Chapter 2
Iconicity in Peirce's Semiotics

> To him who in the love of nature holds / Communion with her visible forms, she speaks / A various language.
>
> William Cullen Bryant, *Thanatopsis*

> The hypothesis of the mathematician is always the conception of a system of relations. In order that they may be reasoned about mathematically, these relations must be conceived as embodied in some kind of objects; but the character of the objects, apart from the relations, is utterly immaterial. They are always made as bare, skeleton-like, or diagrammatic as possible. With mathematicians not born blind, they are always visual objects of the simplest kind, such as dots, or lines, or letters, and the like. The mathematician often passes from one mode of embodiment to another. Such a change is no change in the hypothesis but only in the diagrammatic embodiment of the hypothesis. The hypothesis itself consists in the system of relations alone.
>
> C.S. Peirce, "On the Logic of Quantity" (1895)

2.1 Introduction

We have already treated the most important aspects of Peirce's theory of abduction in the previous chapter. It will suffice here to mention just two of those earlier results: the first is that abduction is conditioned by states and state-transitions of knowledge, ignorance and experience; the second is that abductive reasoning is particularly difficult to formalize. The present chapter continues this line of investigation by focusing on Peirce's semiotic notion of iconicity. We are motivated especially by the general and informal characterization of abduction as the response to a surprising event that instigates a broadening of cognitive horizon, an opening of more or less structured possibilities out of which one or more may be selected as potential "ways forward" in explanatory understanding.

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Peirce's notion of abductive reasoning may be understood to depend on iconicity in at least two respects: (α) to the extent that mathematical reasoning is taken as the paradigm of reasoning – for Peirce, all mathematical reasoning takes place through the use of diagrams and depends upon the discernment of relations internal to the diagram as well as (virtual) mappings from these relations to relations of possible tokens of similar structures; and (β) in the sense of scientific experiment, where experimental science from a pragmatist Peircean perspective is best understood in terms of diagrammatic experimentation on naturally occuring icons. Both (α) and (β) may be subsumed under the general heading of "model-based reasoning". In this way, from a Peircean perspective the relation of iconicity and abduction is coeval with the notion of model-based reasoning.

By closely examining the place of iconicity in Peirce's more general semiotic theory and in particular his notion of propositions as dicisigns, we are led to see how abductive reasoning with models depends on both iconic and indexical components. This relationship between iconic and indexical signs is instantiated both in concrete scientific investigation and abstract mathematical representation, thus indicating a common root to scientific and mathematical modelizations. The present chapter thus culminates with a formal construction based in presheaves that we call the Sheet of Indication, which is meant to serve as a quite general model for various types of diagrammatic experimentation. This construction makes explicit how iconic and indexical elements of diagrammatic reasoning are synthesized in a systematic way.

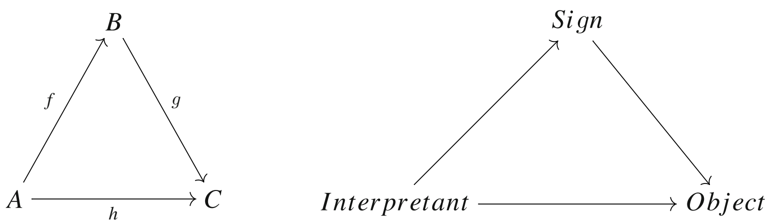
2.2 Peirce's Theory of Signs

2.2.1 Peirce's General SOI Schema

The basic unit of analysis for Peirce's doctrine of semiotics is, of course, the sign. For Peirce, a sign is itself conceived as an integral relation among three sign-components: (S) that which represents, (O) that which is represented, and (I) that in and for which the representational relation of S to O occurs. The letters are chosen in accordance with Peirce's own preferred terminology for these three components, or perhaps better, functional elements, of the sign: *sign* (at times, less intuitively but perhaps more precisely, *representamen*), *object* and *interpretant*.

Our purpose is neither to provide a comprehensive overview nor to attempt a synthetic reconstruction of Peirce's voluminous writings on semiotics. We intend primarily to resituate Peircean semiotics in light of categorical mathematics. In particular, the relational form of the sign as analyzed by Peirce is quite neatly captured by the convention of arrow-composition in category theory. Recall that one of the three axioms of category theory is that for any morphism f with domain A and codomain B and morphism g with domain B and codomain C, there is a unique morphism

h with domain A and codomain C such that h is identical to the composition of g following f. In the diagram below, $h = gf$. It is said that the diagram commutes. Note that in the juxtaposed diagram, the identical arrow-structure is evident and only the letters designating the objects are changed to conform to the S-O-I notation of Peirce's sign-components. What justifies the transference of the categorical notion of composition to the a priori unrelated Peircean notion of the triadic sign? Peirce's own formulations of the triadic sign-relation suggest that the irreducibility of the triadic relation that links sign, object and interpretant is a consequence of an internal coordination of relations between each of the three pairs of terms, such that the sign mediates a relation of interpretant to object.



In a 1909 letter intended for William James but never sent, Peirce describes the fundamental triadic relation in the following way ([1], p. 492):

> A Sign is a Cognizable that, on the one hand, is so determined (i.e., specialized, *bestimmt*) by something *other than itself*, called its Object (or, in some cases, as if the Sign be the sentence "Cain killed Abel," in which Cain and Abel are equally Partial Objects, it may be more convenient to say that which determines the Sign is the Complexus, or Totality, of Partial Objects. And in every case the Object is accurately the Universe of which the Special Object is member, or part), while, on the other hand, it so determines some actual or potential Mind, the determination whereof I term the Interpretant created by the Sign, that Interpreting Mind is therein determined mediately by the Object.

Here, Peirce uses the language of "determination" to order the three terms of the relation: the Object determines the Sign, the Sign determines the Interpreting Mind, and the Object thus "mediately" determines the Mind. The determination itself of the Mind by the Sign is here designated by Peirce as the Interpretant, a perhaps unnecessary complication. A new draft written two weeks later restates the definition like this ([1], p. 497):

> A *Sign* is anything of either of the three Universes which, being *bestimmt* by something other than itself, called its *Object*, in its turn *bestimmt* the mind of an interpreter to a notion which I call the *Interpretant*; *and does this in such a manner* that the Interpretant is, thereby and therein, determined mediately by the Object.

Here and elsewhere, Peirce characterizes signs both in terms of the three object-like components (S, O, I) that are themselves linked by the triadic sign-relation and in terms of the types of sub-relations (here of determination or *Bestimmung*) that in fact hold among these three components.[1] On the one hand, the triadic "mediating" relation that defines the sign as such for Peirce conforms nicely to the abstract relation-composition at the heart of category theory, its basic formal element. On the other hand, however, the specific *content* of each of Peirce's terms (sign, object and interpretant) provides a way to generate more definite characterizations of such triadic mediations, that is, various *types* of signs. It is to these further determinations that we now turn.

2.2.2 Three Trichotomies and Ten Classes of Signs

Peirce distinguishes three trichotomies of sign-types (see [1], Chap. 21 as well as commentaries in [2, 3]). Intuitively, these may be thought of as successive layers of determination proceeding from more simple and general to more complex, specific and differentiated. In effect, the three trichotomies derive from a fractal-like application of each of the three components of the sign-relation to the triadic sign-relation itself. Thus signs may be categorized in terms of their type of representamen, their type of object and their type of interpretant. More precisely, the determination in each case is grounded in distinct types of relation that hold, first of all as an identity-relation of the sign itself, secondly as the type of relation between sign and object, and finally as the type of meta-relation between the relation of the interpretant to the sign on the one hand and the relation of the interpretant to the object on the other.

The First Trichotomy: Qualisign, Sinsign, Legisign

The first of the three trichotomies characterizes types of sign based on the type of being of the sign-component itself. Peirce defines these three terms as follows ([1], p. 291):

> A *Qualisign* is a quality which is a sign. It cannot actually act as a sign until it is embodied; but the embodiment has nothing to do with its character as a sign.
>
> A *Sinsign* (where the syllable *sin* is taken as meaning "being only once," as in *single, simple*, Latin *semel*, etc.) is an actual existent thing or event which is a sign. It can only be so through its qualities; so that it involves a qualisign, or rather, several qualisigns. But these qualisigns are of a peculiar kind and only form a sign through being actually embodied.
>
> A *Legisign* is a law that is a sign. This law is usually established by men. Every conventional sign is a legisign. It is not a single object, but a general type which, it has been agreed, shall be significant. Every legisign signifies through an instance of its application, which may be termed a *Replica* of it. [...] The replica is a sinsign. Thus every legisign requires sinsigns. But these are not ordinary sinsigns, such as are peculiar occurrences that are regarded as significant. Nor would the replica be significant if it were not for the law which renders it so.

[1]The clearest well-known exemplar of this tendency is found in the discussion of triadic relations in Peirce's 1903 Syllabus. See [1], pp. 289–91.

Essentially, this first trichotomy is an ontological one. It corresponds so closely as to be effectively identical with Peirce's phenomenological categories of firstness, secondness and thirdness.[2] It is not clear that by restricting these categories as being applicable only to signs, Peirce is really making a distinction that makes any difference. At the very least Peirce flirts with the idea of equating semiotics with ontology (he speaks of "a universe perfused with signs, if it is not made entirely of signs" (cited in [4], p. 104)), and at any rate we do not seem to encounter anything in actual experience that is not composed solely of constituents of these three types.

Importantly, a qualisign is defined through its independence in principle from any actual embodiment. It is in this sense that it corresponds to sheer possibility as such. A sinsign, on the other hand, only exists as such through its actual instantiation of one or more qualisigns. It thus depends upon some collection of qualisigns to be what it is. Similarly, a legisign depends upon its instantiation through particular replicas, each of which is in itself a sinsign, but whose capacity to signify as they do is grounded not in their actual, particular existences but instead their belonging in common to a general type.

The Second Trichotomy: Icon, Index, Symbol

The next trichotomy is grounded in the type of relation that holds between the sign and the object of the sign. Thus the sign-types that emerge at this level are best understood as classifications of the characteristic dyadic relation that obtains between signs and their objects, namely the general relation of signification or representation as such. This general relation of signification admits, according to Peirce, of three fundamental and exhaustive species. This triad of signifying relation-types is generated by allowing the three basic Peircean categories of firstness, secondness and thirdness (or, equivalently, the essential characters of qualisigns, sinsigns and legisigns respectively) to act in a formally differentiating way on the "raw material" of the generic sign-object signifying relation. What thereby naturally emerge are three types or species of this relation itself: one that is "qualisign-like" or a Peircean "first", one that is "sinsign-like" or a "second", and lastly one that is "legisign-like" or a "third". Peirce's terms for these three types are then, in order, icons, indices and symbols ([1], pp. 291–2):

> An *Icon* is a sign which refers to the Object that it denotes merely by virtue of characters of its own and which it possesses, just the same, whether any such Object actually exists or not. […]
>
> An *Index* is a sign which refers to the Object that it denotes by virtue of being really affected by that Object. […]
>
> A *Symbol* is a sign which refers to the Object that it denotes by virtue of a law, usually an association of general ideas, which operates to cause the Symbol to be interpreted as referring to that Object. […]

[2] For analysis of these fundamental concepts of Peirce's philosophy, see among many similar discussions [1], pp. 4–5 and 160–1.

Thus, the specific type of relation that holds in any particular case between its sign and its object may be, in itself, qualisign-like, sinsign-like, or legisign-like. Crucially, the criterion in play here that distinguishes among icons, indices and symbols is applied not to the sign itself but to the relation between the sign and its object. In this way, the "same" triad of types is generated in a new way when lifted to the level of dyadic sign-object relations.

The Third Trichotomy: Rheme, Dicisign, Argument

Peirce's third trichotomy of signs stands with respect to the second trichotomy much as the second stands to the first. Whereas the second trichotomy lifts the three fundamental types of sign as distinguished by the first trichotomy (qualisign, sinsign, legisign) to the level of the dyadic relation between sign and object (icon, index, symbol), the third trichotomy lifts this same generic triad from the dyadic relation of sign and object to the triadic relation through which the representation of the object by the sign is in fact effective for some interpretant ([1], p. 292):

> A *Rheme* is a sign which for its Interpretant, is a sign of qualitative possibility, that is, is understood as representing such and such a kind of possible Object. […]
>
> A *Dicent Sign* [or *Dicisign*] is a sign which, for its Interpretant, is a sign of actual existence. […]
>
> An *Argument* is a sign which, for its Interpretant, is a sign of law.

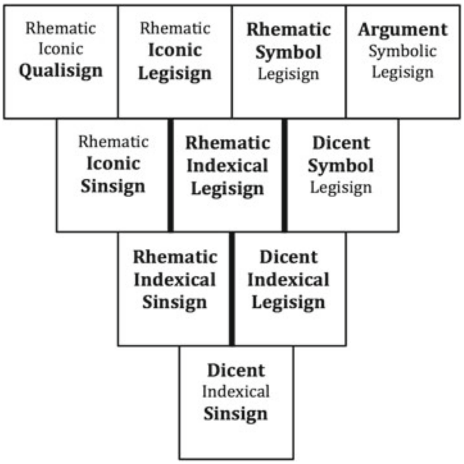
After presenting these three definitions, Peirce immediately restates them in slightly different and perhaps somewhat more helpful terms (ibid.):

> Or we may say that a Rheme is a sign which is understood to represent its Object in its characters merely; that a Dicisign is a sign which is understood to represent its Object in respect to actual existence; and that an Argument is a sign which is understood to represent its Object in its character as sign.

Peirce concedes that "these definitions touch upon points at this time much in dispute" and adjoins "a word […] in defense of them", discussing primarily the relation between the dicent sign and the form of propositional judgments, a proposition being "the matter upon which the act of judging is exercised", and the truth-tending orientation of any argument, the "law, in some shape, which the argument urges" (ibid.). For our purposes, the important point is the overall architectonic that structures the entire hierarchy of types: a reflexive application of the fundamental sign-triad to each of its constituent (and cumulatively determined) relations.

From Three Trichotomoies to Ten Classes

The three trichotomies combine to form ten structurally consistent classes of signs, as organized by Peirce in the following triangular table ([1], p. 296):



Peirce glosses this diagrammatic arrangement as follows (ibid.):

> The affinities of the ten classes are exhibited by arranging their designations in the triangular table here shown, which has heavy boundaries between adjacent squares that are appropriated to classes alike in only one respect. All other adjacent squares pertain to classes alike in two respects. Squares not adjacent pertain to classes alike in one respect only, except that each of the three squares at the vertices of the triangle pertains to a class differing in all three respects from the classes to which the squares along the opposite side of the triangle are appropriated. The lightly printed designations are superfluous.

We leave this fascinating typological puzzle to the reader's perusal without further comment.

2.3 Analyzing Iconicity

In the densely populated thicket of Peirce's semiotic theory, we are interested in the particular role of *iconicity*.

Peirce's notion of iconicity takes shape in the context of his second tripartite division of types of signs into icons, indexes and symbols. It is possible to discern three different kinds of definition of iconicity in Peirce's work, each of which corresponds to a certain point of view taken on iconic signs: as pure quality; as relational structure; and as epistemically determined as a support for abductive reasoning. This last conception will prove to be the most important one for the present argument by providing a natural way to link the semiotic notion of iconicity to the broad notion of model-based reasoning, which in turn serves as a useful framework for treating both scientific experimentation and mathematical theories of nature in a unified fashion.

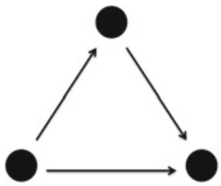
The following sections outline these three distinct conceptions of iconicity and provide a progressive sequence of examples from category theory to illustrate them. The basic notions of category theory needed to follow this section are outlined in the first Appendix. See also the discussion of Peircean semiotics and the categorical expression of diagrammatic reasoning in [4], Chaps. 3–4.

2.3.1 Iconicity as Quality

Peirce sometimes refers to icons as signs that signify what they do independently of any question of the real existence or non-existence of their objects. In other words, icons understood in this fashion are pure possibilities or free-floating qualities. To describe a quality, say redness, as free-floating is to emphasize its essential independence from any particular conditions of instantiation, such as the redness of this fabric here under present lighting conditions, or even any instantiation at all. From a Peircean point of view, this ensures that such qualities are utterly distinct from Lockean or Humean sensations, which are concrete perceptual events with definite qualitative properties, not the pure qualitative essences expressed in and by those events. The Peircean notion of iconicity as pure quality or essential possibility is in this respect quite close to the Husserlian phenomenological concept of *eidos*. See [5] for a detailed analysis of this connection.

The essential contrast here is with indices. An index "is *existent*, in that its being does not consist in any *qualities*, but in its effects – in its actually acting and being acted on, so long as this action and suffering endures" ([6] 6.318 cited in [7], p. 81). An icon, on the contrary, at least as conceived in its purest possible state, would be a quality or collection of qualities whose being consists of those very qualities themselves understood as independent of any actual effects. Without any reference to concrete instantiations in determinate domains, icons at this level are understood as immanent systems of relations. They are in this sense "structures", but structures conceived only in terms of their intrinsic determinations, not as in the next properly structural level which treats such systems of relational determination in terms of their possible instantiations, transformations and comparative mappings.

The following figure illustrates this purely qualitative conception of iconicity within category theory as exemplified by the finite category **3**. The category **3** consists of three otherwise undetermined objects and three non-identity arrows as pictured in the diagram. By the axioms of category theory, the diagram is thus forced to commute. Identity arrows on each of the three objects are implicit, that is, present in the category but for ease of presentation not depicted in the figure. This category naturally represents a linear order on three elements (including identities) with the elements themselves left unspecified.



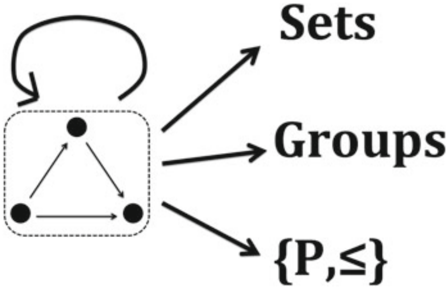
Note that this "abstract" category, that is, the kind of category whose objects are specified only as "mere dots" or structural placeholders rather than individuals of some type (such as, say, sets) with independently determined properties, provides a very natural representation of qualitative iconicity since the system of arrows and arrow-compositions that defines the category also exhausts the category's "content". There is simply nothing else to the category **3** than the linear order on three otherwise unspecified elements itself. In fact, *any* abstract category is qualitatively iconic in precisely this sense. Such categories simply *are* the systems of arrows and arrow-compositions that associate some class of otherwise undetermined objects in a determinate, systematic way.

2.3.2 Iconicity as Structure

A second more elaborated concept of iconicity understands icons in terms of the relational structure composing their parts. This notion of iconicity as structure corresponds quite closely to Peirce's own definition of icons in his second trichotomy in terms of the relation between the iconic sign and its signified object. In order for a sign and its object to share a qualitative or structural homology, it must be possible in principle to abstract the quality or structure at issue and to recognize it as the same (or relevantly similar) in the two semiotic roles.

In practice, it is difficult if not impossible to conceive of this second conception of iconicity as wholly distinct from the previous one. Instead, the conceptions of iconicity as quality and as structure should be understood as existing along a continuum with no absolutely determinate point of rupture separating the one from the other. This is largely because ordinary habits of cognition are immediately situated among complex strata of abstraction and generality. The very idea of a pure quality or a pure structure, that is to say, a structure defined in itself without any reference to its possible homologies, appears as a kind of vanishing limit, like the concept of the unrepresentable or unrepeatable. The very difference between the two is itself dependent upon the second conception. From the standpoint that recognizes quality or structure as such, it is possible to attribute one or another particular structure or quality to some object or sign, retroactively as it were. But from within the perspective that recognizes only qualities or structures in themselves, there is no available framework to distinguish and compare one with another. Each is like a universe unto itself.

To continue with the categorical example introduced above, the difference between the notions of iconicity as quality and iconicity as structure can be shown by "lifting" the previous category **3** from the level of its immanent or intrinsic relational determination to the level of the functors it supports into other categories, particularly those "concrete" categories whose objects are independently determined. The arrows in the diagram below from **3** into the categories indicated represent functors that take **3** as domain and the designated categories (sets and functions, groups and group homomorphisms, and some partial order $\leq$ defined on a given set P) as codomain.



Here the functors may be understood to represent selections of the generic structure-type represented by the category **3** as instantiated in the concrete categories at issue. For example, the functor shown from **3** into **Sets** selects three sets s_1, s_2 and s_3 and three functions $f : s_1 \rightarrow s_2$, $g : s_2 \rightarrow s_3$ and $h : s_1 \rightarrow s_3$ that compose in the codomain category **Sets** according to the schematic conditions provided by the arrow-relations in the domain category **3**. That is, $gf = h$. Composition of set-functions in the "concrete" codomain mirrors composition of arrows in the "abstract" domain. Identity functions in **Sets**, like identity arrows in **3** are here implicit and unremarked upon.

Similarly, the functor represented by the arrow from **3** into **Groups** designates the selection of a composition of two group homomorphisms, and the functor from **3** into the partial order $P, \leq$ selects three elements of P, say p_1, p_2 and p_3 such that $p_1 \leq p_2 \leq p_3$. In each case, the purely relational structure immanent to the arrows and compositions of **3** is instantiated in the codomain category in terms of the specific type of relation (functions, homomorphisms and order-relation) obtaining among the objects constituting the category.

Common structure serves here as the basis for possible relation and comparison across diverse types of mathematical objects. Importantly, the "space" of possible functors defined in this way includes relatively degenerate mappings in which differences in the domain collapse to identities in the codomain. For instance, the functor from **3** into the partial order on P may take all the objects of **3** to the same element p of P. Thus some of the immanent relational structure of **3** is lost but not contravened.

Interestingly, the close relationship between this structural conception and the previous qualitative one is nicely captured by a straightforward construction in category theory. Every category $\mathcal{C}$ supports an identity functor $1_{\mathcal{C}}$ that takes itself as both domain and codomain and is defined by mapping all objects and arrows of $\mathcal{C}$ to themselves. This identity functor expresses how the internal structure of any category, the relational dynamics of the category as an object in itself, is always also determined from a more external perspective as a functorial *relation* of that category to itself. In a similar way, every element of nature, experience and knowledge of whatever sort may, from a Peircean semiotic point of view, be understood as an iconic sign in which the sign and sign-object coincide, that is, a self-signifying icon.

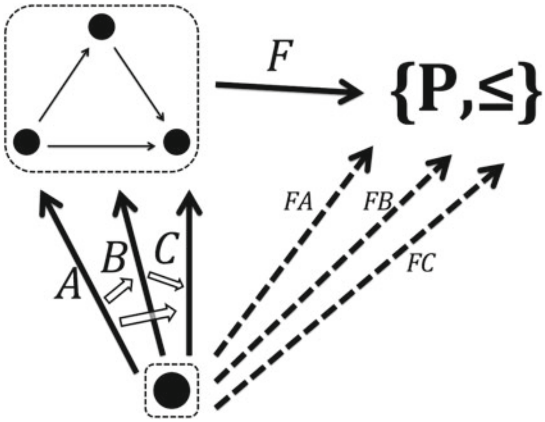
2.3.3 Iconicity as Abductive Support

Finally, we turn to the third and most fully elaborated notion of iconicity as an epistemically determined relation between a sign and its object that is capable of supporting discovery, via investigation of the properties and relations of the sign, of new knowledge concerning the sign's object.

This epistemic conception of iconicity is especially important because although it remains independent of the question of the factual existence of its object, it nonetheless implies a "space" of relational instantiation in which it itself is implicated. In other words, the epistemic conception of iconicity is only intelligible if the notion of knowledge it involves is broad enough to include knowledge of "ideal" or purely structural "objects". Peirce writes, "A pure icon can convey no positive or factual information; for it affords no assurance that there is any such thing in nature" ([6] 4.447 cited in [8], p. 216). How is it possible for knowledge to be generated without any reference to "positive or factual information"? The icon instantiates, in itself, a system of relations, and this system may be investigated as such.

It could be argued that in this way every icon involves some minimal indexicality, namely the indexical relationships among its relational parts. On this view, icons would only subsist as models. In Peirce's terminology, what is relevant here is the difference and the relation between a "qualisign" and an "iconic sinsign". As Short points out, some of Peirce's 1903 writings introduce the technical term "hypoicon" to represent the specific way that an actual existent (a sinsign) serves as the "substratum supporting an icon" ([8], p. 216; see also [9]). This terminology is useful for present purposes, as it suggests the existential "basis" for iconic properties or values that "float over" it. This picture is obviously similar to the construction that we employed at the beginning of chapter one. This picture is at once closely related to and distinct from the traditional formalization in first-order logic between "individuals" as elements of a domain over which quantifiers are allowed to range and the properties that adhere to these. It is distinct insofar as the icons or properties that are linked to particular individuals are not regulated by logical determinations such as binary truth-values. Instead, they are treated as "occasions" for concrete experimentation.

In the figure below, a functor F from the category **3** into $P, \leq$ is investigated by means of functors from the one-object, one-(identity)arrow category **1** into **3**.



The three functors A, B, C: $\mathbf{1} \longrightarrow \mathbf{3}$ select the initial, medial and terminal objects respectively of **3** and are related to one another via the natural transformations depicted by the hollow arrows near the base of the diagram (which compose in the obvious way). These three functors from **1** into **3** induce three functors into the partial order, namely FA, FB, FC: $\mathbf{1} \longrightarrow P, \leq$ via composition with F. Here, the structural mapping provided by the functor F is taken as given, and this mapping itself is investigated by probing **3**, the domain of F, directly with functors from a third category and ascertaining the results of such probing in the codomain of F as entailed by composition with F. The structure of the codomain of the functor is "known" mediately via direct functorial relations with the functor's domain.

The three ways of understanding Peirce's concept of iconicity are in this way both instantiated and clarified by examples drawn from the mathematics of category theory. In short, to recapitulate:

1. An icon conceived in itself is an immanently determined relational structure.
2. An icon conceived in relation to its object(s) is a structure determined by its possible translations and transformations.
3. An icon conceived in relation to its interpretant is an instantiated structure that may be investigated abductively so as to generate potential knowledge about some objective domain that it models.

2.4 Iconicity in Peirce's Theory of Dicisigns

A crucial role is played in Peirce's system of semiotic classification by the second term of his third trichotomy, the proposition, or *dicisign*. For Peirce, a dicisign is a sign-type defined by its interpretant's orientation towards the signified object of the sign as being one of actual existence. Thus a dicisign is understood as a way of treating a sign such that the object of the sign is conceived to exist in reality. In this respect, dicisigns are essential elements of abductive reasoning to the extent that abductive inferences are concrete "existential" responses to genuine epistemic problems.

The role of iconicity in Peirce's theory of propositions is especially striking. Stjernfelt's study *Natural Propositions* [7] synthesizes Peirce's widely scattered writings on this notion and extracts a definite theory of propositions that aligns very closely with Peirce's more general theory of semiotics as well as his philosophy of science. What Peirce calls a dicisign is what the tradition after Peirce will usually call a proposition, although it seems clear after the work of Stjernfelt that the Peircean dicisign is in fact a generalization of the more common notion of proposition, that Peirce's concept envelops not only ordinary linguistic propositions but also certain types of images and other sign-forms.

A dicisign is a type of sign that coordinates an iconic and an indexical function. Peirce's unusual view with respect to the analysis of propositional semiotics amounts to a coordination of syntactical with semantical structure. Yet the coordination at issue is not what might be expected: a homology between the relational composition of the relevant parts of the representing domain and the relations among the semantic elements in the domain of what is represented. Such would be the structural homology view of propositional meaning advocated, for instance, by the early Wittgenstein of the *Tractatus*. Instead of such a view, Peirce proposes a theory of propositions in which the indexical component is just as important as the iconic one in generating propositional sense.

Recall that Peirce introduces dicisigns (his own *terminus technicus* for what are now commonly called propositions) in the context of his third trichotomy of signs. Peirce understands what he calls a "rhema" as a fragmentary part of a logical discourse. It is closely related to what in a Scholastic vocabulary would be called a "term" with the important difference that a rhema is intrinsically fragmentary in character. It is best expressed by a notation such as "—is mortal" rather than the mere word or term "mortal". It includes the copula (the relation or connection to a subject) as an intrinsic yet underdetermined part of itself (see [1], pp. 308–9). Peirce's rhema is thus quite similar in form and function to what Frege develops as the notion of an "unsaturated" concept.

The following excerpt establishes the basic framework for Peirce's theory of propositions ([1], pp. 310–11):

> It may be asked what is the nature of the sign which joins "Socrates" to "—is wise" so as to make the proposition "Socrates is wise." I reply that it is an index. But, it may be objected, an index has for its object a thing *hic et nunc*, while a sign is not such a thing. This is true, if under "thing" we include singular events, which are the only things that are strictly *hic et nunc*. But it is not the two signs "Socrates" and "wise" that are connected, but the *replicas* of them used in the sentence. We do not say that "—is wise", as a general sign, is connected specially with Socrates, but only that it is so as here used. The two replicas of the words "Socrates" and "wise" are *hic et nunc*, and their junction is a part of their occurrence *hic et nunc*. They form a pair of reacting things which the index of connection denotes in their present reaction, and not in a general way; although it is possible to generalize the mode of this reaction like any other. There will be no objection to a generalization which shall call the mark of junction a *copula*, provided it be recognized that, in itself, it is not general, but is an *index*. No other kind of sign would answer the purpose; no general verb "is" can express

> it. For something would have to bring the general sense of that general verb down to the case in hand. An index alone can do this. But how is this index to signify the connection? In the only way in which any index can ever signify anything; by involving an *icon*. The sign itself is a connection. […] It is, thus, clear that the vital spark of every proposition, the peculiar propositional element of the proposition, is an indexical proposition; an index involving an icon. The rhema, say "—love—", has blanks which suggest filling; and a concrete actual connection of a subject with each blank monstrates the connection of ideas.

The basic method, then, of abductive reasoning with respect to propositions may be characterized in terms of the controlled variation of the iconic and indexical components of dicisigns. This coordination of abductive inference with the Peircean conception of propositions has the virtue of providing theoretical continuity between the analysis of abduction with respect to propositions on the one hand and with respect to diagrams on the other. Indeed from this perspective propositions appear as a special case of diagrams.

What generally makes a diagram successful as an instrument of abduction is the epistemic advantage gained by transferring investigation into some object of inquiry from a relatively inaccessible or experimentally intransigent domain to one that is more readily manipulated. This strategy is typically that of "model-building" or "model-based reasoning".

In *Natural Propositions*, Stjernfelt introduces a distinction between "operational iconicity" and "optimal iconicity". Operational iconicity is essentially Stjernfelt's reformulation of the epistemological determination of iconicity discussed above. This conception gives rise to an "operational criterion" for identifying iconic signs: "icons [are] the only sign type able to provide information" ([7], pp. 207–8). By this, Stjernfelt means that only iconic signs carry information about their objects when investigated independently of their signifying relation itself. Clearly, an index, say, a weather vane, carries information about its object, the direction of the wind. The point at stake here, however, concerns the relative externality or internality of the information thus carried. The "internal" representation of information within some medium is essentially diagrammatic.

In distinction from this "operational" determination of icons, which as Stjernfelt emphasizes tends to extend the notion of diagram "to such a degree that the common-sense notion of diagrams vanishes in the haze and seems to constitute only a small subset of the new, enlarged category" (ibid., p. 211), a stronger criterion, "optimal iconicity", is introduced. Sternfelt's own example is taken from Peirce's EG_β level of Existential Graphs and its "line of identity" that represents the existential quantifier in that logical system in an iconic way.

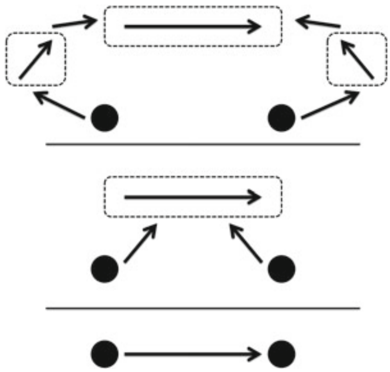
The problem here is that Stjernfelt attempts to ground the distinction between operational and optimal iconicity in an "absolute", or decontextualized determination of degrees of iconicity. But this cannot be done, or at least very significant obstacles present themselves. A very suggestive symptom of the issues at stake is provided by Stjernfelt's own characterization of the EG_β example he cites. Sternfelt writes (ibid., p. 218):

> The substitution of selectives for the line of identity is less iconic because it requires the symbolic convention of identifying different line segments by means of attached identical

> symbols. The line of identity, on the other hand, is immediately an icon of identity because it makes use of the continuity of the line that, so to speak, just stretches the identity represented by the spot – and that is, at the same time, a natural iconical representation of a general concept […].

The problem may be highlighted simply by drawing attention to the phrase "so to speak" in the citation just given.

Nonetheless the most important point about Peirce's interpretation of propositions as dicisigns is that it makes essential use of a particular theory of relations. Specifically, it depends upon the claim that relations imply infinitely iterable repetitions that nonetheless remain semantically inert in the sense that they do not enter into the interpretation of the meaning of the relation as such. The idea is illustrated by the following figure:



Peirce‘s theory effectively internalizes the synthetic operation of the relation (its active "joining" of its respective relata in its own characteristic manner) to the relation's own objecthood. Without further analysis, we merely note here that this semiotic and ultimately metaphysical conception of relations is well-captured by the the role of identity-arrows in category theory.

2.5 Representation and Hypostatic Abstraction

> So [human beings] maintained it as certain that the judgments of the Gods far surpass man's grasp. This alone, of course, would have caused the truth to be hidden from the human race to eternity, if Mathematics, which is concerned not with ends, but only with the essences and properties of figures, had not shown men another standard of truth.
>
> Spinoza, *Ethica* I

Dicisigns are the semiotic bread and butter of representations of fact. They are signs taken as describing what some real state of affairs is actually like. For Peirce, it is the iconic aspect of the dicisign that enables the semiotic relation of sign to reality to hold *for* some interpretant. In particular, in scientific knowing it is the iconic character of whatever discourse is employed for expressing scientific statements that on this view subtends the potential ability of those statements adequately to represent the real. Thus iconicity grounds the very possibility of scientific representation. When the discourse at stake for representing natural reality is essentially mathematical, as for instance throughout physics, it is the iconic character of mathematics that makes its representational function with respect to nature possible. For this reason, only a genuine understanding of iconicity in mathematics will support a robust scientific epistemology and, in particular, a clear grasp of abductive mathematical reasoning in natural science.

It is helpful here to look internally to mathematics, as the standard notion of representation in mathematics is itself an iconic one. Mathematics of course employs various modes of representation in its notational conventions, including for instance multiple sorts of variables (essentially indexical signs) and a variety of notations for representing formulas, operations and equations (essentially symbols). But within this very large space of mathematical representations, there is the relatively narrow conception of "representation" in the distinctly mathematical sense.

A standard mathematical expedient is to represent an abstract structure (typically algebraic or order-theoretic) within the more "concrete" mathematical domain of sets. Of course, sets are by no means concrete in any ordinary or absolute sense. They are mathematically concrete, however, precisely insofar as the theory that governs the overall structure of their common domain is quite well-studied and well-understood, and moreover to the extent that this domain may be conceived as a common space for determining the relationships among effectively all other types of mathematical objects and structures (as in the monumental twentieth-century synthesis arranged by Bourbaki). The "structure" of sets may be determined in various ways, most commonly by axiomatizations of the primitive epsilon-relation (as in Zermelo–Fraenkel or Gödel–Bernays) or through categorical characterization of the category **Sets** of sets and functions.

In many cases, a simple way to translate abstract, purely relational structures into sets is to convert each relevant abstract term into the set of the relations that hold between it and all the other relevant abstract terms at issue. The relations among the abstract terms are then carried over into perfectly isomorphic relations among the newly formed concrete objects just defined.

How? Expressed mathematically, it consists of the establishment of a kind of mapping between two domains that is guaranteed to preserve the relevant structure of one inside another. In category theory, this type of mapping is typically modeled by a functor. Illuminating examples of this are found in Cayley's representation theorem and categorically in the Yoneda lemma. What goes by the name representation theory in contemporary mathematics is a rich elaboration and extension of the sort of method used by Cayley in demonstrating his important result. Rather than representing algebraic structures in sets, representation theory typically translates abstract algebraic structures into linear transformations of vector spaces, thus allowing relatively "concrete" operations on matrices to capture the relevant "abstract" structures at issue.

The very existence within mathematics of Cayley's theorem and such branches of research as representation theory naturally raises the question: What relation, if any, holds between this type of representation as determined internally to mathematics and the general notion of representation at work in the world outside of mathematics that structures, for instance, Peircean semiotics? More specifically, we may now distinguish three broad problems of "representation", each inhabiting its own self-determined space of inquiry:

1. What is representation and how does it function within mathematics across various mathematical domains?
2. What is representation and how does it function between various areas of mathematics and various worldly domains, particularly as employed by the sciences?
3. What is representation and how does it function in a general way in the natural world itself (including of course that most dear part of the natural world that includes us human beings, our cultural products, sciences and so forth)?

We have ordered these questions in a perhaps counter-intuitive way, from the seemingly specific and regional (internal to mathematics) to the apparently most general (semiotics as the general theory of natural and cultural representation via signs), precisely so as to suggest that the otherwise obvious subsumption of mathematical knowledge and truth as a subclass of all kinds of knowledge and truth might be fundamentally mistaken. The undeniable and, at least from perspectives like that of Wigner, astonishing success of mathematics in scientific modeling of the world might in fact reflect the essential and fundamental role of mathematical thinking in *all* forms of abductive reasoning rather than its status as a special case. Perhaps, then, it is possible within mathematics to develop a theory of relations that would include the scientific usages of mathematics and the natural processes of semiotics under its proper purview. What is wanted then is a mathematical theory of representation general enough and flexible enough to accommodate both purely mathematical examples – Cayley's theorem, representations of algebraic structures as linear transformations of vector spaces, etc. – as well as the relatively impure cases of scientific measurement and experimentation of natural processes themselves.

2.5.1 Hypostatic Abstraction

Peirce's semiotic notion of *hypostatic abstraction* provides this needed bridge between mathematical method and the role of representation in mathematics on the one hand and abductive reasoning in natural science on the other. Whereas pure mathematical reasoning, on Peirce's view, considers nothing other than the universe of sheer iconic possibility, scientific inquiry is essentially propositional, that is, it coordinates iconic and indexical elements.

The notion of diagrammatic abduction already complicates this all-too-easy distinction. Peirce maintains that all mathematical reasoning is intrinsically diagrammatic. It thus requires reasoning not only about pure icons, but reasoning concerning pure icons on the basis of *instantiated* icons, that is, diagrams that involve both iconic and indexical dimensions. The mathematical reasoner must be able to *experiment* on the icons that serve as the theoretical objects of her inquiry, and such experimentation requires that the relations pertaining to the icons at stake be manipulable and thus indexically available for concrete interaction. Thus the semiotic "secondness" of actual, reactive and resistant existence is not so easily subtracted from the essence of mathematical reasoning. If it does not enter directly into the ultimate theoretical *objects* of such reasoning, it nonetheless remains an ineluctable aspect of the reasoning process as such.

In a somewhat different fashion, Peirce also keenly understood the importance of the unique and unusual role played by the concept of existence in the sphere of mathematical reasoning, even at the level of mathematical objects as such. The specifically mathematical, or iconic, notion of existence is in contrast to the standard idea of existential secondness as passive, event-like reaction. Instead, in a purely iconic register the objects or "entities" of mathematics are thoroughly constructive, as befits their status as qualisigns or pure possibilities. In this regard, Peirce was at pains to emphasize the role of *hypostatic abstraction* in mathematical reasoning.

The method of hypostatic abstraction is typically abductive in character. It posits or constructs an "existential" hypothesis that tends to play an explanatory or elucidating role in its given cognitional context. This abductive role of hypostatic abstraction links mathematical reasoning directly to scientific inquiry, where the "existential" frame is not in general one of pure iconic possibility but rather of the reactive secondness of causes and effects linked in sequences of real events. In natural science, explanation is typically causal and concerns relations between actual entities. For Peirce, however, the mathematically constructive method of hypostatic abstraction is nonetheless the primary means for conducting scientific inquiry into such relations and the patterns they exhibit.

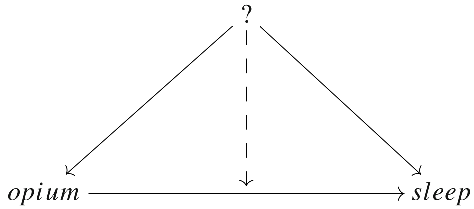
Peirce's own preferred example, to which he returns in a variety of texts and contexts, is that of the "dormitive virtue" attributed to opium by the Scholastically trained doctor in Molière's satirical comedy. For Peirce, the doctor's seemingly vacuous pretension is in fact a theoretical technique of tremendous subtlely and power. Let us analyze this canonical instance of "empty" reasoning more closely:

1. The reasoning begins with the recognition of a causal relation between the substance opium and the property of somnolence:

   $$\text{opium} \rightarrow \text{sleep}$$

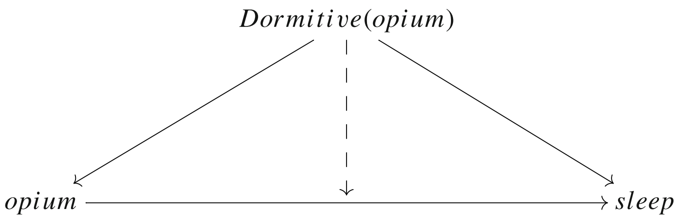
   This step is already in its own way a kind of hypothesis. As the Humean tradition rightly insists, causal relations as such are not observable; they are posited. Nonetheless, one starts here with what is effectively an induction sufficiently provocative of surprise or wonder.
2. The surprise at "seeing" the causal connection between opium and sleep raises the question of the *explanation* of this causal relation. What stands in relation to

this relation as such, in such manner that it would become “a matter of course”? Informally:



This question opens an enormous space, unrestricted in principle, of possible explanations. The second stage of the abductive procedure is thus introduced: how might this too-large space be appropriately reduced to something more manageable, that is, less questionable and more positively suggestive?

3. What immediately suggests itself is the theoretically “minimal” solution that would simply repeat the causal arrow posited in step 1 as a new “object”, that is, an “external” cause that is at the same time a property attributable to the term “opium”.



This procedure induces an implicit argument-scheme very much in accord with the general Peircean model of abductive inference as “affirming the consequent”:

$$\frac{\begin{array}{l}\text{opium} \longrightarrow \text{sleep}\\ \text{Dormitive(opium)} \rightarrow (\text{opium} \longrightarrow \text{sleep})\end{array}}{\text{Dormitive(opium)}}$$

4. From a slightly different point of view, the previous abductive step may be understood not as positing “dormitive virtue” as an external cause, but rather as *internalizing* the causal arrow (the relation) in step 1 to its first term “opium” (its “initial” relatum).

No matter whether the abductive process of hypostatic abstraction as just sketched is interpreted more “externally” (as in step 3) or in a more “internal” manner (as suggested in step 4), the hypothesis thereby formulated that opium possesses dormitive virtue is not merely a restatement of the original datum that opium causes somnolence. What the abductive-conclusion-as-hypothesis provides is a newly enriched space of reasoning that supports new kinds of questions ("Is dormitive virtue somehow like the liveliness virtue of caffeine, even if opposite in effect?") and new forms of inquiry ("Perhaps looking at opium through a microscope will reveal the mechanism of dormitive virtue").

2.5.2 Iconicity and Axiomatics

An analogue of hypostatic abstraction appears in its own way in modern mathematics. In Hilbert's notion of axiomatic method, any system of axioms that does not imply a contradiction may be understood to be satisfied by purely ideal "objects" that are determined only by the axioms themselves. Such an axiom system is required only to be consistent. The mathematical "formalism" that results rejects any limitation of mathematics by natural ontology. The mere fact that a model of an arbitrary axiom system cannot be given in terms of some domain that actually exists is irrelevant to the question of the mathematical truth of that axiom system. More strongly, the axiom system itself guarantees the objects it discerns.

Importantly, however, the elements of the syntax in an axiomatic system are intrinsically variable. They are indifferent as to everything but their syntactical character, which is nonetheless dependent upon pre-existing systems of determinable content. The very possibility of designating a regime of variables with a characterization such as "$a, b, c \ldots$" requires the prior grouping of the lower-case Roman alphabet as well as the cognitive organization of a "Roman-letter type". Rodin, in [10], examines in detail the history of axiomatic method from its earliest emergence in Greek geometry through its initial modern treatment in Hilbert and his followers to its contemporary employment in nearly all branches of mathematics as well as many fields of science.

Rodin follows Lawvere (see especially [11, 12]) in noting a deep connection between the categorical approach to logic and axiomatics on the one hand and the Hegelian notion on the other of an objective logic that would remain immanent to the real dialectic of the "thing itself" understood as an epistemic process in which objective and subjective aspects would ultimately cohere. From the in-itself of purely objective being and the for-itself of simple subjective interiority, a process at once real and epistemically oriented would culminate in the in-and-for-itself of subjective and objective reconciliation. See also [13] in this regard.

In particular, Rodin translates the distinction made by Hegel himself in [14] between a merely "subjective logic" such as that proposed by Kantian idealism and Hegel's own innovative notion of an "objective logic" of dialectical reality into the difference between the traditional use of logical deduction as an *external* framework for examining the consequences of mathematical axioms and the structure, particular to topos theory, of an *internal* logic. Among other things, Rodin shows how the "bootstrapping" approach to topos theoretical axiomatization in McClarty [15] solves in a strictly mathematical register the philosophical problem of immanent logical grounding that appears in Hegel as the problem of overcoming the division of idealism and realism.

Rodin is at pains to show that what goes by the name "formalism" in philosophy of mathematics and is commonly attributed to Hilbert as its originator is in fact rooted in a misreading of the father of modern axiomatic method. Furthermore, in this regard the logicist program that followed Frege and Russell in aiming to ground mathematics in logic also represents a misreading and misapplication of logic in the context of axiomatics as understood by Hilbert. Rodin points out that just as the geometric axioms of Euclid do not simply express logical relationships among truth-claims but in fact underwrite the possibility of concrete *constructions* in the ideal geometrical plane, the formal axiomatic method of Hilbert does not represent the application of independently conceivable logical operations to syntactical data. Rather, the syntax-semantics relation (the core of model theory) is already internal to formal logic as such.

He cites Hilbert's well-known 1927 paper on "Foundations of Mathematics" (cited in [10], p. 61):

> No more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed; no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not.

When Hilbert claims here that "as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation", we may prescind from the Kantian language of "faculties" and attend only to the basic point at stake, namely that logical reasoning (even in its most formal, non-psychologistic sense) operates necessarily on some sort of "material", some sort of non-logical data. Even and especially when this data is purely symbolic, that is, formal, it must nonetheless be instantiated in a system of possible relations (a generic "space") and that space itself must be in its own way *given as evident and available*. The point, philosophically speaking, is an Aristotelian one: no forms without matter; *hylē* and *morphē* are mutually implicative concepts. Yet, from a purely mathematical perspective it may *also* be compatible with a sort of Platonism. The "matter" of mathematical reasoning may itself be purely ideal (eidetic), as in the standard treatment of ZFC-consistent sets as a kind of raw material for building mathematical structures. Interestingly, Hilbert indicates a trio of conditions that appear to govern the very possibility of any such logically operable "material":

1. Existence: "that they occur".
2. Difference: "that they differ from one another".
3. Synthetic relation: "that they follow each other, or are concatenated".

The mathematics of category theory may be understood in this way as a distillation of Hilbert's program, one that deploys it *internally* to mathematics itself.

2.6 Iconicity and Diagrammatic Abduction

At the heart of abductive reasoning there is a sort of reconstruction problem. Abduction is indeed characterized by the gluing of "local", context-dependent information, into a synthetic, coherent global organization of possibilities. In mathematics, the notion of *sheaf* plays a similar role. Since we believe that the role of sheaves is fundamental in the understanding of the notion of iconicity within abduction, in this section we present a short introduction to presheaves and sheaves.

Following an example presented in [16], consider the role of a judge whose job is to evaluate the artistic quality of painted hard-boiled eggs. Clearly, given the shape of the object being evaluated, the judge faces a subtle challenge since she – at any given time – only has a partial view on the whole object (she can only see half of the egg from any given angle).

Obviously, the judge reasons under the assumption that, by rotating the eggs, she will have enough partial points of view that, all taken together, will give her enough information to reconstruct the global picture and therefore make an informed decision on the artistic quality of the painted egg.

In mathematics we can find a very general version of this process: first, the notion of presheaf formalizes the notion of "locality", whereas the notion of sheaf corresponds to the conditions that need to be fulfilled to glue together local information in order to get the global picture.

Definition 1 Let X be a topological space. A *presheaf* of abelian groups on X, say F, is an object F which associates an abelian group $F(U)$ to every open subset $U \subset X$ such that

1. for any inclusion of open sets $V \subset U$ we have a group homomorphism

$$\rho_{VU} : F(U) \longrightarrow F(V)$$

2. for every open U, ρ_{UU} is the identity map on U
3. $W \subset V \subset U \Rightarrow \rho_{WU} = \rho_{WV} \circ \rho_{VU}$

The group $F(U)$ is called the group of *sections* of U and the map ρ_{VU} is called the restriction of $F(U)$ onto $F(V)$.

Example 2 Real (topological) manifolds are topological spaces that, locally, look like the Euclidean space of a fixed dimension. Topologists often study the presheaf of *smooth functions* over a manifold. In other words, given an open set U, $F(U)$ is the ring (hence an abelian group) of all smooth functions over U. In this case the effect of the restriction map ρ_{VU} on a section $f \in V$, is just the actual restriction function $f|_V$.

We remark that, instead of abelian groups, we could have used any category and the definition of presheaf would still be the same. The existence of a unique gluing procedure characterizes the sheaf condition.

Definition 2 A *sheaf* F is a presheaf that satisfies the following conditions. Let $\{U_i\}_{i \in I}$ be a collection of open sets in X and let U be the union of $\{U_i\}_{i \in I}$.

1. (Existence of gluing) Whenever, for every $i \in I, s_i \in F(U_i)$ and for every $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a section $s \in F(U)$ with $s|_{U_i} = s|_{U_j}$ for all $i \in I$.
2. (Uniquiness of gluing) Let $s, t \in F(U)$ such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.

In the subsections below we develop a formal framework aligned with presheaves to characterize the intrinsic syntax-semantics relation of iconicity as it appears in diagrammatic representations quite generally. The clearest motivating intuition will come from considering ready-to-hand, concrete instances. As an example, consider a computerized touchscreen, perhaps as found on a contemporary handheld tablet device. On the screen are several icons, windows, tabs and so forth. The user is able to select from among the various objects available on the screen by touching the screen itself with his or her finger and adopting various simple conventions (tapping twice to select an object, etc.). In particular, he or she may be able to select arbitrary collections of objects on the screen (perhaps by holding the SHIFT key down while selecting individual objects in succession).

Imagine now a generalization of this function of selecting parts by indicating. Think of a sheet that may be inscribed with any image, picture or complex figure and imagine a weblike overlay that may be set to the precise contours of useful, selectable parts of this image, picture or figure enabling reference to some external domain in some communicative context. For instance, someone might select a certain area of a projected map and say, “The suspect is hiding somewhere in here.”

What we call the Sheet of Indication is meant to function something like this idealized map or touchscreen. It is intended to function as a formal semantics that is applicable to a wide variety of diagrams and, as we will see in the next chapter, to the logical diagrams of Peirce’s Existential Graphs (EG) in particular.

2.6.1 Introducing the Sheet of Indication

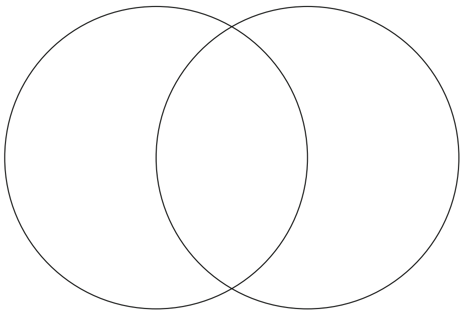
What is the Sheet of Indication (SI)? It is any sheet inscribed with some image or diagram, certain determinate parts of which may be indicated in a given communicative context. Thus we have two basic components:

- the inscribed sheet itself
- a system of indication that depends upon a conventional and rule-governed semiotic relation (*symbolic*, in Peirce's terms) on the basis of which it becomes possible to indicate (*indexically*) certain determinate parts of the inscription.

We require that the parts that may be indicated or selected in this fashion determine amongst themselves a partial order (that is, a relation that is reflexive, transitive and anti-symmetric) and that there be a privileged part, called *terminal*, that is greater than or equal to all the parts according to the order. Together, these conditions ensure a certain synthetic and tractable coherence among the parts.

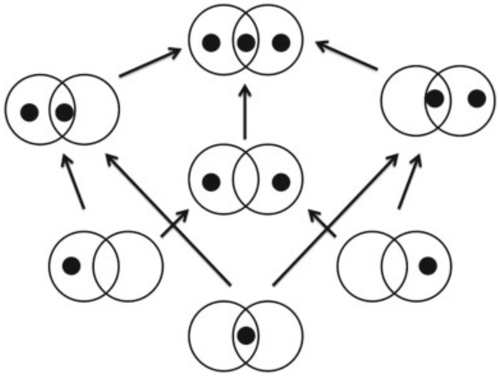
We then associate to each selectable part of the diagram – that is, to each element of the partial order – a set of possible values in some determinate domain. In this way, each selectable part functions somewhat like a variable in a formal syntax. Next we define models for the diagram as a whole in terms of the coordination of sets of value-assignments to the various parts of the diagram that respect in an appropriate way the order among these parts, depending on the specific features and intended uses of the diagram at issue.

In order to motivate the formalization of this semantics, we begin with a pair of informal examples that will allow us to clarify the main features of the construction. Our first example is a simple Venn diagram consisting of two overlapping circles.



In order to conceive this diagram as a Sheet of Indication, we must first come to an agreement concerning which parts of the diagram may be distinctly indicated. We must then ensure that these parts are ordered amongst themselves by some relation that is reflexive, transitive and antisymmetric. Also, there must be some part at the "top" of this order that stands in this relation to all the other parts. In the present case, we may stipulate that any bounded area of the diagram and any collection of such areas may be uniquely indicated (for instance, by touching one or more areas of the diagram simultaneously). The parts that may be selected in this fashion are then themselves naturally ordered by the relation of inclusion: collections of bounded areas include the areas they collect.

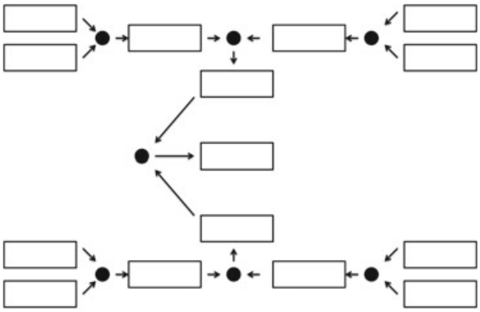
The following "exploded" diagram makes explicit what such parts are and how the order-relation works in this particular case. All the selectable parts are individually represented by collections of areas marked by dots, and an arrow from one such part to another represents that the part at the "tail" of the arrow is "below" the part at the "head" of the arrow in the partial order. For simplicity's sake, we have omitted arrows that may be inferred from the reflexivity and transitivity of this order-relation.



Our concern is to formulate a diagrammatic semantics. How might we actually use the original two-circle diagram in some given context to represent values in the relevant domain of reference? Let us assume we wish to classify teachers working at a particular school. We might naturally assign to each of the parts of the diagram some collection of these teachers. It is clear that according to the standard interpretation of Venn diagrams, we would want the inclusion relations among those subsets to correspond exactly to the arrows (both drawn and implied) in the exploded diagram above, that is, to the partial order defined over the selectable parts of the diagram.

More generally, in using this diagram and above all in employing it in a regular way across various contexts, we would want to introduce certain fixed general conditions on possible assignments of subsets of some set S to a particular part of the diagram that would depend on the assignments to certain other parts. A complete model for the diagram in any given context would then consist of some definite assignment of values to every part that respects these conditions. The information contained in such a model could then be "read off" the largest part, that is, the area containing all the other areas. Some particular distribution of math and science teachers, for instance, could be represented by associating the math teachers with the left circle and the science teachers with the right circle, with those teachers who teach both math and science associated with the lozenge-shaped intersection of the two circles. All of this information would be contained and properly differentiated by the complete determination of the most inclusive area of the diagram, in this case the diagram as a whole.

A second example shows how the same basic idea might be applied to a different sort of diagram with a different mode of semantic determination. In the figure below, we have a diagram representing a sports playoff bracket. Here, the rectangles are meant to represent teams and the dots to represent games. Individual dots and rectangles are the parts that may be selected, and the order defined over these parts is given by the arrows shown. Once again the order relation is reflexive and transitive, so arrows from each part to itself and arrows corresponding to every possible path of drawn arrows are implied. Whereas in the previous example all the selectable parts of the diagram were of the same kind, namely, bounded areas, in this diagram we have two different types of part corresponding to two different types of assignment: dots corresponding to games and rectangular boxes corresponding to teams.



Given some collection of teams, say those constituting a particular conference division, we would want to build a possible model for the diagram by assigning one of the teams to each rectangle and a pair of teams to each dot. Obviously, to maintain the coherence of the semantics, we would want in addition to impose a set of conditions on such assignments. For instance, we would want to ensure that no team can play against itself, that the game assigned to a given dot will consist of precisely the two teams assigned to the rectangles immediately preceding it, that the winner of any game assigned to a dot will be assigned to the rectangle immediately succeeding it, and so on. A complete determination of the diagram will then be given by the assignment of a unique champion team to the rectangle in the center, together with an assignment to all the parts "below" this part of a team or game consistent with the specified conditions.

2.6.2 Formal Definition of the Sheet of Indication

We now define a Sheet of Indication formally.

The *Sheet of Indication*, or SI, is a diagram consisting of:

1. a finite set P of *parts* together with a partial order $(P, \leq)$ endowed with a terminal object p^* (that is, $\forall q \in P \ q \leq p^*$)
2. a finite set of *types* T and a function $t : P \longrightarrow T$ that assigns a type to every part.

A semantics for the SI is then given by specifying functions from *lower sets* of the parts of the diagram into some *domain of interpretation*. The domain of interpretation is any non-empty set D, and a lower set of any part $p \in P$ is defined as the set of all parts less than or equal to p, that is, $\{q \in P | q \leq p\}$. Thus the semantics ranges over functions that we call *models at* p, or m_p, for all $p \in P$:

$$m_p : \{q \in P | q \leq p\} \longrightarrow D$$

We designate the set of all such specified models at a given part p with the notation μ_p.

For any $m_p \in \mu_p$, we define the *value at q* for any $q \leq p$ to be $m_p(q)$, that is, the value assigned to the part q in the model m_p. We denote the value at q in the model m_p by $val_{m_p}(q)$, although we usually drop the index when the model m_p is clear from the context.

Constraints on the functions m_p are specified by a set C of *conditions*, that is, statements that stipulate values (or ranges of values) in D that must or may not be assigned in models m_p to parts $q \leq p$, depending on the order-relation $\leq$ and the type-function t. Any function not explicitly required or prohibited by a condition is implicitly allowed. The only restriction on sets of conditions themselves is that they must never permit some model at p, $m_p \in \mu_p$, while also prohibiting m_p as restricted to the lower set at any $q \leq p$ to be a model at q, $m_q \in \mu_q$.

Let us see how our previous examples fit this formal framework. Parts and types in both cases are obvious. Equally clear are the domains D of the interpretation. What really specifies a particular SI are the conditions that constrain the values in D that the models may assign to distinct parts and typed parts. In the case of the Venn diagram, for instance, there is only one type (all the parts are collections of bounded areas), so only the order-relation $\leq$ is relevant to the conditions on the models. Clearly, we want to ensure that the relations between values at each part in all models correspond to the appropriate set inclusion and intersection operators. This may be guaranteed by a set of three conditions that apply to values in all models at all parts:

- $q \leq p \Rightarrow val(q) \subseteq val(p)$;
- $q \nleq p$ and $p \nleq q \Rightarrow val(q) \bigcap val(p) = \emptyset$;
- $\exists q\ (q \leq p$ and $q \neq p) \Rightarrow val(p) = \bigcup_{r \leq p, r \neq p} val(r)$.

In a similar fashion, we can recover the relevant conditions on models for the playoff bracket diagram. The main difference there is that its semantics involves two different types of parts: games and teams. Presumably with respect to these types at least two of the conditions we would demand of any model would be:

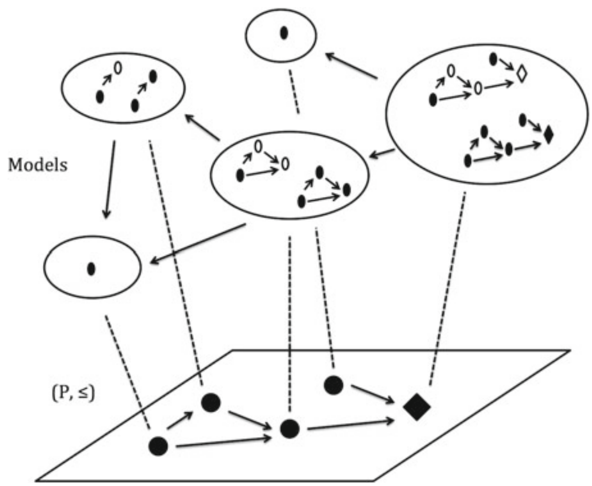
- $t(p) = game \Rightarrow val(p) = \bigcup_{r \leq p, r \neq p, \forall q | r \leq q \leq p \Rightarrow r = q} val(r)$
- $t(p) = game \Rightarrow \exists x, y | x \in val(p)$ and $y \in val(p)$ and $x \neq y$

Thus if there are no two-element sets in the domain D of interpretation, there will be no models for the diagram and the resulting semantics may be said to be *empty*. Assuming an appropriate domain of interpretation, however, here too an adequate set of conditions would function not only for this graph in particular, but for any similarly formed graph (such as a similarly structured playoff bracket with a greater number of rounds). In this way, a set of conditions C functions roughly as a classifier for diagrams that are to be interpreted semantically "in the same way".

The most important thing to notice in both examples is the way that the models in the semantics may be conceived roughly as being built up layer by layer along the contours of the partial order $\leq$ from the "bottom" to the "top." What that means formally is that for any model at $p \in P$ and for any $q \leq p, q \in P$ there is always a model at q which is the model at p restricted to q (the two functions agree on all values). Because the collection of models over each part form a set, we then have a naturally defined function $f_{p,q}$ from the set of models at p, μ_p, to the set of models at q, μ_q for any $q \leq p$. This function takes each model in μ_p to the corresponding model in μ_q that is simply its restriction to q. In addition, such functions compose in a natural way. That is, given any three parts $r \leq q \leq p$ and the three functions $f_{q,r}$ and $f_{p,q}$, we have that $f_{p,q} \circ f_{q,r} = f_{p,r}$. Defined in this way, the models over P have the structure of a *presheaf*.

In [17] the distinctively philosophical importance of presheaves (and the sheaves they induce) is stressed insofar as the most philosophically interesting aspects of advanced contemporary mathematics are both generated by and best represented in terms of synthetic local/global relations across various mathematical domains. Because presheaves and sheaves lend themselves naturally to modeling such relations, they are especially useful in studying structures that vary over the elements of other structures and the way such systems may be layered or "glued" to one another (see [18] and especially [19] for details and references). In [20] this approach via sheaves and presheaves is applied specifically to the philosophy of Peirce and the distinctive understanding of continuity that underlies the "Synechism" of his later philosophy. The present diagrammatic semantics aligns nicely with this general approach to both a wide range of issues in contemporary philosophy of mathematics and logic and to Peirce's philosophy and semiotics in particular. Anticipating our analysis in chapter three, related research that interprets Peirce's EG_α in terms of different category theoretic constructions is found in [21, 22].

For those readers who may be unfamiliar with presheaves, the following diagram may help make the notion more vivid:



The presheaf structure in the above diagram is evident by noticing how the natural restriction maps between the models "reverses" the order given in the base space. The reader may him or herself trace how each of the two models in the diagram at the terminal part represented by the diamond on the sheet corresponds naturally to a model at each of the other parts that "looks exactly the same" when restricted to the "piece" of the model at and below that part. Tracing these restrictions simply involves following the arrows connecting the sets of models one to another.

In general, there may be models at a part p for some sheets of indication that are *not* included in any model at a part $q > p$. Thus there could have been two distinct models in the set over the "least" part of the diagram above (the southwesternmost part) and the restriction maps would have remained exactly the same and the overall structure would still be a pre-sheaf. Nonetheless in practice many if not most interpretations of diagrams induce restriction maps between models that also happen to be surjective. What is impossible in any case because of the required pre-sheaf structure is that there be any model m at a part p such that there is a part $q \leq p$ and there is no model at q that takes exactly the same value as m at each part $r \leq q$. This entails that a restriction map from the set μ_p of models at p to the set μ_q of models at q for any $q \leq p$ is always well-defined.

We thus have a general formal framework (the SI) that quite naturally represents the syntactical and semantical dynamics of representation with respect to the mereological (part-whole) relations of a wide variety of different types of diagrams. The interaction of iconic and indexical components of such diagrams is reflected in the identity conditions and structural relations of the parts themselves.

2.7 Conclusion

Let us review the steps we've taken in this chapter:

- We sketched the main contours of Peirce's general semiotic theory and focused on the notion of iconicity in particular.
- We distinguished three levels of iconicity corresponding to quality, structure and abductive support. Each of these levels found a natural representation in the terms of category theory, thus suggesting a link between semiotic iconicity and categorical mathematics.
- We examined Stjernfelt's analysis of Peirce's doctrine of dicisigns, which establishes a coordination of iconic and indexical semiotic elements as the primary mechanism whereby propositions become capable of referring to the world.
- We considered Peirce's notion of hypostatic abstraction relative to the general problem of propositional and mathematical representation on the one hand and that of abduction on the other.
- We developed the formal theory of Sheets of Indication to model systems of diagrammatic experimentation in a general way.

Thus, by examining the concept of iconicity in detail we have been led to reformulate abductive reasoning as a diagrammatic process involving both iconic and indexical components, thus bridging the gap between purely "ideal" iconic qualities and the "real" structures where such icons are instantiated and experimented upon. This intrication of iconic and indexical elements of abductive reasoning may be analyzed in its own right with the use of tools drawn from the mathematics of category theory. In the following chapter, we show how the general syntax-semantics relation of iconic diagrams applies in the particular case of Peirce's logical notation of Existential Graphs by recasting Peirce's graphs in terms of the Sheet of Indication formalization defined above.

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