Preface to the Second Edition

We were very pleased with the response to the first edition of this book and we were very happy to do a second edition. In this second edition, we cleaned up various typos pointed out by readers and added some new material suggested by them. We have also included important new results that have appeared since the first edition came out. These results include results on the gaps between primes and the twin primes conjecture.

We have added a new chapter, Chapter 7, on p-adic numbers, p-adic arithmetic, and the use of Hensel's Lemma. This can be included in a year-long course.

We have extended the material on elliptic curves in Chapter 5 on primality testing.

We have added material in Chapter 4 on multiple-valued zeta functions.

As before, we would like to thank the many people who read or used the first edition and made suggestions. We would also especially like to thank Anja Moldenhauer and Anja Rosenberger who helped tremendously with editing and LATEX and made some invaluable suggestions about the contents.

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Preface to the First Edition

Number theory is fascinating. Results about numbers often appear magical, both in their statements and in the elegance of their proofs. Nowhere is this more evident than in results about the set of prime numbers. The Prime Number Theorem, which gives the asymptotic density of the prime numbers, is often cited as the most surprising result in all of mathematics. It certainly is the result which is hardest to justify intuitively.

The prime numbers form the cornerstone of the theory of numbers. Many, if not most, results in number theory proceed by considering the case of primes and then pasting the result together for all integers by using the Fundamental Theorem of Arithmetic. The purpose of this book is to give an introduction and overview of number theory based on the central theme of the sequence of primes. The richness of this somewhat unique approach becomes clear once one realizes how much number theory and mathematics in general is needed to learn and truly understand the prime numbers. The approach provides a solid background in the standard material as well as presenting an overview of the whole discipline. All the essential topics are covered the fundamental theorem of arithmetic, theory of congruences, quadratic reciprocity, arithmetic functions, and the distribution of primes. In addition, there are firm introductions to analytic number theory, primality testing and cryptography, and algebraic number theory, as well as many interesting side topics. Full treatments and proofs are given to both Dirichlet's Theorem and the Prime Number Theorem. There is a complete explanation of the new AKS algorithm that shows that primality testing is of polynomial time. In algebraic number theory, there is a complete presentation of primes and prime factorizations in algebraic number fields.

The book grew out of notes from several courses given for advanced undergraduates in the United States and for teachers in Germany. The material on the Prime Number Theorem grew out of seminars also given both at the University of Dortmund and at Fairfield University. The intended audience is upper level undergraduates and beginning graduate students. The notes upon which the book was based were used effectively in such courses in both the United States and

Germany. The prerequisites are a knowledge of Calculus and Multivariable Calculus and some Linear Algebra. The necessary ideas from Abstract Algebra and Complex Analysis are introduced in the book. There are many interesting exercises ranging from simple to quite difficult. Solutions and hints are provided to selected exercises. We have written the book in what we feel is a user-friendly style with many discussions of the history of various topics. It is our opinion that it is also ideal for self-study.

There are two basic facts concerning the sequence of primes that are focused on in this book and from which much of the theory of numbers is introduced. The first fact is that there are infinitely many primes. This fact was of course known since at least the time of Euclid. However, there are a great many proofs of this result not related to Euclid's original proof. By considering and presenting many of these proofs, a wide area of modern number theory is covered. This includes the fact that the primes are numerous enough so that there are infinitely many in any arithmetic progression $an + b$ with a, b relatively prime (Dirichlet's Theorem). The proof of Dirichlet's Theorem allows us to first introduce analytic methods.

In distinction to there being infinitely many primes, the density of primes thins out. We first encounter this in the startling (but easily proved) result that there are arbitrarily large gaps in the sequence of primes. The exact nature of how the sequence of primes thins out is formalized in the Prime Number Theorem, which as already mentioned, many people consider the most surprising result in mathematics. Presenting the proof and the ideas surrounding the proof of the Prime Number Theorem allows us to introduce and discuss a large portion of analytic number theory.

Algebraic Number Theory arose originally as an attempt to extend unique factorization to algebraic number rings. We use the approach of looking at primes and prime factorizations to present a fairy comprehensive introduction to algebraic number theory.

Finally, modern cryptography is intimately tied to number theory. Especially crucial in this connection is primality testing. We discuss various primality testing methods, including the recently developed AKS algorithm and then provide a basic introduction to cryptography.

There are several ways that this book can be used for courses. Chapter 1 together with selections from the remaining chapters can be used for a one-semester course in number theory for undergraduates or beginning graduate students. The only prerequisites are a basic knowledge of mathematical proofs (induction, etc.) and some knowledge of Calculus. All the rest is self-contained, although we do use algebraic methods so that some knowledge of basic abstract algebra would be beneficial. A year-long course focusing on analytic methods can be done from Chapters 1, 2, 3, and 4 and selections from 5 and 6, while a year-long course focusing on algebraic number theory can be fashioned from Chapters 1, 2, 3, and 6 and selections from 4 and 5. There are also possibilities for using the book for one semester introductory courses in analytic number theory, centering on Chapter 4, or for a one semester introductory course in algebraic number theory, centering on Chapter 6. Some suggested courses:

Basic Introductory One Semester Number Theory Course: Chapter 1, Chapter 2, Sections 3.1, 4.1, 4.2, 5.1, 5.3, 5.4, 6.1

Year-Long Course Focusing on Analytic Number Theory: Chapter 1, Chapter 2, Chapter 3, Chapter 4, Sections 5.1, 5.3, 5.4, 6.1

Year-Long Course Focusing on Algebraic Number Theory: Chapter 1, Chapter 2, Chapter 3, Chapter 6, Sections 4.1, 4.2, 5.1, 5.3, 5.4

One-Semester Course Focusing on Analytic Number Theory: Chapter 1, Chapter 2 (as needed), Sections 3.1, 3.2, 3.3, 3.4, 3.5, Chapter 4

One-Semester Course Focusing on Algebraic Number Theory: Chapter 1, Chapter 2 (as needed), Chapter 6

We would like to thank the many people who have read through other preliminary versions of these notes and made suggestions. Included among these people are Kati Bencsath and Al Thaler, as well as the many students who have taken the courses. In particular, we would like to thank Peter Ackermann, who read through the whole manuscript both proofreading and making mathematical suggestions. Peter was also heavily involved in the seminars on the Prime Number Theorem from which much of the material in Chapter 4 comes.

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