Chapter 2
The *K* and *L* Theoretic Farrell-Jones Isomorphism Conjecture for Braid Groups

Daniel Juan-Pineda and Luis Jorge Sánchez Saldaña

Abstract We prove the *K* and *L* theoretic versions of the Fibered Isomorphism Conjecture of F.T. Farrell and L.E. Jones for braid groups on a surface.

Keywords Braid groups · K and L-theory

2.1 Introduction

Aravinda, Farrell, and Roushon in [1] showed that the Whitehead group of the classical pure braid groups vanishes. Later on, in [10], Farrell and Roushon extended this result to the full braid groups. Computations of the Whitehead group of braid groups for the sphere and the projective space were performed by D. Juan-Pineda and S. Millán in [12–14]. In all cases the key ingredient was to prove the Farrell-Jones isomorphism conjecture for the *Pseudoisotopy* functor, this allows computations for lower algebraic *K* theory groups. In this note, we use results by Bartels, Lück and Reich [6] for word hyperbolic groups and Bartels, Lück and Wegner [5, 18] for CAT(0) groups to prove that the algebraic *K* and *L* theory versions of the conjecture holds for braid groups on surfaces.

D. Juan-Pineda (✉)
Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México,
Campus Morelia, Apartado Postal 61-3 (Xangari), 58089 Morelia, Michoacan, Mexico
e-mail: daniel@matmor.unam.mx

L.J. Sánchez Saldaña
Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México,
Apartado Postal 61-3 (Xangari), 58089 Morelia, Michoacan, Mexico
e-mail: luisjorge@matmor.unam.mx

2.2 The Farrell-Jones Conjecture and Its Fibered Version

In this work we are interested in the formulations of the Farrell-Jones Isomorphism Conjecture, in both, its fibered and nonfibered versions. Let $\mathbb{K}$ be the Davis-Lück algebraic K theory functor, see [8, Sect. 2] and let $\mathbb{L}$ be the Davis-Lück $\mathscr{L}^{-\infty}$ theory functor, see [8, Sect. 2] and [9, Sect. 1.3] for the definition of $\mathscr{L}^{-\infty}$-theory (or L theory for short). The validity of this conjecture allows us, in principle, to compute the algebraic K or L theory groups of the group ring of a given group G from (a) the corresponding K or L theory groups of the group rings of virtually cyclic subgroups of G and (b) homological information. More precisely, let $\mathscr{L}_*()$ be any of the functors $\mathbb{K}$ or $\mathbb{L}$.

Conjecture 2.2.1 (Farrell-Jones Isomorphism Conjecture, IC) *Let G be a discrete group. Then for, all $n, \in \mathbb{Z}$ the assembly map*

$$A_{Vcyc} : H_n^G(\underline{\underline{E}}G; \mathscr{L}_*()) \to H_n^G(pt; \mathscr{L}_*()) \tag{2.1}$$

induced by the projection $\underline{\underline{E}}G \to pt$ is an isomorphism, where the groups $H_*^G(--; \mathscr{L}_*())$ build up a suitable equivariant homology theory with local coefficients in the functor $\mathscr{L}_*()$, and $\underline{\underline{E}}G$ is a model for the classifying space for actions with isotropy in the family of virtually cyclic subgroups of G.

A generalization of the Farrell-Jones conjecture is what is known as the *Fibered Isomorphism Conjecture* (FIC). This generalization has better hereditary properties (see [4, 6]).

Definition 2.2.2 Given a homomorphism of groups $\varphi : K \to G$ and a family of subgroups $\mathscr{F}$ of G closed under conjugation and finite intersections, we define the induced family

$$\varphi^* \mathscr{F} = \{H \leq K | \varphi(H) \in \mathscr{F}\}.$$

If K is a subgroup of G and φ is the inclusion we denote $\varphi^* \mathscr{F} = \mathscr{F} \cap K$.

In this note, we will use FIC to refer to the Fibered Isomorphism Conjecture with respect to the family $Vcyc$ (of virtually cyclic subgroups), for either K- or L- theory.

Definition 2.2.3 Let G be a group and let $\mathscr{F}$ be a family of subgroups of G. We say that the pair $(G, \mathscr{F})$ satisfies the **Fibered Farrell-Jones Isomorphism Conjecture (FIC)** if for all group homomorphisms $\varphi : K \to G$ the pair $(K, \varphi^* \mathscr{F})$ satisfies that

$$A_{\varphi^* \mathscr{F}} : H_n^K(E_{\varphi^* \mathscr{F}} K; \mathscr{L}_*()) \to H_n^K(pt; \mathscr{L}_*())$$

is an isomorphism for all $n \in \mathbb{Z}$.

One of the most interesting hereditary properties of FIC is the so called *Transitivity Principle* [6, Theorem 2.4]:

Theorem 2.2.4 *Let G be a group and let $\mathscr{F} \subset \mathscr{G}$ be two families of subgroups of G. Assume that $N \in \mathbb{Z} \cup \{\infty\}$. Suppose that for every element $H \in \mathscr{G}$ the group H satisfies FIC for the family $\mathscr{F} \cap H$ for all $n \leq N$. Then $(G, \mathscr{G})$ satisfies FIC for all $n \leq N$ if and only if $(G, \mathscr{F})$ satisfies FIC for all $n \leq N$.*

Let G be a group, we denote $Vcyc(G)$ for the family or virtually cyclic subgroups of G. The next two theorems are fundamental for later sections, see [6, lemma 2.8].

Theorem 2.2.5 *Let $f : G \rightarrow Q$ be a surjective homomorphism of groups. Suppose that Q, satisfies FIC and for all $f^{-1}(H)$ with $H \in Vcyc(Q)$, $(f^{-1}(H), Vcyc(f^{-1}(H)))$ FIC is true. Then G, satisfies FIC.*

Theorem 2.2.6 *If G satisfies FIC then every subgroup of G satisfies FIC as well.*

2.3   Pure Braid Groups on Aspherical Surfaces

Definition 2.3.1 [1, Definition 1.1] We say that a group G is strongly poly-free if there exists a filtration $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that the following conditions hold:

(a) G_i is normal in G for each i.
(b) G_{i+1}/G_i is a finitely generated free group.
(c) For each $g \in G$ there exists a compact surface F and a diffeomorphism $f : F \rightarrow F$ such that the action by conjugation of g in G_{i+1}/G_i can be geometrically realized, i.e., the following diagram commutes:

$$
\begin{CD}
\pi_1(F) @>{f_{\#}}>> \pi_1(F) \\
@V{\varphi}VV @AA{\varphi^{-1}}A \\
G_{i+1}/G_i @>{C_g}>> G_{i+1}/G_i
\end{CD}
$$

where φ is an suitable isomorphism.

In this situation we say that G has rank lower or equal than n. Now we enunciate some theorems that will be useful later.

Theorem 2.3.2 *Every word hyperbolic group satisfies FIC. In particular, every finitely generated free group satisfies FIC.*

Proof See [6, Theorem 1.1] for K-theory and [5, Theorem B] for L-theory.

Theorem 2.3.3 *Every CAT(0) group satisfies FIC.*

Proof See [18] for K-theory and [5, Theorem B] for L-theory.

Remark 2.3.4 Both Theorems 2.3.2 and 2.3.3 were proven for a more general version of the Farrell-Jones Isomorphism conjecture stated here, namely they were proven for generalized homology theories with coefficients in any additive category, these versions imply the one given here and also Theorems 2.2.4, 2.2.5 and 2.2.6, see [7].

Theorem 2.3.5 *Let M be a simply connected complete Riemannian manifold whose sectional curvatures are all nonpositive and let G be a group. Assume that G acts by isometries on M, properly discontinuously and cocompactly, then G is a CAT(0) group. In particular, G satisfies FIC.*

Proof This is the basic example of a $CAT(0)$ group. See [3].

Lemma 2.3.6 *Let F be a finitely generated free group and $f : F \to F$ be an automorphism that can be geometrically realized, in the sense of Definition 2.3.1(c), then $F \rtimes \mathbb{Z}$, with the action of $\mathbb{Z}$ in F given by f, satisfies FIC.*

Proof It is proven in [1, Lemma 1.7] that under our hypotheses $F \rtimes \mathbb{Z}$ is isomorphic to the fundamental group of a compact Riemannian 3-manifold M that supports in its interior a complete Riemannian metric with nonpositive sectional curvature everywhere and the metric is a cylinder near the boundary, in fact, near the boundary is diffeomorphic to finitely many components of the form $\mathbb{R} \times N$ where N is either a torus or a Klein bottle. Let DM be the double of M, by the description of the boundary, we can endow the closed manifold DM with a Riemannian metric with nonpositive curvature everywhere as in [2, Proposition 6]. As DM is now compact and complete, it follows that the fundamental group $\pi_1(DM)$ is CAT(0), hence it satisfies FIC. Moreover, $F \rtimes \mathbb{Z} \cong \pi_1(M)$ injects into $\pi_1(DM)$ our result follows by Theorem 2.2.6 and Theorem 2.3.5.

Our main theorem is now the following.

Theorem 2.3.7 *Every strongly poly-free group G satisfies FIC.*

Proof Let us proceed by induction on the rank of G. The induction base, when G has rank ≤ 1, is true as in this case G is a finitely generated free group, thus the assertion follows from Theorem 2.3.2.

Assume that strongly poly-free groups of rank $\leq n$ satisfy FIC and let G be a group of rank $\leq n + 1$ with $n > 0$. We apply Theorem 2.2.5 to the surjective homomorphism $q : G \to G/G_n$. Observe that G/G_n is a finitely generated free group, hence it satisfies FIC. Let $C \subset G/G_n$ be a virtually cyclic (and hence cyclic) subgroup, not excluding the possibility of C being G/G_n. We have the following cases:

(a) If $C = \{1\}$ then $q^{-1}(C) = G_n$, which is strongly poly-free of rank $\leq n$, hence it satisfies FIC.
(b) Assume now that C is an infinite cyclic subgroup of G/G_n. Let

$$f : q^{-1}(C) \to \frac{q^{-1}(C)}{G_{n-1}}$$

be the natural projection and observe that

$$\frac{q^{-1}(C)}{G_{n-1}} \cong \frac{G_n \rtimes C}{G_{n-1}} \cong \frac{G_n}{G_{n-1}} \rtimes C.$$

Moreover, the group $\frac{G_n}{G_{n-1}} \rtimes C$ satisfies the hypotheses of Lemma 2.3.6 by the condition (c) for SPF groups, thus $\frac{q^{-1}(C)}{G_{n-1}}$ satisfies FIC and we apply Theorem 2.2.5 to the homomorphism $f : q^{-1}(C) \to \frac{q^{-1}(C)}{G_{n-1}}$. Let $V \subseteq \frac{q^{-1}(C)}{G_{n-1}}$ be a cyclic subgroup, again we have the following cases:

a. $V = 1$ it follows that $f^{-1}(V) = G_{n-1}$ which is an SPF group of rank $\leq n - 1$, and it does satisfy FIC by induction.
b. V is an infinite cyclic subgroup. By the definition of V it fits in a filtration

$$1 = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset f^{-1}(V),$$

which gives that $f^{-1}(V)$ is an SPF group of rank $\leq n$, hence it satisfies FIC by induction.

It follows that $q^{-1}(C)$ satisfies FIC and therefore G satisfies FIC.

We now recall the definition of the pure braid groups on a surface.

Definition 2.3.8 Let S be a surface with boundary (possibly empty) and $P_k = \{y_1, \ldots, y_k\} \subset S$ be a finite subset of interior points. Define the configuration space to be $M_n^k(S) = \{(x_1, x_2, \ldots, x_n) | x_i \in S - P_k, \ x_i \neq x_j \ \text{for} \ i \neq j\}$. The Pure Braid group on S with n-strings $B_n(S)$ is by definition $\pi_1(M_n^0(S))$.

Lemma 2.3.9 *Let S be a surface with boundary. For $n > r \geq 1$ and $k \geq 0$, the projection on the first r coordinates $M_n^k(S^0) \to M_r^k(S^0)$ is a fibration with fiber $M_{m-r}^{k+r}(S)$, where $S^0 = S - \partial S$.*

Proof See [15, Lemma 1.27].

Lemma 2.3.10 *Suppose that $S = \mathbb{C}$ or that S is a compact surface with nonempty boundary. Then for all $m \geq 0$, $n \geq 1$ the manifold $M_n^m(S)$ is aspherical.*

Proof Consider the fibration $M_n^m(S^0) \to M_1^m(S^0) = S^0 - P_m$ with fiber $M_{n-1}^{m+1}(S^0)$ given by previous lemma. The exact sequence associated to this fibration is as follows

$$\cdots \to \pi_{i+1}(S - P_m) \to \pi_i(M_{n-1}^{m+1}(S)) \to \pi_i(M_n^m(S)) \to \pi_i(S - P_m) \to \cdots .$$

Since $S - P_m$ is aspherical, because the boundary is nonempty, $\pi_i(S - P_m) = 0$ for all $i \geq 2$. Hence for all $i \geq 2$, $\pi_i(M_{n-1}^{m+1}(S)) \cong \pi_i(M_n^m(S))$. An inductive argument shows that for all $i \geq 2$ we have

$$\pi_i(M_n^m(S)) \cong \pi_i(M_1^{m+n-1}(S)) \cong \pi_i(S - P_{m+n-1}) = 0.$$

Theorem 2.3.11 *Suppose that $S = \mathbb{C}$ or that S is a compact surface with nonempty boundary different from $\mathbb{S}^2$ or $\mathbb{R}P^2$. Then the pure braid group $B_n(S)$ is strongly poly-free of rank $\leq n$ for all $n \geq 1$.*

Proof See [1, Theorem 3.1]. □

Recall that the braid groups on $\mathbb{C}$ or on a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$ are torsion free. The above Theorem and Theorem 2.3.7 now gives:

Theorem 2.3.12 *Suppose that $S = \mathbb{C}$ or S is a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Then the pure braid groups $B_n(S)$ satisfy FIC for all $n \geq 1$.*

2.4 Full Braid Groups on Aspherical Surfaces

The main goal of this section is to prove that any extension of a finite group by an SPF group satisfies FIC. In order to prove this we shall need some results.

Definition 2.4.1 Let G and H be groups, with H finite. We define de wreath product $G \wr H$ to be the semidirect product $G^{|H|} \rtimes H$, where $G^{|H|}$ is the group of $|H|$-tuples of elements in G indexed by elements in H, H acts on $G^{|H|}$ by permuting the coordinates as the action of H on H by right translation.

Wreath products have been widely studied, the following properties are well known.

Lemma 2.4.2 *Let $1 \to G \to \Gamma \to H \to 1$ be a group extension with H finite. Then there are injective homomorphisms $\delta : \Gamma \to G \wr H$ and $\theta : G \to G^{|H|}$ which together with $\mathrm{id} : H \to H$ define a map to the group extension $1 \to G^{|H|} \to G \wr H \to H \to 1$.*

Proof See [10, Algebraic Lemma]. □

The following lemmas contain standard facts about wreath products.

Lemma 2.4.3 *Let A, B, S and H be groups with S and H finite.*

(a) If A is a subgroup of B, then $A \wr H$ is a subgroup of $B \wr H$.
(b) $A^{|H|} \wr S$ is a subgroup of $A \wr (H \times S)$.

Lemma 2.4.4 ([1, Fact 2.4]) *Let G be a group and H a finite subgroup of $Aut(G)$. We define $G^{|H|} \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on the left factor via $f = \bigoplus_{h \in H} h \in Aut(G^{|H|})$, and $G \rtimes_h \mathbb{Z}$ for each $h \in H$ in the obvious way. Then $G^{|H|} \rtimes \mathbb{Z}$ is a subgroup of $\prod_{h \in H} G \rtimes_h \mathbb{Z}$.*

Theorem 2.4.5 *If G is a $CAT(0)$ group and H is a finite group, then $G \wr H$ is a $CAT(0)$ group, and hence satisfies FIC.*

Proof Let G act properly, isometrically and cocompactly on the $CAT(0)$ space X, then $G \wr H$ acts properly, isometrically and cocompactly on $X^{|H|}$ with $G^{|H|}$ acting coordinate wise on $X^{|H|}$ and H by permuting the coordinates.

Theorem 2.4.6 *Let G be an SPF group, and let H be a finite group. Then $G \wr H$ satisfies FIC.*

Proof Let us proceed by induction on the rank n of G. When $n = 0$ it follows that $G = 1$ and hence $G \wr H$ is finite, thus is hyperbolic and it satisfies FIC by Theorem 2.3.2.

Now assume G has rank $\leq n$, where $n > 1$, and consider the filtration $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ given in the definition of a strongly poly-free group.

Note that $G_1^{|H|}$ is a normal subgroup of $G \wr H$ and hence, we have the group extension

$$1 \to G_1^{|H|} \to G \wr H \to (G/G_1) \wr H \to 1.$$

Let $p : G \wr H \to (G/G_1) \wr H$ denote the above epimorphism. We will apply Theorem 2.2.5, note that this is possible because G/G_1 is an SPF of rank $\leq n - 1$. Hence $(G/G_1) \wr H$ satisfies FIC by induction hypothesis.

Next, let $S \subset (G/G_1) \wr H$ be a virtually cyclic subgroup. We have to prove that $p^{-1}(S)$ satisfies FIC. There are two cases to consider.

Case 1: S is finite. We have an exact sequence $1 \to G_1^{|H|} \to p^{-1}(S) \to S \to 1$. Using Lemmas 2.4.2 and 2.4.3, we get

$$p^{-1}(S) \subset G_1^{|H|} \wr S \subset G_1 \wr (H \times S),$$

Now by Theorems 2.4.5 and 2.2.6, $p^{-1}(S)$ satisfies FIC.

Case 2: S is infinite. S contains a normal subgroup T of finite index such that T is infinite cyclic and $T \subset (G/G_1)^{|H|}$. In fact, we assume $T = S \cap (G/G_1)^{|H|}$. We have the exact sequence $1 \to p^{-1}(T) \to p^{-1}(S) \to S/T \to 1$. By Lemma 2.4.2 we get $p^{-1}(S) \subset T_1 \wr (S/T)$, where $T_1 = p^{-1}(T)$. By Theorem 2.2.6 it suffices to show that $T_1 \wr (S/T)$ satisfies FIC.

Fix $t = (t_h)_{h \in H} \in G^{|H|}$, which maps to a generator of T. Note that each t_h acts geometrically on G_1. Then, by Lemma 2.4.4

$$T_1 = G_1^{|H|} \rtimes_t \mathbb{Z} \subset \prod_{h \in H} (G_1 \rtimes_{t_h} \mathbb{Z}),$$

now using Definition 2.3.1, Lemma 2.3.6, and the fact that the finite product of $CAT(0)$-groups is a $CAT(0)$-group we conclude that $\prod_{h \in H} (G_1 \rtimes_{t_h} \mathbb{Z})$ is a $CAT(0)$-group. Hence by Theorems 2.4.5 and 2.2.6, $T_1 \wr (S/T)$ satisfies FIC. Thus, $p^{-1}(S)$ also satisfies FIC. Thus $G \wr H$ satisfies FIC.

Corollary 2.4.7 *Every extension Γ of a finite group by an SPF satisfies FIC.*

Proof It is immediate from the previous Theorem and Lemma 2.4.2 and Theorem 2.2.6.

Theorem 2.4.8 *Suppose that $M = \mathbb{C}$ or M is a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Let Γ be an extension of a finite group H by $B_n(M)$ for some $n \geq 1$. Then Γ satisfies FIC.*

Proof If M has nonempty boundary or $M = \mathbb{C}$ then $B_n(M)$ is SPF by Theorem 2.3.11, so by the previous Corollary the assertion is true. From now on, we assume that M has empty boundary. By Lemma 2.4.2 and Theorem 2.2.6 it suffices to prove that $B_n(M) \wr H$ satisfies FIC. Considering the fiber bundle projection $p : M_n^0(M) \to M_1^0(M) = M$ with fiber $M_{n-1}^1(M) = M_{n-1}^0(M - pt)$ we have the following short exact sequence

$$1 \to A = B_{n-1}(M - pt) \to B_n(M) \xrightarrow{p} \pi_1(M) \to 1.$$

Consider the exact sequence

$$1 \to A^{|H|} \to B_n(M) \wr H \to \pi_1(M) \wr H \to 1.$$

Let $p : B_n(M) \wr H \to \pi_1(M) \wr H$ be the surjective homomorphism in the above sequence. We proceed to apply Theorem 2.2.5 to p. Note that $\pi_1(M) \wr H$ satisfies FIC by Theorem 2.4.5. Let S be a virtually cyclic subgroup of $\pi_1(M) \wr H$. We claim that $p^{-1}(S)$ contains a SPF of finite index. Let $T = S \cap \pi_1(M)^{|H|}$. Then $p^{-1}(T)$ is of finite index in $p^{-1}(S)$. Now, as A is SPF, it follows that $A^{|H|}$ is also SPF by considering the filtration

$$1 = G_0 \subset \cdots G_n = A \subset A \times G_1 \subset \cdots A \times A \subset \cdots \subset A^{|H|-1} \times G_{n-1} \subset A^{|H|}$$

where $1 = G_0 \subset G_1 \subset \cdots G_n = A$ is an SPF structure on A. On the other hand, we have an exact sequence $1 \to A^{|H|} \to p^{-1}(T) \to T \to 1$. Now, from the monodromy action on the pure braid group of M it can be checked that $p^{-1}(T)$ is SPF. The proof now follows from the previous Corollary.

Definition 2.4.9 We recall from Definition 2.3.8 that

$$M_n^0(M) = \{(x_1, x_2, \ldots, x_n) | x_i \in S, \; x_i \neq x_j \text{ if } i \neq j\}.$$

The symmetric group S_n acts on M_n^0. We define the Full Braid Group $FB_n(M)$ on a surface M to be $\pi_1(M_n^0/S_n)$.

It is not difficult to see that we have an exact sequence

$$1 \to B_n(M) \to FB_n(M) \to S_n \to 1,$$

hence by our previous Theorem we have the following:

Theorem 2.4.10 *Suppose that $M = \mathbb{C}$ or M is a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Then the full braid group $FB_n(M)$ satisfies FIC.*

2.5 Braid Groups on $\mathbb{S}^2$ and $\mathbb{R}P^2$

Theorem 2.5.1 *The pure braid groups $B_n(\mathbb{S}^2)$ satisfy FIC for all $n > 0$.*

Proof We have that $B_1(\mathbb{S}^2) = B_2(\mathbb{S}^2) = 0$, and $B_3(\mathbb{S}^2) \cong \mathbb{Z}_2$ (see [11]), so they satisfy FIC because they are finite. For $n > 3$, we consider the fiber bundle $M_n^0(\mathbb{S}^2) \to M_3^0(\mathbb{S}^2)$ with fiber $M_{n-3}^3(\mathbb{S}^2) \cong M_{n-3}^2(\mathbb{C})$. Applying the long exact sequence of the corresponding fibration and the fact that $\pi_2(M_3^0(\mathbb{S}^2))$ is trivial we get

$$1 \to \pi_1(M_{n-3}^2(\mathbb{C})) \to \pi_1(M_n^0(\mathbb{S}^2)) \to \pi_1(M_3^0(\mathbb{S}^2)) \to 1.$$

Note that $\pi_1(M_{n-3}^2(\mathbb{C}))$ is SPF because it is part of the filtration of $B_{n-3}(\mathbb{C})$, hence by Corollary 2.4.7 we have that $\pi_1(M_n^0(\mathbb{S}^2)) = B_n(\mathbb{S}^2)$ satisfies FIC.

Lemma 2.5.2 *Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that K is virtually cyclic and Q satisfies FIC. Then G satisfies FIC.*

Proof See [6] Lemma 3.4.

Theorem 2.5.3 *The full braid groups $FB_n(\mathbb{S}^2)$ satisfy FIC for all $n > 0$.*

Proof In [16], Theorem 24 it is proven that $FB_n(\mathbb{S}^2)$ fits in an exact sequence

$$1 \to \Gamma_n \to FB_n(\mathbb{S}^2))/\mathbb{Z}_2 \to S_n \to 1$$

where Γ_n is SPF. Hence $FB_n(\mathbb{S}^2)/\mathbb{Z}_2$ satisfies FIC by Corollary 2.4.7. Now consider the exact sequence

$$1 \to \mathbb{Z}_2 \to FB_n(\mathbb{S}^2) \to FB_n(\mathbb{S}^2)/\mathbb{Z}_2 \to 1$$

applying the previous Lemma, we conclude that $FB_n(\mathbb{S}^2)$ satisfies FIC.

Theorem 2.5.4 *The pure braid groups $B_n(\mathbb{R}P^2)$ satisfy FIC for all $n > 0$.*

Proof In [17] it is proven that $B_1(\mathbb{R}P^2) = \mathbb{Z}_2$, $B_2(\mathbb{R}P^2) \cong Q_8$ and $B_3(\mathbb{R}P^2) \cong F_2 \rtimes Q_8$, where Q_8 is the quaternion group with eight elements and F_2 is the free group on two generators. Hence $B_1(\mathbb{R}P^2)$ and $B_2(\mathbb{R}P^2)$ satisfy FIC because they are finite, while $F_2 \rtimes Q_8$ does as it is hyperbolic. In [16, Theorem 27] it is proven that $B_n(\mathbb{R}P^2)$ fits in an exact sequence

$$1 \to \Lambda_n \to B_n(\mathbb{R}P^2) \to Q_8 \to 1$$

where Λ_n is SPF, for all $n > 3$. Hence $B_n(\mathbb{R}P^2)$ satisfies FIC by Corollary 2.4.7.

Theorem 2.5.5 *The full braid groups $FB_n(\mathbb{R}P^2)$ satisfy FIC for all $n > 0$.*

Proof In [17] it is proven that $FB_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and $FB_2(\mathbb{R}P^2)$ is isomorphic to a dicyclic group of order 16. Hence $FB_1(\mathbb{R}P^2)$ and $FB_2(\mathbb{R}P^2)$ satisfy FIC because they are finite. In [16] Theorem 29 it is proven that $FB_n(\mathbb{R}P^2)$ fits in an exact sequence

$$1 \to S_n \to FB_n(\mathbb{R}P^2) \to FB_n(\mathbb{R}P^2)/S \to 1$$

where S_n is a normal SPF subgroup of $FB_n(\mathbb{R}P^2)$ with finite index, for all $n > 2$. Hence by Corollary 2.4.7 we conclude that $FB_n(\mathbb{R}P^2)$ satisfies FIC for all $n > 2$.

Recall that if the groups G and H satisfy FIC then $F \times H$ also satisfies FIC. Therefore we have the following

Theorem 2.5.6 *Let G be a braid group of a surface in any number of strands then* $G \times \mathbb{Z}^n$ *satisfies FIC for all* $n \geq 1$.

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