situations: imposed torque and imposed angular velocity.

For an imposed torque, the stationary problem reduces to

(P): find $\sigma_{r\theta}$ and ω, defined in $]r_1, r_2[$, and p, defined in $]r_2[\times]0, z_0[$, such that

$$\rho\,\omega^2 r - \frac{\partial p}{\partial r} = 0 \quad \text{in }]r_1, r_2[\times]0, z_0[\tag{2.5a}$$

$$-\frac{1}{r^2}\left(r^2\left(K|r\omega'|^{-1+n} r\omega'\right)\right)' = 0 \quad \text{in }]r_1, r_2[\tag{2.5b}$$

$$-\frac{\partial p}{\partial z} = -\rho g \quad \text{in }]r_1, r_2[\times]0, z_0[\tag{2.5c}$$

$$K|r_1\omega'(r_1)|^{-1+n} r_1\omega'(r_1) = -f \text{ and } \omega(r_2) = \omega_2 \tag{2.5d}$$

Here, ω_2 is the angular velocity of the outer cylinder and f is a stress boundary data related to the imposed torque (see Sect. 1.6, p. 21). The problem presents eight physical parameters: $n, K, f, \omega_2, r_1, r_2, \rho, g$. In order to reduce the set of parameters, let us perform a dimensional analysis. As usual, the dimensionless quantities are denoted with tildes:

$$\tilde{r} = \frac{r}{r_2}, \quad \tilde{z} = \frac{z}{r_2}, \quad \tilde{\omega}(\tilde{r}) = \frac{\omega(r_2\tilde{r}) - \omega_2}{W}$$

where the characteristic angular velocity is $W = (f/K)^{\frac{1}{n}}$. After this change of unknowns, the reduced problem becomes:

(P): find $\tilde{\sigma}$ and $\tilde{\omega}$, defined in $]\beta, 1[$, and $\tilde{p}$, defined in $]\beta, 1[\times]0, \gamma[$, such that

$$Re\,\tilde{r}\tilde{\omega}^2 - \frac{\partial \tilde{p}}{\partial \tilde{r}} = 0 \quad \text{in }]\beta, 1[\times]0, \gamma[\tag{2.6a}$$

$$-\frac{1}{\tilde{r}^2}\left(\tilde{r}^{n+2}|\tilde{\omega}'|^{-1+n}\tilde{\omega}'\right)' = 0 \quad \text{in }]\beta, 1[\tag{2.6b}$$

$$-\frac{\partial \tilde{p}}{\partial \tilde{z}} = -\frac{Re}{Fr^2} \quad \text{in }]\beta, 1[\times]0, \gamma[\tag{2.6c}$$

$$\beta^n|\tilde{\omega}'(\beta)|^{-1+n}\tilde{\omega}'(\beta) = -1 \text{ and } \tilde{\omega}(1) = 0 \tag{2.6d}$$

Clearly, in (2.6a)–(2.6d), the computation of ω is decoupled to those of p, that could be performed later, using (2.6a) and (2.6c). The reduced problem (2.6b) and (2.6d) for computing the angular velocity involves two dimensionless numbers: the power index n and the geometry confinement $\beta = r_1/r_2 \in]0, 1[$. The two equations (2.6a) and (2.6c) for the pressure involve three additional dimensionless numbers: the Reynolds $Re = \rho W^2 r_2^2/\Sigma = \rho f r_2^2/\eta^2$, the Froude $Fr = (W r_2)/\sqrt{g r_2} = (f/\eta)\sqrt{r_2/g}$

and the cylinder vertical extension $\gamma = z_0/r_2$. Thus, let us focus on the computation of $\tilde{\omega}$. Integrating (2.6b) yields $\tilde{r}^{n+2}|\tilde{\omega}'|^{-1+n}\tilde{\omega}' = -c$ where c is an integration constant and the Neumann boundary condition (2.6d) at $\tilde{r} = \beta$ leads to $c = \beta^2$. Then $|\tilde{\omega}'|^{-1+n}\tilde{\omega}' = -\beta^2\tilde{r}^{-2-n}$ and thus $\tilde{\omega}' < 0$ in $[\beta, 1]$. Taking the absolute value of the previous relation leads to $|\tilde{\omega}'|^n = \beta^2\tilde{r}^{-2-n}$ or equivalently $|\tilde{\omega}'| = \beta^{2/n}\tilde{r}^{-(2+n)/n}$ and putting back the sign of $\tilde{\omega}'$ yields $\tilde{\omega}' = -\beta^{2/n}\tilde{r}^{-1-2/n}$. A second integration, together with the Dirichlet boundary condition (2.6d) at $\tilde{r} = 1$ leads to

$$\tilde{\omega}(\tilde{r}) = \frac{n\beta^{2/n}}{2}\left(\tilde{r}^{-2/n} - 1\right), \quad \forall \tilde{r} \in [\beta, 1]$$

Going back to dimensional quantities leads to the following expression of the velocity

$$u_\theta(r) = \omega_2 r + \frac{nf^{1/n}r}{2K^{1/n}}\left(\left(\frac{r_1}{r}\right)^{2/n} - \left(\frac{r_1}{r_2}\right)^{2/n}\right), \quad \forall r \in [r_1, r_2]$$

The solution is represented on Fig. 2.3. Note that when n decreases, the solution flattens at the vicinity of the outer cylinder. Conversely, when n becomes large, the solution becomes close to a linear function. The computation of the pressure is similar to those of the Newtonian case (see Sect. 1.6, p. 21). Finally, the case of an imposed angular velocity is similar to the case of an imposed torque and is left as an exercise to the reader.

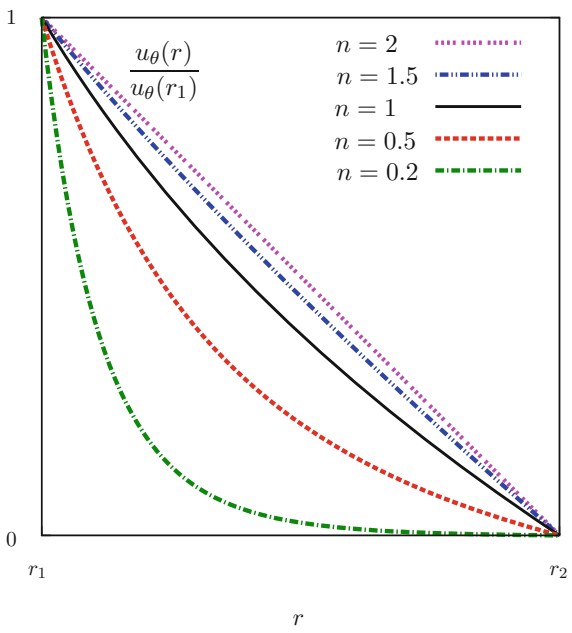

Fig. 2.3 Couette flow with a power law viscosity

For complex geometries or flow conditions, explicit solutions are no more possible and numerical approximation of the solutions is developed: this is the aim of the following sections.

2.5 Time Discretization

The methodology for the time discretization is then similar to those used for the Navier-Stokes equations, as presented in Sect. 1.7, p. 26. The method of characteristics is considered here for the discretization of the Lagrange derivative of the velocity by using a first order implicit scheme.

Algorithm 2.1 (*first order implicit scheme – quasi-Newtonian fluid*)

- $m = 0$: let $\boldsymbol{u}_0$, the initial condition, being given
- $m \geqslant 0$: let $\boldsymbol{u}_m$ being known, find $\boldsymbol{u}_{m+1}$ and p_{m+1} such that

$$\frac{\rho}{\Delta t}\boldsymbol{u}_{m+1} - \mathbf{div}\left\{2\eta\left(|2D\left(\boldsymbol{u}_{m+1}\right)|^2\right) D\left(\boldsymbol{u}_{m+1}\right)\right\} + \nabla p_{m+1} = \rho \boldsymbol{g} + \frac{\rho}{\Delta t}\boldsymbol{u}_m \circ X_m \text{ in } \Omega \tag{2.7a}$$

$$-\operatorname{div} \boldsymbol{u}_{m+1} = 0 \text{ in } \Omega \tag{2.7b}$$

$$\boldsymbol{u}_{m+1} = \boldsymbol{u}_\Gamma(t_{m+1}) \text{ on } \partial\Omega \tag{2.7c}$$

where X_m denotes the first order approximation of the characteristic line, defined by $X_m(\boldsymbol{x}) = \boldsymbol{x} - \Delta t\, \boldsymbol{u}_m(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$. Thus, at each step of this algorithm, there is a nonlinear subproblem to solve. Dropping the $m+1$ subscript for clarity, the subproblem writes:

(S): find $\boldsymbol{u}$ and p, defined in Ω, such that

$$\kappa\boldsymbol{u} - \mathbf{div}\left\{2\eta(|2D(\boldsymbol{u})|^2)\, D(\boldsymbol{u})\right\} + \nabla p = \boldsymbol{f} \text{ in } \Omega \tag{2.8a}$$

$$-\operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \tag{2.8b}$$

$$\boldsymbol{u} = \boldsymbol{u}_\Gamma \text{ on } \partial\Omega \tag{2.8c}$$

where $\kappa \geqslant 0$, $\boldsymbol{f}$ and $\boldsymbol{u}_\Gamma$ are given. For simplicity and without loss of generality, we suppose in the rest of this chapter that $\kappa = 0$: the extension to $\kappa > 0$ do not present any difficulty. The resulting subproblem correspond to the stationary solution of the quasi-Newtonian fluid flow problem when inertia terms are neglected. Instead of a first order scheme, a higher order one could be used, e.g. Algorithm 1.3, p. 29: at each iteration, the subproblem would be similar.

2.6 Minimization Problem

We are looking to reformulate problem (2.8a)–(2.8c) as a minimization one. Let

$$J(\boldsymbol{u}) = \int_\Omega \mathscr{D}\left(|2D(\boldsymbol{u})|^2\right)\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{u}\,\mathrm{d}x$$

where $\mathscr{D} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a scalar differentiable function called the *energy of dissipation* function. We are looking for this function $\mathscr{D}$ such that the solution of the stationary problem (2.8a)–(2.8c) is also the solution of the following minimization problem

$$\boldsymbol{u} = \underset{\boldsymbol{v}\in K(\boldsymbol{u}_\Gamma)}{\arg\inf}\, J(\boldsymbol{v}) \tag{2.9}$$

where $K(\boldsymbol{u}_\Gamma)$ denotes the kernel of the divergence operator, as introduced in (1.46), p. 33, i.e. the set of divergence-free vector fields that satisfies also the Dirichlet boundary condition. As $\mathscr{D}$ is differentiable, the J functional is also differentiable. Let us compute its Gâteaux derivative. For any vector field $\boldsymbol{v}$, we have: by definition

$$J'(\boldsymbol{u}).(\boldsymbol{v}) = \lim_{\varepsilon\to 0} \frac{J(\boldsymbol{u}+\varepsilon\boldsymbol{v}) - J(\boldsymbol{u})}{\varepsilon}$$

Recall that the tensor norm is defined by $|\boldsymbol{\tau}|^2 = \boldsymbol{\tau}:\boldsymbol{\tau}/2$. Then $|2D(\boldsymbol{u})|^2 = 2D(\boldsymbol{u}):D(\boldsymbol{u})$ and

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{u}}\left(|2D(\boldsymbol{u})|^2\right).(\boldsymbol{v}) = 4D(\boldsymbol{u}):D(\boldsymbol{v})$$

By expanding:

$$\begin{aligned} J(\boldsymbol{u}+\varepsilon\boldsymbol{v}) &= \int_\Omega \mathscr{D}\left(2D(\boldsymbol{u}+\varepsilon\boldsymbol{v}):D(\boldsymbol{u}+\varepsilon\boldsymbol{v})\right)\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.(\boldsymbol{u}+\varepsilon\boldsymbol{v})\,\mathrm{d}x \\ &= \int_\Omega \mathscr{D}\left(|2D(\boldsymbol{u})|^2\right)\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{u}\,\mathrm{d}x \\ &\quad + \varepsilon\left(\int_\Omega 4\mathscr{D}'\left(|2D(\boldsymbol{u})|^2\right)\,(D(\boldsymbol{u}):D(\boldsymbol{v}))\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{v}\,\mathrm{d}x\right) \\ &\quad + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Then, the directional derivative is identified as the second term in the expansion:

$$J'(\boldsymbol{u}).(\boldsymbol{v}) = \int_\Omega 4\mathscr{D}'\left(|2D(\boldsymbol{u})|^2\right)\,(D(\boldsymbol{u}):D(\boldsymbol{v}))\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{v}\,\mathrm{d}x \tag{2.10}$$

From (2.8a), multiplying by $\boldsymbol{v}$ such that div $\boldsymbol{v} = 0$ and $\boldsymbol{v} = 0$ on $\partial\Omega$ and integrating, we obtain the following variational formulation of the problem:

$$\int_\Omega 2\eta(|2D(\boldsymbol{u})|^2)\,(D(\boldsymbol{u}) : D(\boldsymbol{v}))\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{v}\,\mathrm{d}x = 0 \tag{2.11}$$

As J is differentiable, its minimum is characterized by $J'(\boldsymbol{u}).(\boldsymbol{v}) = 0$ for all div $\boldsymbol{v} = 0$ and $\boldsymbol{v} = 0$ on $\partial\Omega$. Thus, by identification between (2.10) and (2.11), we get:

$$\begin{aligned} &2\mathscr{D}'(\xi) = \eta(\xi),\ \forall \xi \in \mathbb{R}^+ \\ \Longleftrightarrow\ &2\mathscr{D}(\xi) = \frac{1}{2}\int_0^\xi \eta\,(\xi)\ \mathrm{d}\xi,\ \forall \xi \in \mathbb{R}^+ \end{aligned}$$

Let us compute the energy function for the two previous examples of viscosity functions.

Property 2.1 (power law viscosity)

$$\begin{aligned} \eta(\xi) &= K\xi^{\frac{-1+n}{2}} \\ \mathscr{D}(\xi) &= \frac{1}{2}\int_0^\xi K\xi^{\frac{-1+n}{2}}\,\mathrm{d}\xi = \frac{K}{1+n}\xi^{\frac{1+n}{2}} \end{aligned}$$

Property 2.2 (Carreau's law viscosity)

$$\begin{aligned} \eta(\xi) &= \eta_\infty + (\eta_0 - \eta_\infty)(1+\lambda\xi)^{\frac{-1+n}{2}} \\ \mathscr{D}(\xi) &= \frac{1}{2}\int_0^\xi \left(\eta_\infty + (\eta_0 - \eta_\infty)(1+\lambda\xi)^{\frac{-1+n}{2}}\right)\mathrm{d}\xi \\ &= \frac{\eta_\infty \xi}{2} + \frac{\eta_0 - \eta_\infty}{\lambda(n+1)}\left((1+\lambda\xi)^{\frac{1+n}{2}} - 1\right) \end{aligned}$$

The energy function is represented on Fig. 2.4 versus the shear rate $\dot{\gamma}$.

Let us consider the power low viscosity: the energy functional J involved by the minimization problem writes:

$$J(\boldsymbol{u}) = \frac{K}{1+n}\int_\Omega |2D(\boldsymbol{u})|^{1+n}\,\mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{v}\,\mathrm{d}x \tag{2.12}$$

For this case, we are able to be more precise about the required regularity of the test functions. Our aim is to ensure that the integrals involved in J are convergent, i.e. $J(\boldsymbol{u}) < +\infty$. The regularity assumptions are introduced via some *functional spaces*. First, let us introduce some standard Lebesgue and Sobolev functional spaces (see also [36]).

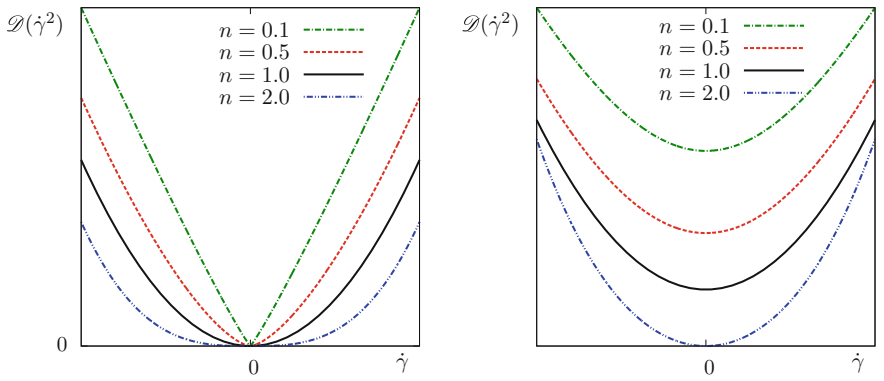

Fig. 2.4 Quasi-Newtonian fluid: energy function for the power law (*left*) and Carreau's law (*right*) viscosity functions

Definition 2.6 (*Lebesgue functional space*)

$$L^r(\Omega) = \left\{ \varphi : \Omega \to \mathbb{R}; \quad \int_\Omega |\varphi|^r \, \mathrm{d}x \; < \; +\infty \right\}, \quad \forall r \in [1, +\infty[$$

Note that this definition extends Definition 1.17, p. 32 to a general index r.

Definition 2.7 (*Sobolev functional space*)

$$W^{1,r}(\Omega) = \left\{ \varphi \in L^r(\Omega); \quad \nabla\varphi \in (L^r(\Omega))^3 \right\}, \quad \forall r \in [1, +\infty[$$
$$W_0^{1,r}(\Omega) = \left\{ \varphi \in W^{1,r}(\Omega); \quad \varphi = 0 \text{ on } \partial\Omega \right\}$$

Note that when $r = 2$ we have $W^{1,2}(\Omega) = H^1(\Omega)$, the Hilbert space introduced at Definition 1.18, p. 32. Note also that, in (2.12), the symmetric part of the gradient of velocity appears with a $r = 1 + n$ power index. Thus, the adequate functional space for the velocities is

$$V(\boldsymbol{u}_\Gamma) = \left\{ \boldsymbol{v} \in \left(W^{1,1+n}(\Omega)\right)^3; \quad \boldsymbol{v} = \boldsymbol{u}_\Gamma \text{ on } \partial\Omega \right\}$$

The associated kernel of the divergence operator writes:

$$K(\boldsymbol{u}_\Gamma) = \left\{ \boldsymbol{v} \in \left(W^{1,1+n}(\Omega)\right)^3; \quad \boldsymbol{v} = \boldsymbol{u}_\Gamma \text{ on } \partial\Omega \text{ and } \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \right\}$$

From the Sobolev trace theorem [36], a necessary and sufficient condition for the boundary condition $\boldsymbol{u}_\Gamma$ to satisfy $\boldsymbol{u}_\Gamma = \boldsymbol{u}_{|\partial\Omega}$ when $\boldsymbol{u} \in \left(W^{1,1+n}(\Omega)\right)^3$ is $\boldsymbol{u}_\Gamma \in \left(W^{1-\frac{n}{1+n},1+n}(\partial\Omega)\right)^3$. This is a regularity assumption on the boundary condition data of the problem. Note that the last term in (2.12), involves the integral

of $\boldsymbol{f}.\boldsymbol{v}$ for any $\boldsymbol{v} \in V(0) = \left(W_0^{1,1+n}(\Omega)\right)^3$. For this integral to be bounded, from a duality argument [36], this requires that the data $\boldsymbol{f} \in \left(W^{-1,1+n}(\Omega)\right)^3$. With this last assumption on the data of the problem, we are sure that all terms in the minimization have a sense.

Theorem 2.1 (existence and uniqueness of the stationary solution)
Assume that the viscosity function is given either by the power law or by the Carreau law. Then, the functional J is convex and problem (2.9) admits a unique solution. Moreover, this solution interprets as a weak solution of the stationary problem (2.8a)–(2.8c).

Proof The convexity of J follows from a Korn inequality [214] in the Sobolev space $W^{1,r}$ showed in 1971. In 1990 by Baranger and Najib [8] proved that the previous problem is well posed for the power law and the Carreau's one, and this result was improved in 2008 by Ervin et al. [95]. □

2.7 Saddle Point and Variational Formulations

In the previous section, we have shown that the minimization problem (2.9) is equivalent to find $\boldsymbol{u} \in K(\boldsymbol{u}_\Gamma)$ such that

$$\begin{aligned} & J'(\boldsymbol{u}).(\boldsymbol{v}) = 0, \ \forall \boldsymbol{v} \in K(0) \\ \Longleftrightarrow\ & a(\boldsymbol{u};\ \boldsymbol{u},\ \boldsymbol{v}) = \ell(\boldsymbol{v}),\ \forall \boldsymbol{v} \in K(0) \end{aligned}$$

where we have introduced the following forms

$$\begin{aligned} a(\bar{\boldsymbol{u}};\ \boldsymbol{u},\ \boldsymbol{v}) &= \int_\Omega 2\eta(|2D(\bar{\boldsymbol{u}})|^2)\, D(\boldsymbol{u}) : D(\boldsymbol{v})\, \mathrm{d}x, \quad \forall \bar{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v} \in \left(W^{1,1+n}(\Omega)\right)^3 \\ \ell(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, \mathrm{d}x, \quad \forall \boldsymbol{v} \in \left(W^{1,1+n}(\Omega)\right)^3 \end{aligned}$$

Let us define the bilinear form

$$b(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v}\, \mathrm{d}x$$

and the Lagrangian

$$\begin{aligned} L(\boldsymbol{u}, p) &= J(\boldsymbol{u}) + b(\boldsymbol{u}, p) \\ &= \int_\Omega \mathscr{D}\left(|D(\boldsymbol{u})|^2\right)\, \mathrm{d}x - \int_\Omega \boldsymbol{f}.\boldsymbol{u}\, \mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{u}\, \mathrm{d}x \end{aligned}$$

The pressure appears in the Lagrangian as a Lagrange multiplier for the incompressibility constraint. The corresponding term requires the integration of $p \operatorname{div} \boldsymbol{u}$ for any $\boldsymbol{u} \in \left(W^{1,1+n}(\Omega)\right)^3$. By duality, a necessary and sufficient condition for this integral to be bounded (see [36]) is $p \in L^{1+\frac{1}{n}}(\Omega)$.

Following Sect. 1.8, in order to circumvent the indetermination of the pressure, defined up to an additive constant when using Dirichlet boundary conditions, let us introduce the space of zero-average pressures:

$$L_0^{1+\frac{1}{n}}(\Omega) = \left\{ q \in L^{1+\frac{1}{n}}(\Omega); \quad \int_\Omega q \, \mathrm{d}x = 0 \right\}$$

Finally, the minimization problem is equivalent to the following saddle point problem:

$$(\boldsymbol{u}, p) = \underset{\boldsymbol{v} \in V(\boldsymbol{u}_\Gamma)}{\arg\inf} \quad \sup_{q \in L_0^{1+n}(\Omega)} L(\boldsymbol{v}, q)$$

As the Lagrangian is differentiable, convex in $\boldsymbol{u}$ and linear in p, its unique saddle point is characterized by:

find $(\boldsymbol{u}, p) \in V(\boldsymbol{u}_\Gamma) \times L_0^{1+\frac{1}{n}}(\Omega)$ such that

$$\frac{\partial L}{\partial \boldsymbol{u}}(\boldsymbol{u}, p).(\boldsymbol{v}) = \frac{\partial L}{\partial p}(\boldsymbol{u}, p).(q) = 0, \quad \forall (\boldsymbol{v}, q) \in V(0) \times L_0^{1+\frac{1}{n}}(\Omega)$$

that writes equivalently:

(FV): find $(\boldsymbol{u}, p) \in V(\boldsymbol{u}_\Gamma) \times L_0^{1+\frac{1}{n}}(\Omega)$ such that

$$\begin{cases} a(\boldsymbol{u};\, \boldsymbol{u},\, \boldsymbol{v}) + b(\boldsymbol{v},\, p) = \ell(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(0) \\ \quad b(\boldsymbol{u},\, q) \qquad\qquad\qquad\quad = 0, \quad \forall q \in L_0^{1+\frac{1}{n}}(\Omega) \end{cases}$$

This is the variational formulation of the problem. This problem is still non-linear: the form $a(.;\, .,\, .)$ is non-linear with respect to its first argument, due to the non-linearity of the viscosity function. The numerical treatment of this non-linearity requires some specific algorithms. Two of them are presented in the rest of this chapter: the fixed point algorithm and the Newton method.

2.8 Fixed Point Algorithm

Algorithm 2.2 (*fixed point*)

- $k = 0$: let $\boldsymbol{u}_0$ being given, e.g. the Newtonian solution
- $k \geqslant 0$: let $\boldsymbol{u}_{k-1}$ being known,

find $\boldsymbol{u}_k \in V(\boldsymbol{u}_\Gamma)$ and $p_k \in L_0^{1+\frac{1}{n}}(\Omega)$ such that

$$\begin{cases} a(\boldsymbol{u}_{k-1};\, \boldsymbol{u}_k,\, \boldsymbol{v}) + b(\boldsymbol{v},\, p_k) = (\boldsymbol{f},\, \boldsymbol{v}), & \forall \boldsymbol{v} \in V(0) \\ b(\boldsymbol{u}_k,\, q) = 0, & \forall q \in L_0^{1+\frac{1}{n}}(\Omega) \end{cases}$$

At the k-th step of this algorithm, there is a linear Stokes-like subproblem to solve. This subproblem is associated to a non-constant viscosity:

$$\eta_{k-1} = \eta\left(|2D(\boldsymbol{u}_{k-1})|^2\right)$$

where $\boldsymbol{u}_{k-1}$ is known from the previous step. Then, the form simply writes:

$$a(\boldsymbol{u}_{k-1};\, \boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega 2\eta_{k-1}\, D(\boldsymbol{u}) : D(\boldsymbol{v})\, \mathrm{d}x, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \left(W^{1,1+n}(\Omega)\right)^3$$

This form is linear with respect to its second and third arguments. All algorithms developed in the Chap. 1, such as the direct and the iterative one, are easily adaptable to this problem. Thus, the sequence $(\boldsymbol{u}_k)_{k \geqslant 0}$ is practically computable. Is this sequence convergent, i.e.

$$\lim_{k \to +\infty} \boldsymbol{u}_k = \boldsymbol{u} \ ?$$

The answer to this question is difficult and requires some additional tools. Let G be the following operator:

$$\begin{aligned} G : V(\boldsymbol{u}_\Gamma) &\longrightarrow V(\boldsymbol{u}_\Gamma) \\ \bar{\boldsymbol{u}} &\longmapsto \hat{\boldsymbol{u}} = G(\bar{\boldsymbol{u}}) \end{aligned}$$

where $\hat{\boldsymbol{u}} \in V(\boldsymbol{u}_\Gamma)$ is defined, together with $\hat{p} \in L_0^{1+\frac{1}{n}}(\Omega)$, for any $\bar{\boldsymbol{u}} \in V(\boldsymbol{u}_\Gamma)$ as the unique solution of the following linear Stokes-likes system:

$$\begin{cases} a\left(\bar{\boldsymbol{u}};\, \hat{\boldsymbol{u}},\, \boldsymbol{v}\right) + b\left(\boldsymbol{v},\, \hat{p}\right) = (\boldsymbol{f},\, \boldsymbol{v}), & \forall \boldsymbol{v} \in V(0) \\ b\left(\hat{\boldsymbol{u}},\, q\right) = 0, & \forall q \in L_0^{1+\frac{1}{n}}(\Omega) \end{cases}$$

Note that the operator G, which associate to each $\bar{\boldsymbol{u}}$ the solution $\hat{\boldsymbol{u}}$ of the previous linear system, is not itself a linear operator. Indeed, when replacing $\bar{\boldsymbol{u}}$ by $\lambda\bar{\boldsymbol{u}}$ for any $\lambda \in \mathbb{R}$, we do not get $\lambda\hat{\boldsymbol{u}}$ as solution of the corresponding problem, since the form $a(.;\, .,\, .)$ is not linear with respect to its first argument. This is due to the non-linearity of the viscosity function. The non-linear problem is then reformulated in a concise way, as

$$(P) : \text{find } \boldsymbol{u} \in V(\boldsymbol{u}_\Gamma) \text{ such that } G(\boldsymbol{u}) = \boldsymbol{u}$$

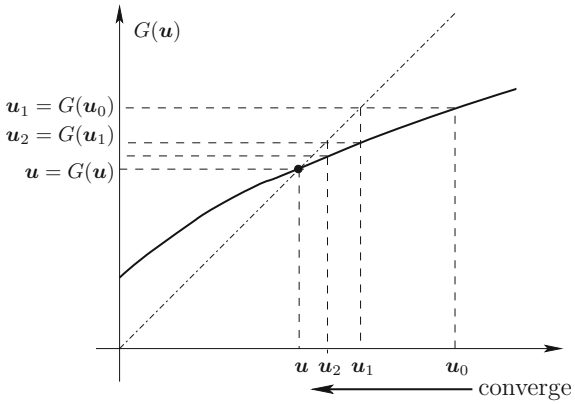

Fig. 2.5 Fixed point iteration: a convergent sequence

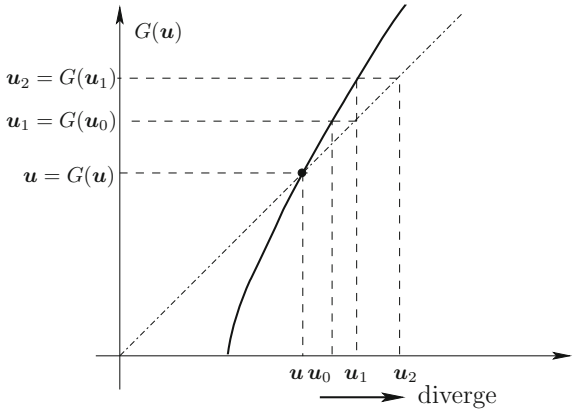

Fig. 2.6 Fixed point iteration: a divergent sequences

We says that the solution is a fixed point of G. Note also that $\boldsymbol{u}_k = G(\boldsymbol{u}_0)$ as G is the operator involved during one iteration. By recurrence, we have $\boldsymbol{u}_k = G^k(\boldsymbol{u}_0)$, i.e. the k-th iterate is obtained after k iterations of G.

Figures 2.5 and 2.6 present two typical sequences of the fixed point algorithm: one is convergent and the other is divergent. Let us try to elucidate the situation and find a necessary condition for the sequence to be convergent. Consider the following simple example:

$$\begin{aligned} G : \mathbb{R} &\longrightarrow \mathbb{R} \\ u &\longmapsto G(u) = \alpha u \end{aligned}$$

where $\alpha \in \mathbb{R}$. We have $u_k = G^k(u_0) = \alpha^k u_0$ and the sequence is convergent if and only if $|\alpha| < 1$. For the general case, the sequence is convergent when there exists a vicinity of $\boldsymbol{u}$ and containing $\boldsymbol{u}_0$ such that $|G'| < 1$ in this vicinity (see [319, p. 19]). In that case, we say that G is *contractive* in this vicinity. This result was due to Banach

in 1920. Nevertheless, in practice, and especially for our problem, it is difficult to prove the convergence of the sequence by this way. Numerical experimentations [284, Chap. 8] shows that the fixed point algorithm converges when the power index $n \in]0, 2[$ for a power law fluid. Thus, no all values of n are covered by this method. The following variant of the fixed point algorithms leads to improve the convergence rate of the algorithm and to build a convergent sequence for all power index.

Algorithm 2.3 (*Relaxed fixed point algorithm – abstract version*)

- $k = 0$: let $\boldsymbol{u}_0$ being given, e.g. the Newtonian solution
- $k \geqslant 0$: let $\boldsymbol{u}_{k-1}$ being known, compute

$$\boldsymbol{u}_k = \omega G\left(\boldsymbol{u}_{k-1}\right) + (1-\omega)\boldsymbol{u}_{k-1} \tag{2.13}$$

where $\omega > 0$ is a relaxation parameter.

This abstract algorithm involves the G operator: let us now unpack the G notation and get a practical version of the algorithm.

Algorithm 2.4 (*Relaxed fixed point algorithm – practical version*)
Let $\omega > 0$ be the relaxation parameter.

- $k = 0$: let $\boldsymbol{u}_0$ being given, e.g. the Newtonian solution
- $k \geqslant 0$: let $\boldsymbol{u}_{k-1}$ being known, let:

$$\eta_{k-1} = \eta\left(|2D\left(\boldsymbol{u}_{k-1}\right)|^2\right)$$
$$a\left(\boldsymbol{u}_{k-1};\, \boldsymbol{u},\, \boldsymbol{v}\right) = \int_{\Omega} 2\eta_{k-1}\, D(\boldsymbol{u}) : D(\boldsymbol{v})\, \mathrm{d}x$$

find $\boldsymbol{u}_* \in V\left(\boldsymbol{u}_\Gamma\right)$ and $p_k \in L_0^{1+\frac{1}{n}}(\Omega)$ such that

$$\begin{cases} a\left(\boldsymbol{u}_{k-1};\, \boldsymbol{u}_*,\, \boldsymbol{v}\right) + b\left(\boldsymbol{v},\, p_k\right) = \left(\boldsymbol{f},\, \boldsymbol{v}\right), & \forall \boldsymbol{v} \in V(0) \\ b\left(\boldsymbol{u}_*,\, q\right) = 0, & \forall q \in L_0^{1+\frac{1}{n}}(\Omega) \end{cases}$$

and set

$$\boldsymbol{u}_k = \omega \boldsymbol{u}_* + (1-\omega)\boldsymbol{u}_{k-1}$$

The last relation, that compute $\boldsymbol{u}_k$ as a linear combination of $\boldsymbol{u}_*$ and $\boldsymbol{u}_{k-1}$, interprets as an interpolation or extrapolation procedure, depending upon the value of ω. When, $0 < \omega < 1$, it is an *under-relaxation* method that slow down the changes from $\boldsymbol{u}_{k-1}$ to $\boldsymbol{u}_k$ by using and interpolation between $\boldsymbol{u}_k$ and $\boldsymbol{u}_*$. Conversely, when $\omega > 1$, it is an **over-relaxation** method, that increases the changes by using an extrapolation. Finally, when $\omega = 1$, we obtain the previous Algorithm 2.2 without relaxation. The relaxation could be used when $n > 2$ and then $0 < \omega < 1$ may be adjusted in order to obtain convergence. Also, when $n < 2$, adjusting $\omega > 1$ leads to improve the convergence rate, for an optimal efficiency. Numerical experimentations

[284, Chap. 8] shows that the optimal relaxation parameter ω_{opt} depends upon the power index n as $\omega_{\text{opt}} = \dfrac{2}{1+n}$. This value allows an optimal convergence rate of the relaxed fixed point algorithm for all values of the power index. Finally, observe that (2.13) writes equivalently

$$\frac{\boldsymbol{u}_k - \boldsymbol{u}_{k-1}}{\omega} + (I - G)(\boldsymbol{u}_{k-1}) = 0$$

Thus, the relaxation parameter interprets as a time step for an explicit first order scheme for the discretization of an evolutionary problem related to the $I - G$ operator. Increasing this time step as possible leads to reach faster the stationary solution, also solution of the fixed point problem. As the explicit scheme is only conditionally stable, there is a constraint upon the time step that should be taken sufficiently small.

2.9 Example: Poiseuille Flow in a Square Section

We consider the Poiseuille flow in pipe with a square cross section, as represented on Fig. 2.7 for a quasi-Newtonian fluid with a power law viscosity function. Assume that the flow is laminar: the pressure is given by $p(x, y, z) = -fz + p_0$ where $f > 0$ and $p_0 \in \mathbb{R}$ are given. The stationary velocity is parallel to the plates and depends only upon (x, y) in the section: $\boldsymbol{u}(x, y) = (0, 0, u_z(x, y))$. Its gradient is given by

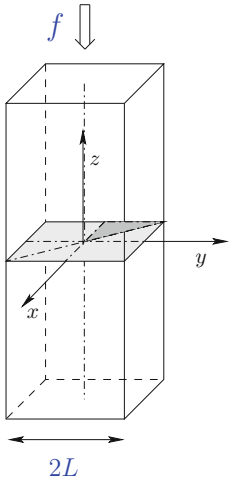

Fig. 2.7 Poiseuille flow in a square pipe section

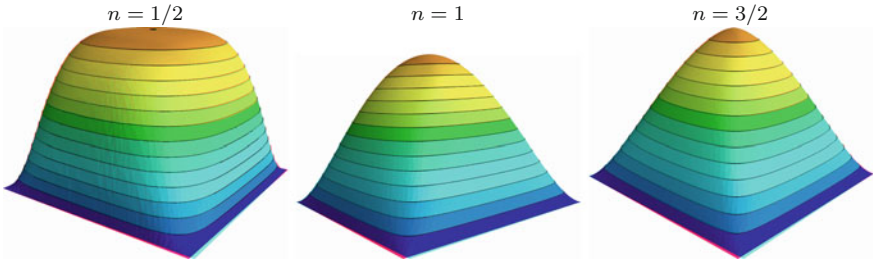

Fig. 2.8 Quasi-Newtonian fluids: Poiseuille flow in a square pipe section (from [284, Chap. 8])

$$\nabla \boldsymbol{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & 0 \end{pmatrix}$$

and the shear rate by $\dot{\gamma} = |2D(\boldsymbol{u})| = |\nabla u_z|$. Note that the inertia term $(\boldsymbol{u}.\nabla)\boldsymbol{u}$ is zero. Indeed, the velocity is in the z direction while ∇ contains just two components $\partial/\partial x$ and $\partial/\partial y$ in the (O, x, y) plane. Then the operator $\boldsymbol{u}.\nabla$ is zero. The stationary problem (2.8a)–(2.8c) reduces to a two-dimensional one and the computational domain is the section of the pipe, denoted by Ω.

(P): find u_z defined in Ω such that

$$-\operatorname{div}\left(K|\nabla u_z|^{-1+n}\right)\nabla u_z\Big) = f \ \text{ in } \ \Omega \tag{2.14a}$$

$$u_z = 0 \ \text{ on } \ \partial\Omega \tag{2.14b}$$

This is problem is often referred as the p-Laplacian problem, with $p = 1 + n$. The dimensional analysis is similar to those of Sect. 2.3 and it remains only one relevant parameter in this problem, the power index n. Figure 2.8 presents the solution: it shares some similarities with the Poiseuille flow between two parallel plates (see Fig. 2.2, p. 68). Here the solution is no more explicit and it has been approximated by a finite element method and the non-linear problem has been solved by the relaxed fixed point Algorithm 2.4. The implementation uses the finite element library `Rheolef` and the C++ code is presented with details for this computation in [284, Chap 8].

2.10 Newton Algorithm

Figure 2.9 plots on the left the convergence to zero of the residual terms versus iterations in logarithmic scale. The fixed point algorithm converges linearly. The aim of this section is to study another method that presents superior convergence properties: Fig. 2.9 plots on the right the convergence of the Newton method. The curve is no more a strait line and the convergence is said super-linear. This very fast

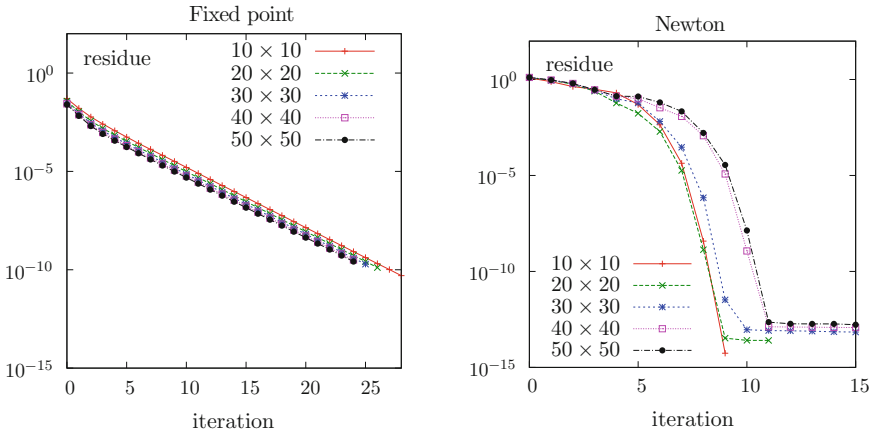

Fig. 2.9 Quasi-Newtonian fluids: convergence rate of the fixed point algorithm (*left*) and the Newton method (*right*), from [284, Chap. 8]

convergence of the residual terms to the machine precision is a definitive advantage of the Newton method, as compared with the fixed point algorithm. Note also that, for both algorithms, the number of iteration for reaching a prescribed precision is roughly mesh invariant.

Let:

$$F(\boldsymbol{u},p)=\begin{pmatrix}F_u(\boldsymbol{u},p)\\F_p(\boldsymbol{u},p)\end{pmatrix}=\begin{pmatrix}-\mathbf{div}\left\{2\eta(|2D(\boldsymbol{u})|^2)\,D(\boldsymbol{u})\right\}+\nabla p-\rho\boldsymbol{g}\\-\operatorname{div}\boldsymbol{u}\end{pmatrix} \tag{2.15}$$

With this notation, the problem is rewritten as

(P): find $\chi=(\boldsymbol{u},p)$ such that $F(\chi)=0$.

The principle of the Newton method is to build by recurrence a sequence $(\chi_k)_{k\geqslant 0}$. Figure 2.10 represents such a sequence, by plotting $F(\chi)$ versus χ. At step $k\geqslant 0$, assume that χ_k is known, and let us define χ_{k+1}. The tangent at the point $(\chi_k,F(\chi_k))$ is a line, i.e. the set of points (X,Y) such that

$$Y-F(\chi_k)=F'(\chi_k).(X-\chi_k)$$

where $F'(\chi).(\delta\chi)$ denotes the Gâteaux derivative at χ in a direction $\delta\chi$, defined by

$$F'(\chi).(\delta\chi)=\lim_{\varepsilon\to 0}\frac{F(\chi+\varepsilon\,\delta\chi)-F(\chi)}{\varepsilon}$$

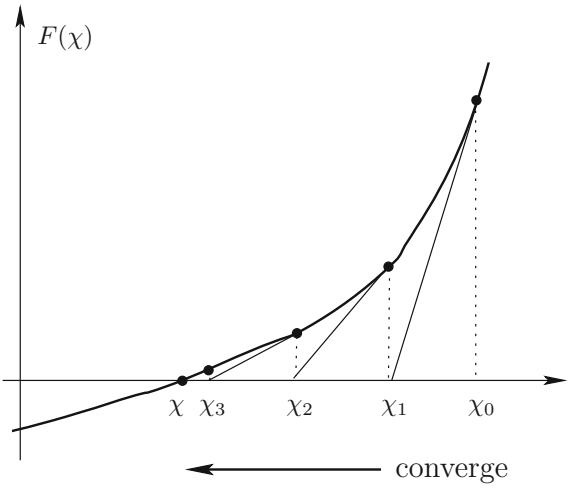

Fig. 2.10 Newton method: construction of a convergent sequence

The next iterate χ_{k+1} is defined such that the tangent line intersects the horizontal axis $Y = 0$ at $(X, Y) = (\chi_{k+1}, 0)$. Thus, χ_{k+1} is characterized by:

$$F'(\chi_k) \,.\, (\chi_{k+1} - \chi_k) = -F(\chi_k)$$

We are able to define an abstract version of the Newton method.

Algorithm 2.5 (*Newton algorithm – abstract version*)

- $k = 0$: let χ_0 being given
- $k \geqslant 0$: let χ_{k-1} being known, find $\delta\chi_k$ such that

$$F'(\chi_k).(\delta\chi_k) = -F(\chi_k)$$

and then

$$\chi_{k+1} = \chi_k + \delta\chi_k$$

While Fig. 2.10 presents a convergent sequence, Fig. 2.11 shows that the sequence is not always convergent: it seems to depend upon its starting point χ_0 and the properties of F. A necessary condition is that $F'(\chi)$ is non-singular for each element χ of the sequence: otherwise the sequence is not computable. This condition is not sufficient: in order to study the convergence properties, some second order derivative information is needed. The classical standard form to include this second order information is via a Lipschitz condition on F' (see e.g. [79, p. 11] or [3]). Nevertheless, in practice, it is difficult to prove the convergence of the sequence by this way and numerical experimentations show that the convergence of the method is very affected by the choice of the starting point χ_0. An efficient remedy is a variant called

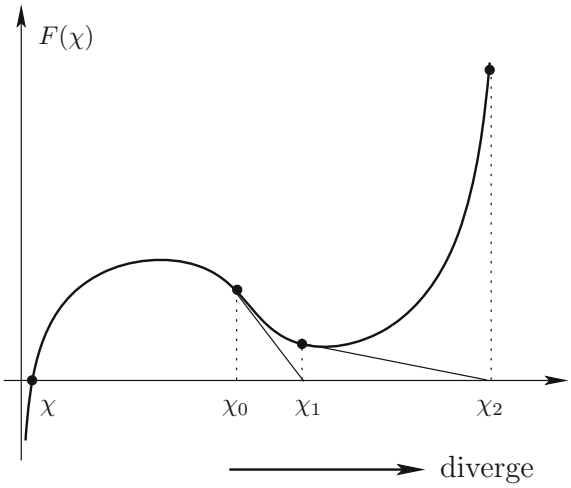

Fig. 2.11 Newton method: a divergent sequence

the *globalized Newton algorithm*, that guaranties that the sequence always converges from any starting point χ_0. This method simply modifies the last step of the algorithm as:

$$\chi_{k+1} = \chi_k + \lambda_k \delta\chi_k$$

where $\lambda_k \in]0, 1]$ is a damp coefficient. This coefficient is adjusted at each iteration and a practical and generic implementation can be found in the `Rheolef` library [284, Chap. 8]. Let us go back to our quasi-Newtonian fluid problem: the function F is defined by (2.15) and it remains to compute F', also called the Jacobian.

$$F'(\boldsymbol{u}, p).(\delta\boldsymbol{u}, \delta p) = \begin{pmatrix} \dfrac{\partial F_u}{\partial u}(\boldsymbol{u}, p).(\delta\boldsymbol{u}) + \dfrac{\partial F_u}{\partial p}(\boldsymbol{u}, p).(\delta p) \\ \dfrac{\partial F_p}{\partial u}(\boldsymbol{u}, p).(\delta\boldsymbol{u}) + \dfrac{\partial F_p}{\partial p}(\boldsymbol{u}, p).(\delta p) \end{pmatrix}$$

There are four directional derivatives to compute. Three of them are obtained without difficulty:

$$\begin{aligned} \frac{\partial F_p}{\partial p}(\boldsymbol{u}, p).(\delta p) &= 0 \\ \frac{\partial F_u}{\partial p}(\boldsymbol{u}, p).(\delta p) &= \nabla(\delta p) \\ \frac{\partial F_p}{\partial u}(\boldsymbol{u}, p).(\delta\boldsymbol{u}) &= -\operatorname{div}(\delta\boldsymbol{u}) \end{aligned}$$

The last one, $\dfrac{\partial F_u}{\partial u}(\boldsymbol{u}, p).(\delta \boldsymbol{u})$, requires more attention. We have

$$F_u(\boldsymbol{u}, p) = -\mathbf{div}\left\{2\eta(|2D(\boldsymbol{u})|^2)\, D(\boldsymbol{u})\right\}$$

Recall the definition of the tensor norm via the double dot product: $|\boldsymbol{\tau}|^2 = (\boldsymbol{\tau} : \boldsymbol{\tau})/2$. Then, we introduce an auxiliary function

$$f(\boldsymbol{u}) = |2D(\boldsymbol{u})|^2 = 2D(\boldsymbol{u}) : D(\boldsymbol{u})$$

Its directional derivative is

$$f'(\boldsymbol{u}).(\boldsymbol{v}) = 4D(\boldsymbol{u}) : D(\boldsymbol{v})$$

Let us introduce a second auxiliary function:

$$g(\boldsymbol{u}) = \eta\left(|2D(\boldsymbol{u})|^2\right) \;=\; (\eta \circ f)(\boldsymbol{u})$$

Its directional derivative is obtained by a derivation of the composition:

$$\begin{aligned} g'(\boldsymbol{u}).(\delta \boldsymbol{u}) &= (\eta' \circ f)(\boldsymbol{u})\; f'(\boldsymbol{u}).(\delta \boldsymbol{u}) \\ &= 4\eta'\left(|2D(\boldsymbol{u})|^2\right)\; (D(\boldsymbol{u}) : D(\delta \boldsymbol{u})) \end{aligned}$$

Finally, F_u expresses as:

$$F_u(\boldsymbol{u}, p) = -\mathbf{div}\left\{2g(\boldsymbol{u})\, D(\boldsymbol{u})\right\}$$

Its directional derivative is obtained by a derivation of the product:

$$\begin{aligned} \frac{\partial F_u}{\partial u}(\boldsymbol{u}, p).(\delta \boldsymbol{u}) &= -\mathbf{div}\left\{(2g'(\boldsymbol{u}).(\delta \boldsymbol{u}))\, D(\boldsymbol{u}) + 2g(\boldsymbol{u})\, D(\delta \boldsymbol{u})\right\} \\ &= -\mathbf{div}\left\{8\eta'(|2D(\boldsymbol{u})|^2)\, (D(\boldsymbol{u}) : D(\boldsymbol{v}))\, D(\boldsymbol{u})\right. \\ &\qquad \left. + 2\eta(|2D(\boldsymbol{u})|^2)\, D(\boldsymbol{v})\right\} \end{aligned}$$

Then, grouping, we obtain an expression of the Jacobian:

$$\begin{aligned} &F'(\boldsymbol{u}, p)(\delta \boldsymbol{u}, \delta p) \\ &= \begin{pmatrix} -\mathbf{div}\left\{2\eta(|2D(\boldsymbol{u})|^2)\, D(\delta \boldsymbol{u}) + 8\eta'(|2D(\boldsymbol{u})|^2)\, (D(\boldsymbol{u}) : D(\delta \boldsymbol{u}))\, D(\boldsymbol{u})\right\} + \nabla(\delta p) \\ -\operatorname{div}(\delta \boldsymbol{u}) \end{pmatrix} \end{aligned}$$

From the abstract version of the Newton algorithm, we obtain a concrete version for quasi-Newtonian fluids.

Algorithm 2.6 (*Newton algorithm – quasi-Newtonian fluids*)

- $k = 0$: let $(\boldsymbol{u}_0, p_0)$ being given, e.g. a Newtonian solution
- $k \geqslant 0$: let $(\bar{\boldsymbol{u}}, \bar{p}) = \left(\boldsymbol{u}^k, p_k\right)$ being the known current solution

1) compute the residue $(\boldsymbol{r}_u, r_p) = F(\bar{\boldsymbol{u}}, \bar{p})$
2) then, find the correction $(\delta\boldsymbol{u}, \delta p)$ such that

$$\begin{aligned} -\mathbf{div}\left\{2\eta(|2D(\bar{\boldsymbol{u}})|^2)\, D(\delta\boldsymbol{u}) + 8\eta'(|2D(\bar{\boldsymbol{u}})|^2)\, (D(\bar{\boldsymbol{u}}) : D(\delta\boldsymbol{u}))\; D(\bar{\boldsymbol{u}})\right\} & \\ +\nabla(\delta p) &= -\boldsymbol{r}_u \text{ in } \Omega \\ -\operatorname{div}(\delta\boldsymbol{u}) &= -r_p \text{ in } \Omega \\ \delta\boldsymbol{u} &= 0 \text{ on } \partial\Omega \end{aligned}$$

3) finally, apply the correction

$$\begin{aligned} \boldsymbol{u}_{k+1} &= \boldsymbol{u}_k + \delta\boldsymbol{u} \\ p_{k+1} &= p_k + \delta p \end{aligned}$$

This algorithm performs at each iteration three step. The first one is the computation of the residue, that could be used for a stopping criterion. The second step solves a linear system in order to compute the correction $(\delta\boldsymbol{u}, \delta p)$: this is the more computing time consuming step of the algorithm. The last step is simply an update, applying the correction.

The linear system concentrates now all our attention. In order to easily discretize this system, e.g. by a finite element method, we aim at obtaining a variational formulation. Let us introduce the following forms:

$$\begin{aligned} \tilde{a}(\bar{\boldsymbol{u}}; \delta\boldsymbol{u}, \delta\boldsymbol{v}) = & \int_\Omega 2\eta(|2D(\bar{\boldsymbol{u}})|^2)\, (D(\delta\boldsymbol{u}) : D(\delta\boldsymbol{v}))\, \mathrm{d}x \\ & + \int_\Omega 2\eta'(|2D(\bar{\boldsymbol{u}})|^2)\, (2D(\bar{\boldsymbol{u}}) : D(\delta\boldsymbol{u}))\, (2D(\bar{\boldsymbol{u}}) : D(\delta\boldsymbol{v}))\, \mathrm{d}x \\ b(\delta\boldsymbol{v}, \delta q) = & -\int_\Omega \delta q \operatorname{div}(\delta\boldsymbol{v})\, \mathrm{d}x \\ \ell_u(\delta\boldsymbol{v}) = & -\int_\Omega \boldsymbol{r}_u . \delta\boldsymbol{v}\, \mathrm{d}x \\ \ell_p(\delta q) = & -\int_\Omega r_p\, \delta q\, \mathrm{d}x \end{aligned}$$

The variational formulation of the linear problem involved by the Newton method writes:

(FV): find $\delta\boldsymbol{u} \in \left(H_0^1(\Omega)\right)^3$ and $\delta p \in L^2(\Omega)$ such that

$$\begin{cases} \tilde{a}(\bar{\boldsymbol{u}};\ \delta\boldsymbol{u},\ \delta\boldsymbol{v}) + b(\delta\boldsymbol{v},\ \delta p) = \ell_u(\delta\boldsymbol{v}), & \forall \delta\boldsymbol{v} \in \left(H_0^1(\Omega)\right)^3 \\ b(\delta\boldsymbol{u},\ \delta q) = \ell_p(\delta q), & \forall \delta q \in L^2(\Omega) \end{cases}$$

This problem appears as Stokes-like one. As the $a(.;.,.)$ involves some complex expressions, it is not clear whether is expression is well defined and if the problem is well posed. In order to give an answer to this question, we introduce some additional mathematical tools.

Definition 2.8 (*Lebesgue space*)

$$L^\infty(\Omega) = \left\{ \varphi : \Omega \to \mathbb{R};\ \sup_{\boldsymbol{x}\in\Omega} \text{ess}\ |\varphi(\boldsymbol{x})| < +\infty \right\}$$

This is the space of bounded functions in Ω, at least almost everywhere. Note that this definition extends the space $L^r(\Omega)$, introduced at Definition 2.6, p. 74, to the case $r = \infty$.

Definition 2.9 (*Sobolev space*)

$$W^{1,\infty}(\Omega) = \left\{ \varphi \in L^\infty(\Omega);\ \nabla\varphi \in \left(L^\infty(\Omega)\right)^3 \right\}$$

This is the space of functions with bounded gradients in Ω, at least almost everywhere. Also, this definition extends the space $W^{1,r}(\Omega)$, introduced at Definition 2.7, p. 74, to the case $r = \infty$. See [36] for complements about these spaces. Note that when $\bar{\boldsymbol{u}} \in \left(W^{1,\infty}(\Omega)\right)^3$, then $\tilde{a}(\bar{\boldsymbol{u}};\ \delta\boldsymbol{u},\ \delta\boldsymbol{v})$ is well defined, i.e. the integrals involved in the definition of $\tilde{a}$ are bounded for any $\delta\boldsymbol{u},\ \delta\boldsymbol{v} \in \left(H_0^1(\Omega)\right)^3$.

Theorem 2.2 (well-posedness of the subproblem)
Let $\bar{\boldsymbol{u}} \in \left(W^{1,\infty}(\Omega)\right)^3$. Suppose that $\tilde{a}(\bar{\boldsymbol{u}};.;.)$ is coercive in $\left(H_0^1(\Omega)\right)^3$ with respect to its second and third arguments. Then the sequence defined by the Newton method is well-defined.

Proof Each iteration of the Newton method involves a Stokes-like linear system, that falls intro the framework of mixed problems. A sufficient condition for this system to be well-posed [38] is that the $\tilde{a}$ form is coercive with respect to its second and third arguments. As this property is assumed by the theorem, the linear system are always well-posed and the sequence defined by the Newton method is then well-defined. □

It remains to study the coercivity of the $\tilde{a}$ form.

Lemma 2.1 (coercivity for shear-thickening fluids)
Let $\bar{\boldsymbol{u}} \in \left(W^{1,\infty}(\Omega)\right)^3$. Assume $\eta' > 0$ and either $\eta(0) > 0$ or $\inf \text{ess}_\Omega\ |D(\bar{\boldsymbol{u}})| > 0$. Then the $\tilde{a}(\bar{\boldsymbol{u}};.;.)$ form is coercive with respect to its second and third arguments.

Proof Recall the definition of the tensor norm via the double dot product: $|\boldsymbol{\tau}|^2 = (\boldsymbol{\tau}:\boldsymbol{\tau})/2$. Then, from the definition of $\tilde{a}$, we have:

$$\begin{aligned}\tilde{a}(\bar{\boldsymbol{u}};\delta\boldsymbol{u},\delta\boldsymbol{u}) = &\int_\Omega \eta(|2D(\bar{\boldsymbol{u}})|^2)\ \left(\frac{2D(\delta\boldsymbol{u}):2D(\delta\boldsymbol{u})}{2}\right)\mathrm{d}x\\ &+\int_\Omega 2\eta'(|2D(\bar{\boldsymbol{u}})|^2)\ \left(\frac{2D(\bar{\boldsymbol{u}}):2D(\delta\boldsymbol{u})}{2}\right)^2\mathrm{d}x\\ = &\int_\Omega \eta(|2D(\bar{\boldsymbol{u}})|^2)\ |D(\delta\boldsymbol{u})|^2\,\mathrm{d}x\\ &+\int_\Omega 2\eta'(|2D(\bar{\boldsymbol{u}})|^2)\ \left(\frac{2D(\bar{\boldsymbol{u}}):2D(\delta\boldsymbol{u})}{2}\right)^2\mathrm{d}x \qquad (2.16)\end{aligned}$$

As $\eta' > 0$ we get

$$\begin{aligned}\tilde{a}(\bar{\boldsymbol{u}};\delta\boldsymbol{u},\delta\boldsymbol{u}) &\geqslant \int_\Omega \eta(|2D(\bar{\boldsymbol{u}})|^2)\ |D(\delta\boldsymbol{u})|^2\,\mathrm{d}x\\ &\geqslant \inf_{\boldsymbol{x}\in\Omega}\operatorname{ess}\eta(|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2)\ \ \|2D(\delta\boldsymbol{u})\|^2_{L^2(\Omega)}\\ &\geqslant c_0^2\ \ \inf_{\boldsymbol{x}\in\Omega}\operatorname{ess}\eta(|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2)\ \ \|\delta\boldsymbol{u}\|^2_{H^1(\Omega)}\end{aligned}$$

where $c_0 > 0$ is the constant of the Lemma 1.2, p. 34. As η is a strictly increasing and non-negative function, we have

$$\tilde{a}(\bar{\boldsymbol{u}};\delta\boldsymbol{u},\delta\boldsymbol{u}) \geqslant c_0^2\ \eta(\xi_0)\ \ \|\delta\boldsymbol{u}\|^2_{H^1(\Omega)}$$

with $\xi_0 = \inf\operatorname{ess}_{\boldsymbol{x}\in\Omega}|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2$. By the assumption of the lemma, either $\xi_0 > 0$ and then $\eta(\xi_0) > 0$ or $\xi_0 = 0$ and $\eta(0) > 0$. In all cases, the form is coercive, with a coercivity constant $\alpha = c_0^2\eta(\xi_0) > 0$. □

Lemma 2.2 (coercivity for shear-thinning fluids)
Let $\bar{\boldsymbol{u}} \in \left(W^{1,\infty}(\Omega)\right)^3$.

Let

$$\nu(\xi) = \eta(\xi) + 2\eta'(\xi)\,\xi,\ \ \forall\xi\in\mathbb{R}^+$$

Assume $\eta' < 0$ *and either* $\nu > 0$ *or* $\nu \geqslant 0$ *and* $\nu' < 0$. *Then the* $\tilde{a}(\bar{\boldsymbol{u}};.;.)$ *is coercive with respect to its second and third arguments.*

Proof From Cauchy-Schwartz inequality:

$$\begin{aligned}\left(\frac{2D(\bar{\boldsymbol{u}}):2D(\delta\boldsymbol{u})}{2}\right)^2 &\leqslant \left(\frac{2D(\bar{\boldsymbol{u}}):2D(\bar{\boldsymbol{u}})}{2}\right)^2\left(\frac{2D(\delta\boldsymbol{u}):2D(\delta\boldsymbol{u})}{2}\right)^2\\ &= |2D(\bar{\boldsymbol{u}})|^2\ \ |2D(\delta\boldsymbol{u}))|^2\end{aligned}$$

Then, from (2.16) and using $\eta' < 0$, we get:

$$\begin{aligned}\tilde{a}(\bar{\boldsymbol{u}};\delta\boldsymbol{u},\delta\boldsymbol{u}) &\geqslant \int_\Omega \left\{\eta(|2D(\bar{\boldsymbol{u}})|^2) + 2\eta'(|2D(\bar{\boldsymbol{u}})|^2)\ |2D(\bar{\boldsymbol{u}})|^2\right\}\ |2D(\delta\boldsymbol{u})|^2\,\mathrm{d}x\\ &= \int_\Omega \nu(|2D(\bar{\boldsymbol{u}})|^2)\ |2D(\delta\boldsymbol{u})|^2\,\mathrm{d}x\\ &\geqslant \inf_{\boldsymbol{x}\in\Omega}\operatorname{ess}\nu(|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2)\ \ \|2D(\delta\boldsymbol{u})\|^2_{L^2(\Omega)}\\ &\geqslant c_0^2\ \inf_{\boldsymbol{x}\in\Omega}\operatorname{ess}\nu(|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2)\ \ \|\delta\boldsymbol{u}\|^2_{H^1(\Omega)}\end{aligned}$$

Let $\xi_0 = \inf \operatorname{ess}_{\boldsymbol{x}\in\Omega}|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2$ and $\xi_1 = \sup \operatorname{ess}_{\boldsymbol{x}\in\Omega}|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2$. Since $\bar{\boldsymbol{u}} \in \left(W^{1,\infty}(\Omega)\right)^3$, we have $\xi_1 = \|2D(\bar{\boldsymbol{u}})\|^2_{L^\infty(\Omega)} < +\infty$. When $\nu > 0$ then the coercivity is immediate. Otherwise, assume $\nu \geqslant 0$ and $\nu' < 0$. Then, as ν is a strictly decreasing function:

$$\inf_{\boldsymbol{x}\in\Omega}\operatorname{ess}\nu(|2D(\bar{\boldsymbol{u}})(\boldsymbol{x})|^2) \geqslant \nu(\xi_1)$$

As ν is strictly decreasing and positive, and ξ_1 is finite, then $\nu(\xi_1) > 0$ and the proof is complete. □

Note that Lemma 2.2 for shear-thickening fluids requires a strong assumption on $\bar{\boldsymbol{u}}$, namely $\inf \operatorname{ess}_\Omega |D(\bar{\boldsymbol{u}})| > 0$. It means that the rate of deformation tensor may be always nonzero, which is not physically relevant. For instance $D(\bar{\boldsymbol{u}})$ could vanishes on boundaries or symmetry axis. Conversely, Lemma 2.2 do not need such assumption and the sequence of the Newton method to be well-defined with more restrictive assumptions. Recall that the most physically relevant case is the shear-thinning case and this is a good news for practical applications. Let us study whether these assumptions lemma are satisfied by the two classical examples of viscosity functions, namely the power law and the Carreau's law.

Corollary 2.1 (power law)
For the power law (2.2)*, when* $n < 1$*, the sequence defined by the Newton method is well-defined.*

Proof From (2.2), we have

$$\begin{aligned}\eta'(\xi) &= \frac{(-1+n)K}{2}\xi^{\frac{-3+n}{2}} < 0, \quad \forall \xi > 0\\ \nu(\xi) &= nK\xi^{\frac{-1+n}{2}} \geqslant 0, \quad \forall \xi \geqslant 0\\ \nu'(\xi) &= \frac{n(-1+n)K}{2}\xi^{\frac{-3+n}{2}} < 0, \quad \forall \xi > 0\end{aligned}$$

Then, assumptions of Lemma 2.2 applies and the result follows from Theorem 2.2. □

Corollary 2.2 (Carreau's law)
For the Carreau's law (2.2)*, when* $n < 1$ *or* $n > 1$ *and* $\eta_0 > 0$*, the sequence defined by the Newton method is well-defined.*

Proof From (2.3), we have

$$\begin{aligned}
\eta'(\xi) &= \frac{(-1+n)\lambda}{2}(\eta_0 - \eta_\infty)(1+\lambda\xi)^{\frac{-3+n}{2}} \\
\nu(\xi) &= \eta_\infty + n\lambda(\eta_0 - \eta_\infty)(1+\lambda\xi)^{\frac{-1+n}{2}} \geqslant 0, \quad \forall \xi \in \mathbb{R}^+ \\
\nu'(\xi) &= \frac{n(-1+n)\lambda^2}{2}(\eta_0 - \eta_\infty)(1+\lambda\xi)^{\frac{-3+n}{2}}
\end{aligned}$$

When $n > 1$ and $\eta_0 > 0$, assumptions of Lemma 2.1 applies. Otherwise, when $n < 1$, assumptions of Lemma 2.2 applies. In both cases, the form is coercive and the result follows from Theorem 2.2. □

2.11 Notes

Starting from a software that solves the Navier-Stokes equation, quasi-Newtonian models can be easily implemented via the fixed point algorithm by updating the viscosity function at each iteration. The relaxed version of the fixed point algorithm enables some improvements of the convergence. Nevertheless, for serious application, the Newton method is more efficient. Since the viscosity is expected to vary in a very large range, particularly for the power law viscosity function that is unbounded, the linear Stokes subproblems of both the fixed point and the Newton algorithms are very ill conditioned. Thus, the direct Stokes solver is the method of predilection for medium sized problems. However, for some large scale problems, e.g. three-dimensional ones, iterative solvers could be preferred: they should be combined with an efficient preconditioner that takes into account the large variations of the viscosity. Such iterative solvers are proposed by many authors [128, 147, 219] for the nonlinear Stokes problem or the Jacobian of the Newton method. Section 2.9 showed that quasi-Newtonian power law fluids could also be viewed as a generalization of the well-known p-Laplacian problem. This link with the p-Laplacian problem is fruitful, as many numerical methods was tested on the p-Laplacian problem. In 1983, Fortin and Glowinski [107, p. 173] suggested to solve this problem by an augmented Lagrangian approach, but, as convergence properties of this method bases on the fixed point theorem, there is few hope to improve significantly the fixed point algorithm. Recent works focused on the Newton method. For instance, inexact Newton methods, where the Jacobian is approximately solved, are investigated in 2013 by Ern and Vohralík [91, 92] for the p-Laplacian problem.

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