# The Genesis of Geometry

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A. Meskens, P. Tytgat, *Exploring Classical Greek Construction Problems with Interactive Geometry Software*, Compact Textbooks in Mathematics, DOI 10.1007/978-3-319-42863-5_2

## 2.1 Greeks and their Geometry

Many elementary geometry books for secondary schools contain exercises in which a construction with compass and straightedge is called for. In many cases, these are exercises whose history goes back to Ancient Greece.

In the history of western civilisation, Greek Antiquity occupies a prominent place. This civilisation is sometimes called the cradle of western civilisation, philosophy and mathematics. Greek civilisation can not be reduced to the Greek mainland however. At its zenith, it stretched from Sicily across the eastern Mediterranean to Asia Minor (now Turkey) and the Black Sea coasts (see Fig. 2.1).

Next to Athens and Sparta, the main Greek cities of the era were Alexandria on the Nile delta, Syracuse on Sicily and Milete on the coast of Asia Minor. Greek identity was defined through common trade interests, not as a nation state. Politically, the *polis* or the city state was the most important political entity.

Despite the impression to the contrary, our knowledge about the lives of Greek mathematicians is limited to say the least. Moreover it is hard to separate fact from fiction as most biographical information about them was written several decades, sometimes even centuries, after their death. This applies even more to pre-Socratic thinkers who lived before the fifth century BC.

The first Greek mathematician we can identify with some confidence is Thales of Milete (ca. 624–548/45 BC). Thales lived in Ionia (on the coast of Turkey). It was there that classical Greek science emerged, which would go on to have such a profound impact on our science. It was in Ionia that the idea took root that complex, natural phenomena can be explained by a set of basic rules.

Some of the theorems that are attributed to Thales include:

1. A circle is bisected by its diameter
2. The base angles in an isosceles triangle are equal
3. When two straight lines intersect then the opposite angles are equal
4. An angle inscribed in a semicircle is a right angle (Thales' circle theorem)
5. If two intersecting lines are cut by a two parallel lines then two similar triangles are produced (Thales' intercept theorem)

Fig. 2.1 Greek colonisation about 500 BC (after Vermaseren (1977), p. 9)

Plato (Athens ca. 427 BC–ca. 348 BC) was a philosopher, as well as a mathematician. He founded the *Academy* in Athens, the first institution of higher learning in the Western world. Plato, Socrates and his most famous student, Aristotle, are seen as the founders of Western philosophy and science. From the fourth century onwards, Plato's philosophical ideas gained ground. Their influence on mathematics became visible in geometry, in which hierarchies of classes of constructions emerged.

The first class was the most abstract, the third and last was the most "earthly and mechanical":

1. constructions with only straight lines and/or circles, i.e. constructions with compass and straightedge
2. constructions in which conic sections (parabolae, hyperbolae and ellipses) are used
3. constructions with other (mechanical) construction means

Of course these constructions need to be accomplished in a finite number of steps.

## 2.2 Examples from Plato's hierarchy

In Plato's hierarchy, compass and straightedge constructions are the most abstract. Here is one example of such a construction.

To draw a hexagon inscribed in a circle, we observe the following procedure (see Fig. 2.2):

» Select the width of the compass equal to the radius of the given circle
Select a point $A$ on the circle
Place the compass point in $A$
Draw arcs which intersect the circle at $B$ and $C$
Place the compass point, with the same width, in $B$ and draw an arc
This arc intersects the circle at $A$ and $D$

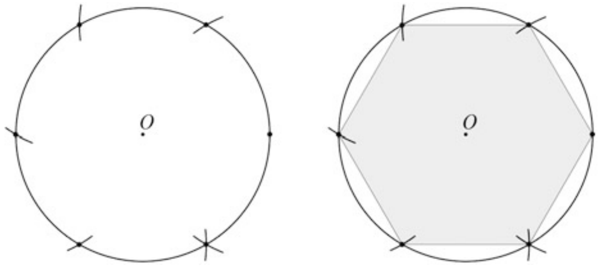



**Fig. 2.2** The construction of a regular hexagon inscribed in a circle

Place the compass point, with the same width, in $D$ and repeat the above procedure until an arc intersects the circle at $C$
The intersection points which you have constructed are the vertices of a regular hexagon inscribed in the circle.

Plato's second category contains constructions which can be carried out with the aid of conic sections. A circle, which can be drawn with the aid of a compass, is a special type of conic section. It is therefore natural to consider the conic sections as additional construction aids. To show why they cannot be included in the first category, let us first take a look at the parabola.

A parabola is defined as the locus of all points having the same distance to a given line, directrix $d$, and to a given point, focus $F$.

We can construct points of a parabola with this procedure (see Fig. 2.3):

» Draw the perpendicular on $d$ and through $F$
This line intersects $d$ in $D$
Draw straight lines $d_i$ parallel to $d$ at regular intervals, beginning at a distance $\frac{\|DF\|}{2}$ to $d$
Let the intersection of $d_i$ with $DF$ be $D_i$
For each of these lines set the compass width at $\|DD_i\|$, place the compass point in $F$

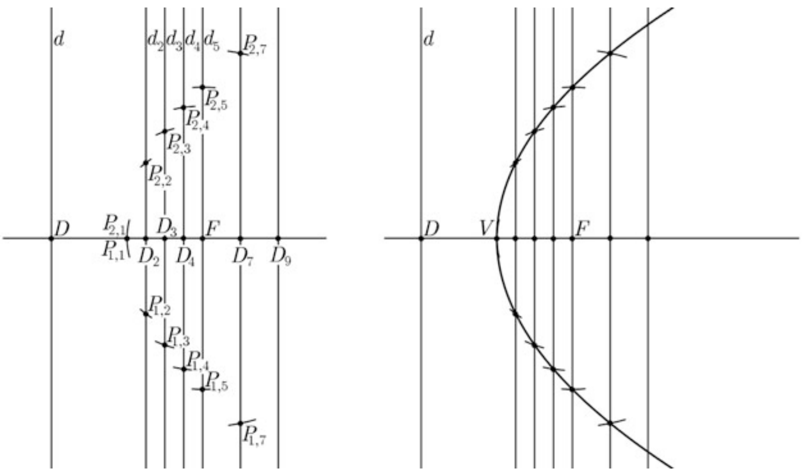



**Fig. 2.3** The construction of a parabola with focus $F$ and directrix $d$

Draw arcs which intersect $d_i$
Let these intersection points be denoted by $P_{i,1}$ and $P_{i,2}$
$P_{i,1}$ and $P_{i,2}$ are points of the parabola
$D_1$ is a special point, here $P_{1,1}$ and $P_{1,2}$ coincide. $D_1$ is called the vertex of the parabola, and is often referred to with the letter $V$. The line $DF$ is called the axis of symmetry.

We notice that it is possible to construct individual points of the parabola with compass and straightedge, but that it is not possible to construct the curve itself in this way.

Plato's second category therefore is a natural extension of the first category.

**Exercise 15**
Create an IGS file in which you define two sliders $a$ and $b$ both ranging from 0 to 10. Draw the straight line $y = -a$ and the point $A\,(0, a)$. Draw the straight line $y = b$ and the circle centered at $A$ and with radius $a + b$ and determine their intersections $B$ and $C$. For $B$ and $C$ select Trace and move the slider $b$. You will see a parabola appear. Alternatively click on Locus, select the mover, the slider $b$, and the tracer, $B$ and $C$ respectively. You will now see a parabola being traced out. What happens when you move the slider $a$ (do NOT use Animation On)?

**Exercise 16**
Select a system of axes $x$ and $y$, in which $x$ coincides with $DF$ and $y$ intersects $[DF]$ at the midpoint. Let the coordinates of $F$ be $\left(\frac{p}{2}, 0\right)$, then the Cartesian equation of d is $x = -\frac{p}{2}$. Use the formulae for the distance between two points and the distance of a point to a straight line to prove that the Cartesian equation for the parabola in this configuration is $y^2 = 2px$.

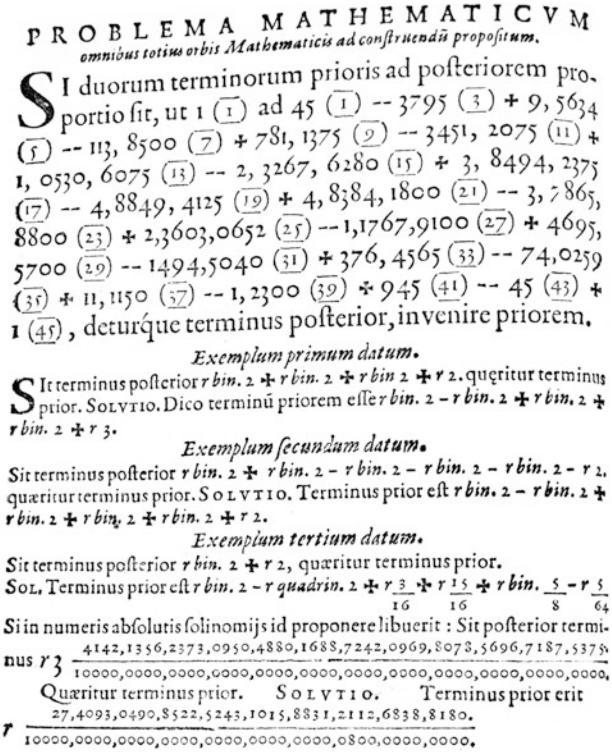
## 2.3 Adrian van Roomen and Apollonius' Problem

Some problems require ingenious methods to find a compass and straightedge solution, but can be easily solved using conic sections. One such problem is Apollonius' problem, for which a solution with conic sections, i.e. belonging to Plato's second category, is straightforward. Later we will also discuss the compass and straightedge construction (see ▶ Sect. 3.3).

Apollonius (ca. 262 BC–ca. 190 BC) is best known for his work on conic sections. His book *Conics* is preserved partly in Greek with commentaries by Eutocius and partly in an Arab translation by Thabit ibn Qurra. Apollonius' other work has unfortunately been lost, but we know of his exploits through Pappus of Alexandria's commentaries[1]. In his book Ἐπαφαί (Epaphaí, "*Tangencies*") Apollonius posed, and solved, the problem which is now known as Apollonius' problem: find a circle which is tangent to three given circles[2]. Three given circles have at most eight circles which are tangent to all of them. Each solution encloses or excludes the circles in a different way.

Adriaan van Roomen, also known as Adrianus Romanus (1561–1615) was a Flemish physician and mathematician who solved the problem using conic sections. Van Roomen became professor of mathematics and medicine first in Leuven then at Würzburg (1593). In 1604 he was ordained priest. He died in 1615 in Mainz en route to Leuven. He corresponded with the foremost mathematicians of his age, Ludolff van Ceulen and François Viète among them[3].

[1] On Apollonius and *Conics* see Heath (1961); Fried and Unguru (2012).
[2] For a reconstruction of Apollonius's solution see Heath (1981) vol. 2, p. 182–185.
[3] Bockstaele (1976, 2009).


PROBLEMA MATHEMATICVM
omnibus totius orbis Mathematicis ad construendū propositum.

SI duorum terminorum prioris ad posteriorem proportio sit, ut 1 (1) ad 45 (1) -- 3795 (3) + 9,5634 (5) -- 113,8500 (7) + 781,1375 (9) -- 3451,2075 (11) + 1,0530,6075 (13) -- 2,3267,6280 (15) + 3,8494,2375 (17) -- 4,8849,4125 (19) + 4,8384,1800 (21) -- 3,7865,8800 (23) + 2,3603,0652 (25) -- 1,1767,9100 (27) + 4695,5700 (29) -- 1494,5040 (31) + 376,4565 (33) -- 74,0259 (35) + 11,1150 (37) -- 1,2300 (39) + 945 (41) -- 45 (43) + 1 (45), deturque terminus posterior, invenire priorem.

*Exemplum primum datum.*

Sit terminus posterior r bin. 2 + r bin. 2 + r bin 2 + r 2. quęritur terminus prior. SOLVTIO. Dico terminū priorem esse r bin. 2 – r bin. 2 + r bin. 2 + r bin. 2 + r 3.

*Exemplum secundum datum.*

Sit terminus posterior r bin. 2 + r bin. 2 – r bin. 2 – r bin. 2 – r bin. 2 – r 2. quæritur terminus prior. SOLVTIO. Terminus prior est r bin. 2 – r bin. 2 + r bin. 2 + r bin. 2 + r bin. 2 + r 2.

*Exemplum tertium datum.*

Sit terminus posterior r bin. 2 + r 2, quæritur terminus prior.
SOL. Terminus prior est r bin. 2 – r quadrin. 2 + r 3/16 + r 15/16 + r bin. 5/8 – r 5/64

Si in numeris absolutis solinomijs id proponere libuerit: Sit posterior terminus r 3 4142,1356,2373,0950,4880,1688,7242,0969,8073,5696,7187,5375 / 10000,0000,0000,0000,0000,0000,0000,0000,0000,0000,0000,0000.
Quæritur terminus prior. SOLVTIO. Terminus prior erit r 27,4093,0490,8522,5243,1015,8831,2112,6838,8180. / 10000,0000,0000,0000,0000,0000,0000,0800,0000,0000.

**Fig. 2.4** Adriaan van Roomen's challenge: to find a solution to a 45th degree polynomial equation (van Roomen (1593), MPM 8 533)

In 1593 he published *Ideæ Mathematica* in which he challenged his contemporaries to solve a 45th degree polynomial equation[4] (see Fig. 2.4):

$$\begin{aligned}
&45x - 3795x^3 + 95\,634x^5 - 1\,138\,500x^7 + 7\,811\,375x^9 - 34\,512\,075x^{11} \\
&+ 105\,306\,075x^{13} - 232\,676\,280x^{15} + 384\,942\,375x^{17} - 488\,494\,125x^{19} \\
&+ 483\,841\,800x^{21} - 378\,658\,800x^{23} + 236\,030\,652x^{25} - 117\,679\,100x^{27} \\
&+ 46\,955\,700x^{29} - 14\,945\,040x^{31} + 3\,764\,565x^{33} - 740\,259x^{35} \\
&+ 111\,150x^{37} - 12\,300x^{39} + 945x^{41} - 45x^{43} + x^{45} = C
\end{aligned}$$

$$C = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

At one point an ambassador from the Low Countries boasted to the French King Henry IV that van Roomen had listed all European mathematicians but did not mention any French mathematicians. He also referred to van Roomen's challenge. Henry IV had Viète summoned and had him read van Roomen's book. Viète recognised that the polynomial expresses the length of a chord $d_{45} = \text{chord}(45\alpha) = 2\sin\frac{45\alpha}{2}$ given that $x = \text{chord}(\alpha) = 2\sin\frac{\alpha}{2}$. He reported to the King that same day and gave him two solutions, of which one was

[4] Bockstaele (1993).

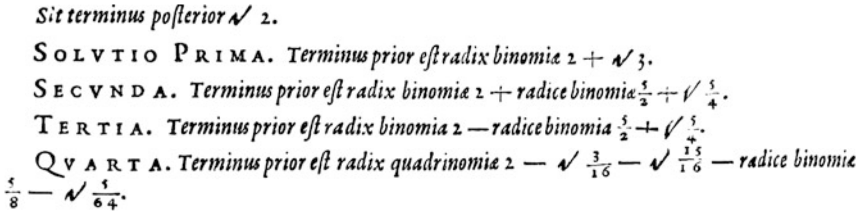

*Sit terminus poſterior* $\surd$ 2.
SOLVTIO PRIMA. *Terminus prior eſt radix binomiæ* $2 + \surd 3$.
SECVNDA. *Terminus prior eſt radix binomiæ* 2 + *radice binomiæ* $\frac{5}{2} + \surd \frac{5}{4}$.
TERTIA. *Terminus prior eſt radix binomia* 2 — *radice binomia* $\frac{5}{2} + \surd \frac{5}{4}$.
QVARTA. *Terminus prior eſt radix quadrinomiæ* $2 - \surd \frac{3}{16} - \surd \frac{15}{16}$ — *radice binomiæ* $\frac{5}{8} - \surd \frac{5}{64}$.

**Fig. 2.5** Viète's solution to van Roomen's problem (Viète (1646), EHC G 4858)

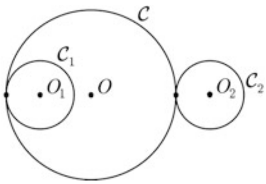



**Fig. 2.6** Circle $C$ touches circle $C_1$ internally and circle $C_2$ externally

$x_0 = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}$, the edge of a regular 96-gon (see ▶ Sect. 6.7). He claimed that he could calculate all other solutions as well (see Fig. 2.5).

Viète published his solution in a book *Ad problema quod omnibus mathematicis totius orbis construendum proposuit Adrianus Romanus, Francisci Vietae responsum* (1595). In this book he challenged van Roomen to a new problem: this was nothing else than Apollonius' problem. Viète maintained that, because all curves have a degree two it should have a compass and straightedge solution (also see ▶ Sect. 3.3). van Roomen was able to solve the problem, but not with compass and straightedge. He used hyperbolae to solve the problem, his solution thus belongs to Plato's second category[5].

Suppose the circles $C_1$, $C_2$ and $C_3$, centered at $O_1$, $O_2$ and $O_3$ and with radii $r_1$, $r_2$ and $r_3$ respectively are given. Van Roomen wants to find the locus of the centre of all circles which are tangent to $C_1$ and $C_2$ and the locus of the centre of all circles which are tangent to $C_1$ and $C_3$. In both cases $C_1$ has to be touched in the same way by the tangent circle, i.e. either internally or externally (see Fig. 2.6). The intersection of these two loci then yields the centre of a circle which is tangent to the three circles and hence a solution to Apollonius' problem.

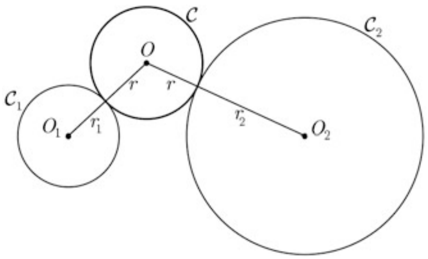
**Exercise 17**

Prove that if a circle $C$ centered at $O$ and with radius $r$ is tangent to a circle $C_1$ centered at $O_1$ and with radius $r_1$, then $\|O_1O\| = r_1 + r$.

Suppose a circle $C$ centered at $O$ and with radius $r$ is tangent to $C_1$ and $C_2$ (see Fig. 2.7). We notice that $\|O_1O\| - \|O_2O\| = (r_1 + r) - (r_2 + r) = r_1 - r_2 =$ constant. We therefore have to determine the locus of all points for which the differences of the distances to the centres of the circles is $r_1 - r_2$. This is a hyperbola $\mathcal{H}_{12}$ with foci $O_1$ and $O_2$. All points on this hyperbola are the centre of circles which are tangent to both circles $C_1$ and $C_2$. We can now do the same for $C_2$ and $C_3$ and we again find a hyperbola $\mathcal{H}_{23}$. The intersections

[5] Barbin and Boyé (2005), p. 10, 24–27, Bos (2001), p. 110–112.

**Fig. 2.7** The locus of the centres of the circles tangent to two given circles is a hyperbola. $\|O_1O\| - \|O_2O\| = (r_1 + r) - (r_2 + r) = r_1 - r_2 = \text{constant}$




of these two hyperbolae yield two centres of tangent circles (see Fig. 2.8). To see why not all intersections yield a centre for a tangent circle we need to look at the branches of the hyperbolae. For instance for $\mathcal{H}_{12}$ one branch yields all the centres of circles $C$ which are externally tangent to $C_1$ and $C_2$ – i.e. $C$ contains neither $C_1$ nor $C_2$ – the other branch yields the centres of circles $C$ for which $C_1$ and $C_2$ are internally tangent to $C$ (see Fig. 2.9).

Constructing e.g. $\mathcal{H}_{13}$ yields other solutions to the problem (see Fig. 2.10).

**Exercise 18**

Create an IGS file in which you solve Apollonius' problem. Select three points $A$, $B$ and $C$ and draw circles $C_A, C_B, C_C$ centered at these points and with chosen radii $r_A, r_B, r_C$. Draw a line through $A$ and $B$ on the one hand and $B$ and $C$ on the other. Determine the intersection points of these lines with the respective circles. For the line through $A$ and $B$, select one of the intersection points on each circle, $I_A$ and $I_B$ and determine their midpoint $M_{AB}$ (see Fig. 2.11). $A$ and $B$ are the foci of a hyperbola and $M_{AB}$ is on the hyperbola. Repeat this procedure for $B$ and $C$. $B$ and $C$ are the foci of the second hyperbola, and $M_{BC}$ is on the hyperbola. Determine the intersections of the hyperbolae. Select one of the intersection points as the centre of a circle. Select the option

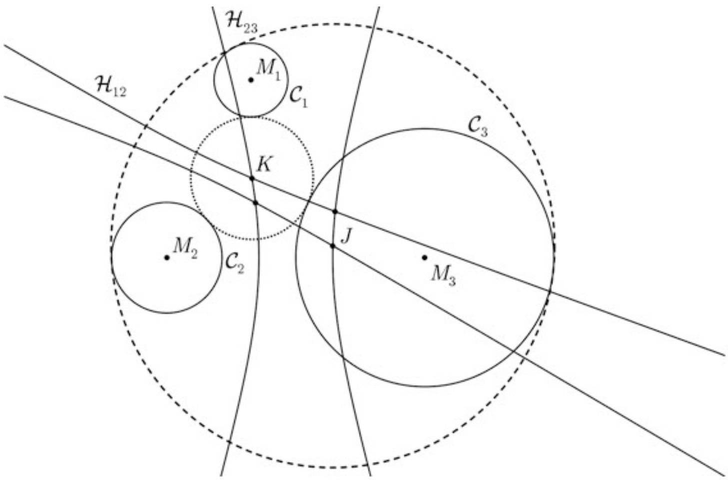



**Fig. 2.8** Three circles, in *solid lines*, are given. The hyperbolae which are the loci of the centre of circles tangent to $C_1$ and $C_2$ on the one hand and $C_2$ and $C_3$ on the other are drawn. The intersection point $A$ is the centre of a circle externally tangent to the three circles (*dotted line*). The intersection point $B$ is the centre of a circle for which the given circles are internally tangent (*dashed line*)

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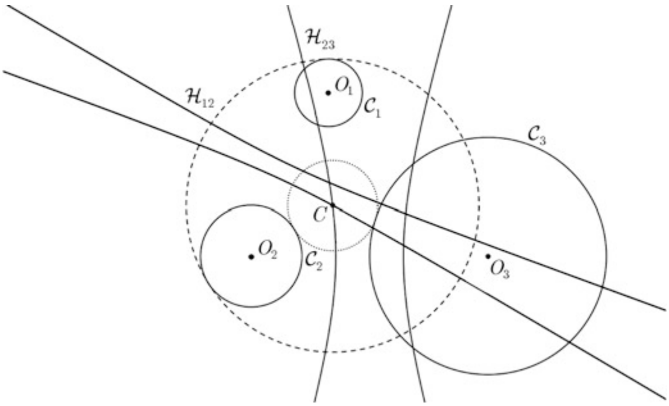



**Fig. 2.9** In this figure intersection point $C$ is used to construct the tangent circle. We clearly see that one circle centered at $C$ is externally tangent to two circles (*dotted line*), while another is internally tangent to two circles (*dashed line*). The branch of $\mathcal{H}_{12}$ through $C$ yields all the centres of circles $C$ which are internally tangent to $C_1$ and $C_2$, the branch of $\mathcal{H}_{23}$ yields the centres of circles $C$ which are externally tangent to $C_2$ and $C_3$. Obviously a tangent circle to the three circles cannot be tangent to $C_2$ internally and externally at the same time

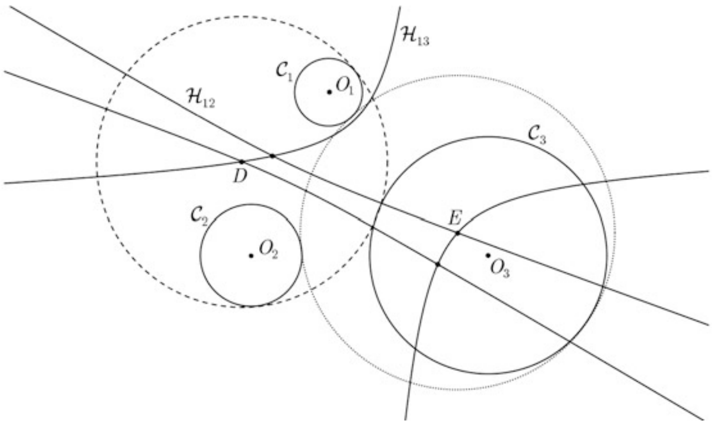
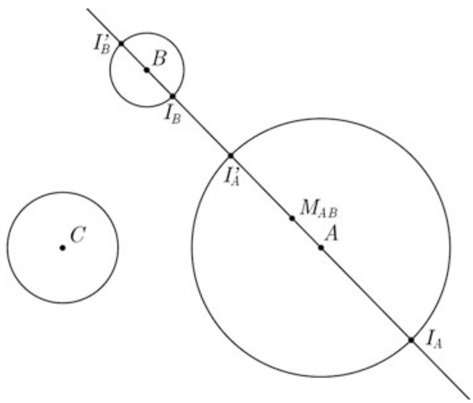



**Fig. 2.10** Another solution to Apollonius' problem. The hyperbolae which are the loci of the centres of circles tangent to $C_1$ and $C_2$ on the one hand, and $C_1$ and $C_3$ on the other, are drawn. The intersection point $D$ is the centre of a circle externally tangent to one circle and internally tangent to two circles (*dashed line*). The intersection point $E$ is the centre of a circle externally tangent to two circles and internally to one (*dotted line*)

**Fig. 2.11** $M_{AB}$ is the midpoint of $[I_A, I_B]$. We can distinguish between four line segments for which the endpoints are on the respective circles: $[I_A, I_B], [I_A, I'_B], [I'_A, I_B], [I'_A, I'_B]$



Circle with centre and through point, draw the circle in such a way that it is tangent to one of the given circles. You will notice that either the circle is tangent to only two circles or that it is tangent to the three circles. Depending on which of the intersection points of the line with the circle you have chosen the tangency will either be internal or external.

**Exercise 19**

Use the IGS file of Exercise 18 to identify all possible solutions to Apollonius' problem. The problem has a maximum of eight solutions.

## 2.4 Mechanical construction aids

Although conic sections cannot be constructed with compass and straightedge, they can be constructed using mechanical means, methods from Plato's third category. Consider for instance an ellipse. An ellipse is defined as the locus of the points of which the sum of the distances to two given points (the foci) is a constant. Figure Fig. 2.12 shows how an ellipse can be constructed with the aid of two drawing pins and a piece of rope.

**Exercise 20**

Create an IGS file in which you choose two numbers, $a$ and $c$, $c < a$. Define a slider $t$ ranging from $a - c$ to $a + c$. Draw the points $F'(-c, 0)$ and $F(c, 0)$. Draw the circles $C_1$, centered at $F$ and with radius $t$, and $C_2$, centered at $F'$ and with radius $2a - t$. Determine the intersection points $D$ and $E$ of the circles $C_1$ and $C_2$. Determine the locus of $D$ and $E$ respectively with the slider $t$.

There are other ways to describe an ellipse with mechanical aids (see Fig. 2.13 right and Fig. 2.14). One is Archimedes' trammel for which we use a ruler, a set square, and a pencil (see Fig. 2.13 left):

» Draw two perpendicular lines $x$, $y$ on paper; these will be the major ($x$) and minor ($y$) axes of the ellipse
Mark three points $A$, $B$ and $C$ on the ruler such that $\|AC\|$ is the length of the semi-major axis and $\|BC\|$ is the length of the semi-minor axis
With one hand, move the ruler across the paper, turning and sliding it so as to keep point $A$ on line $y$, and $B$ on line $x$ at all times
With the other hand, keep the pencil's tip on the paper, following point $C$ of the ruler
The tip will trace out an ellipse.

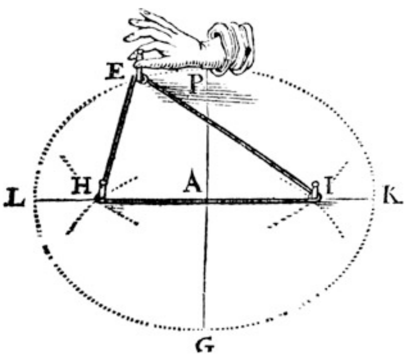



**Fig. 2.12** Drawing an ellipse using two tacks and piece of rope (van Schooten (1659), KBR VH 8.040 A)

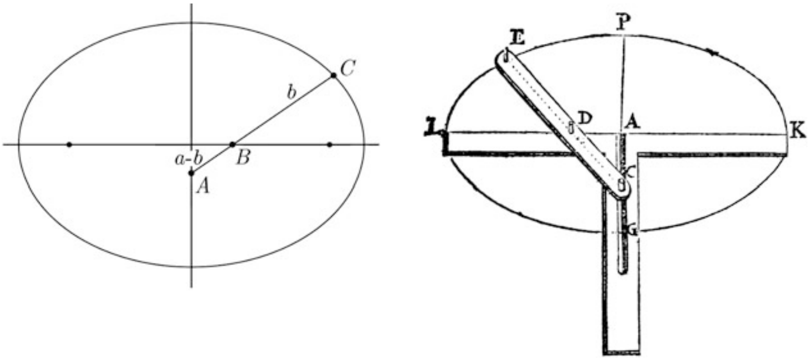



**Fig. 2.13** Archimedes' trammel (*right*: van Schooten (1659), KBR VH 8.040 A)

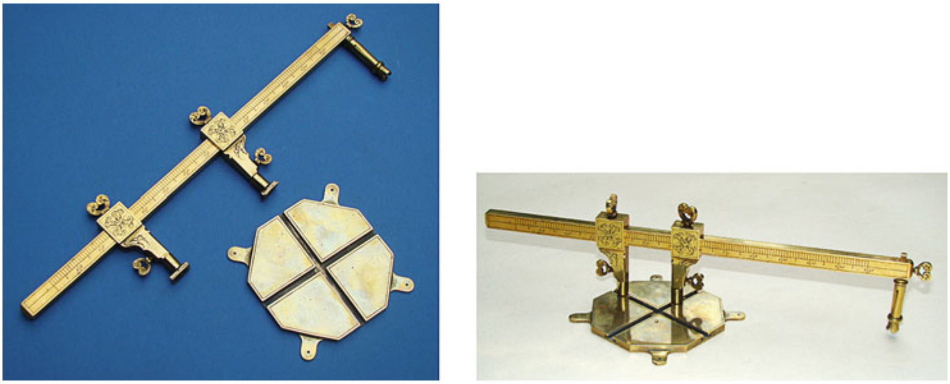


**Fig. 2.14** Italian ellipsograph signed on the arm "Dominicus Lusuerg F. Romae 1700". The Lusuerg family had some of the most remarkable craftsmen building scientific instruments of the late 17th and early 18th centuries, such as Dominicus (1669–1744) who built a wide number of mathematical instruments. Photos *Tesseract – Early Scientific Instruments (Hastings-on-Hudson)*

**Exercise 21**

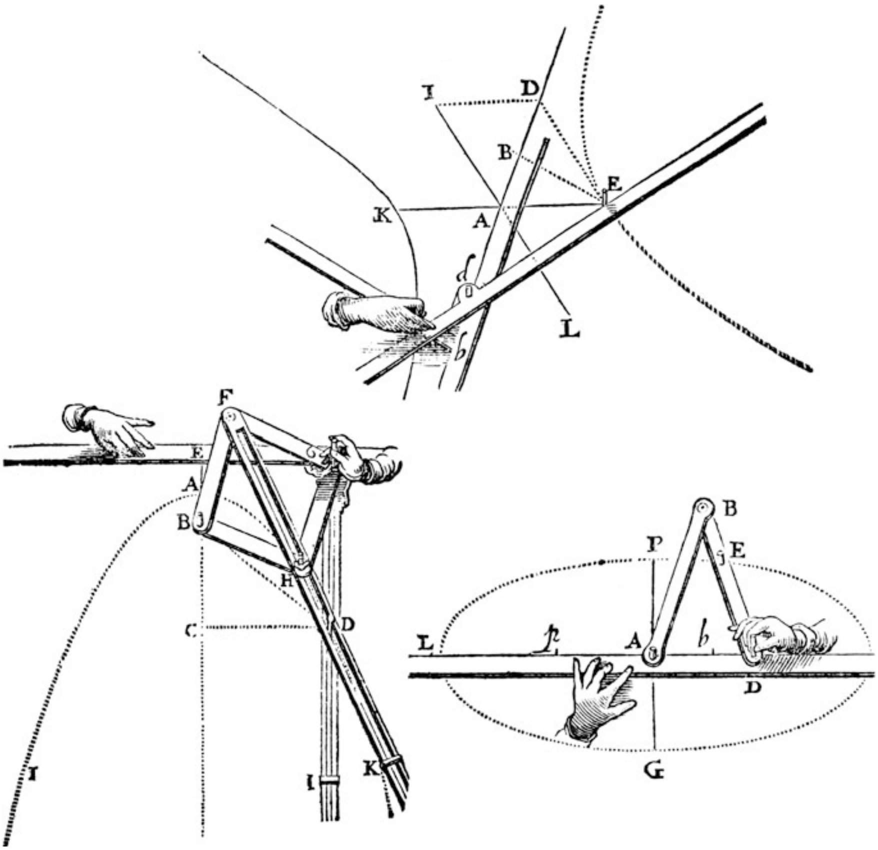
Prove the procedure of Archimedes' trammel (hint: determine the coordinates of $C$ when $AB$ makes an angle $\theta$ with the $x$-axis). Prove that $[AB]$ is the diameter of a circle $\mathcal{C}$ which passes through the origin. If $[OD]$ is a diameter of $\mathcal{C}$, prove that $D$ traces out a circle as $A$ moves along the $x$-axis (the locus of $D$ with $A$).

**Exercise 22**

Create an IGS file in which you mimic Archimedes' trammel.

Quite a number of drawing devices with which conic sections can be drawn were proposed by Frans van Schooten in his book *Exercitationum Mathematicorum libri quinque* (Leiden, 1657) (see Fig. 2.15). Frans van Schooten was the grandson of Franchois Verschooten, a Fleming who had fled the religious intolerance of the Spanish Netherlands in 1584 and had settled in the Northern Netherlands, which was fighting for independence from Spain[6].

[6] van Maanen (1987), p. 212–213.



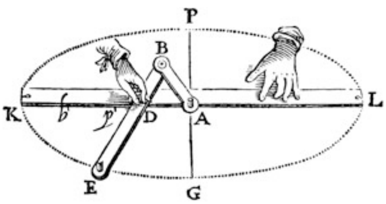

■ **Fig. 2.15** Some of Frans van Schooten's drawing devices to draw a conic section (van Schooten (1659), KBR VH 8.040 A)

His father, Frans van Schooten sr., was a professor at the Leiden engineering school. Frans jr. followed in his footsteps. He championed the ideas of Descartes and made fundamental contributions to analytic geometry.

**Exercise 23**[7]

One of van Schooten's devices can be seen in the figure (van Schooten (1659), KBR VH 8.040 A).

On a long slat $KL$ a small slat $AB$, with length $a$, can rotate about $A$. Another slat $BE$ can rotate about $B$. On $BE$ a point $D$ is determined for which $\|BD\| = \|AB\| = a$.




[7] van Maanen (1987), p. 215.

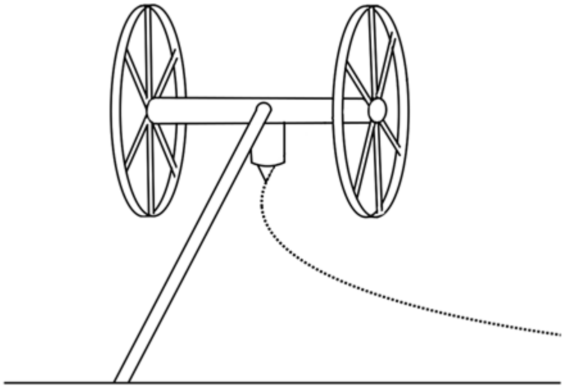


**Fig. 2.16** The "construction" of a tractrix. After a drawing in Huygens' manuscript on tractional motion (University Library Leiden, Ms Hug. 6, f64*r*)

Point $D$ is attached to $KL$ and can move along this slat. Put $\|DE\| = b$. As $D$ is moved along $KL$ $E$ traces out an ellipse.

Prove this assertion by finding the equation of the ellipse in a system in which $KL$ is the $x$-axis and $A$ the origin.

**Exercise 24**

Create an IGS file in which you mimic van Schooten's device of Exercise 23.

An archetypical example of a curve which is drawn mechanically is the *tractrix* (see Fig. 2.16). In the seventeenth century, watches were rather heavy, and were usually kept on a chain in a pocket. The problem of which curve the watch would observe if the end of the chain was dragged along a straight line, was put by Claude Perrault (1613–1688) to Gottfried Leibniz (1646–1716) in 1676. Perrault is best remembered as the brother of Charles Perrault, author of such classic children's tales as "Cinderella" and "Puss-in-Boots". Trained as a physician, in 1666 Claude was invited to become a founding member of the Académie des Sciences, where he earned a reputation as an anatomist.

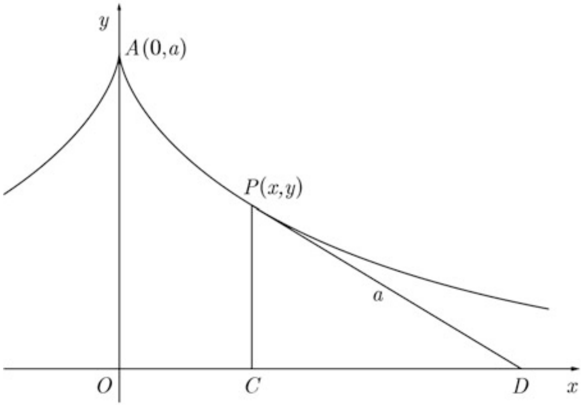
The first known solution to Perrault's problem was given by Christian Huygens (1629–1695), who named the curve the *tractrix* from the Latin "tractus" which denotes something that is pulled along. In Huygens' presentation, it is not a watch but a cart of which the drawbar is being drawn along a line. At the rear of the cart a pencil is attached which traces out the tractrix[8].

Although solving this problem is just beyond the scope of this book we will give an outline here for those who have a knowledge of differential equations.

Take a look at Fig. 2.17. Suppose $P\ (x, y)$ is a point on the curve. Suppose $C$ has co-ordinates $(x, 0)$, suppose $D$ is the other end of the drawbar $[PD]$, moving along the $x$-axis. The drawbar, which has length $a$, has to be tangent to the curve. If the height is $y$, then using Pythagoras' theorem, we know that $\|CD\| = \sqrt{a^2 - y^2}$. Therefore the slope of the straight line along the drawbar is $-\dfrac{y}{\sqrt{a^2 - y^2}}$. On the other hand, calculus proves that the slope is given by $\dfrac{dy}{dx}$.

[8] Bos (1989), p. 10–12.

**Fig. 2.17** The tractrix




The problem reduces to solving the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}.$$

This is an equation which can be solved by separating the variables to produce the following solution:

$$x = \int_y^a \frac{\sqrt{a^2 - t^2}}{t}\, dt = \pm \left( a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \right).$$

Unfortunately, the nature of this problem, involving differentials, does not allow for a construction to be created in an IGS using elementary techniques.

## 2.5 Neusis

A particular construction method belonging to Plato's third category is a *neusis* (from the Greek νεῦσις from νεύειν or "neuein", meaning "inclining or pointing towards"; plural: νεύσεις "neuseis"). In a neusis construction, a line segment $[AB]$ with length $a$ is fitted between two curves, $l$, the directrix, and $m$, the captrix (from the Latin *captare*, meaning "to catch"), in such a way that the straight line $AB$ passes through a given point $P$, the pole of the neusis (see Fig. 2.18). Neusis is also called *verging* in English. To obtain this solution one verges a ruler, on which $[AB]$ is marked off, through $P$ in such a way that $A$ follows the directrix. The solution is obtained at the position where $B$ is on the captrix. Note that the roles of the directrix and the captrix can be interchanged. If $B$ follows line $m$, then the solution is reached when $A$ is on $l$ (see Fig. 2.19).

When the line segment is fitted in between the curves, a solution for a particular problem is found (e.g. the trisection of an angle, the duplication of the cube, ...). Implicitly, by using a *neusis* we have defined a new curve: the locus of point $B$ while $A$ moves along the directrix. The question is thereby reduced to finding the intersection(s) of this locus with the captrix $m$. Note that we have already defined such a curve as "the locus of $B$ with $A$" (see ► Sect. 1.2).

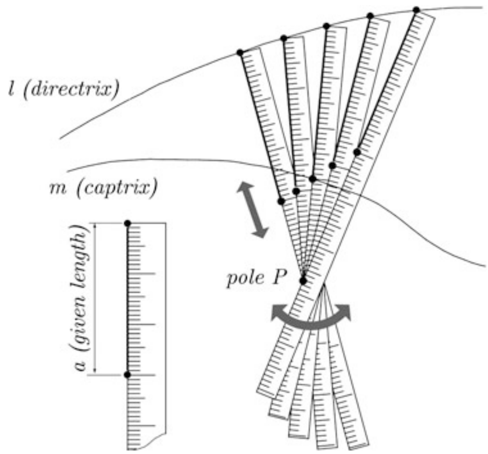



**Fig. 2.18** Performing a neusis (drawing by Paul Tytgat)

**Exercise 25**
Create an IGS file in which you select a pole $P$ for a neusis. Draw a circle centered at $O$, not coincident with $P$, and with radius 5. Draw a straight line $a$ which intersects the circle. Select a point $A$ on $a$. Draw $PA$ and determine the points $B$ and $C$ for which $\|AB\| = \|AC\| = 3$. Determine all positions for which either $[AB]$ or $[AC]$ can be fitted in between $a$ and the circle by dragging $A$ along $a$. Remember the point $A$ does not have to lie within the circle.

**Exercise 26**
Use the IGS file from Exercise 25 and draw the locus of $B$ with $A$ and of $C$ with $A$. The intersections of both branches of the locus give the positions for either $B$ or $C$ when the line segment is fitted in between the straight line $a$ and the circle. You can determine these positions by sliding $A$ over $a$. Let the position of the pole $P$ and the orientation of the line $a$ differ. Can you find an orientation for which there are 2, 3, 4, 5, 6 intersections of the branches with the circle? Can you find a configuration with only one intersection?

We can make an instrument to draw such curves. Take a ruler with a slot in the middle. Put a cursor in the pole $P$ and put the slot of the ruler over it. From the end of the ruler $A$

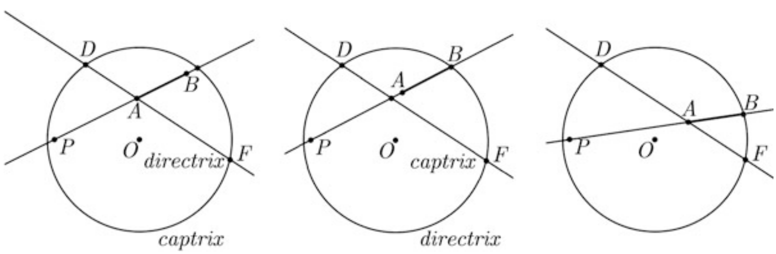



**Fig. 2.19** An example of a neusis. Suppose that given a pole $P$ a line segment $[AB]$ has to be fitted in between the straight line and the circle. In the *left figure* the straight line is used as directrix, in the *middle one* the circle is the directrix. In the *right figure* (one of) the final position(s) is reached. If we use the circle as directrix $A$ has to coincide with the intersection of the straight lines. If the straight line is used a directrix, then $B$ has to coincide with the intersection of the line $PA$ and the arc of a circle $DF$

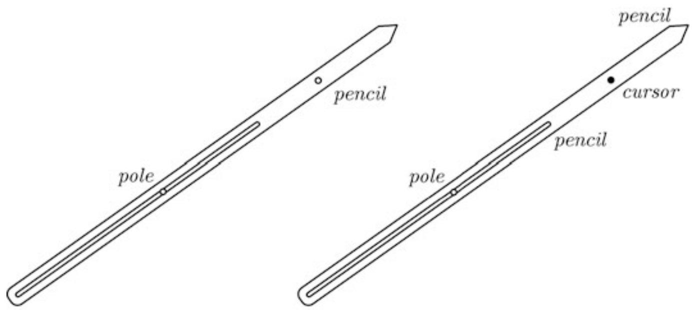



**Fig. 2.20** An instrument to draw conchoids with (drawing by Paul Tytgat)

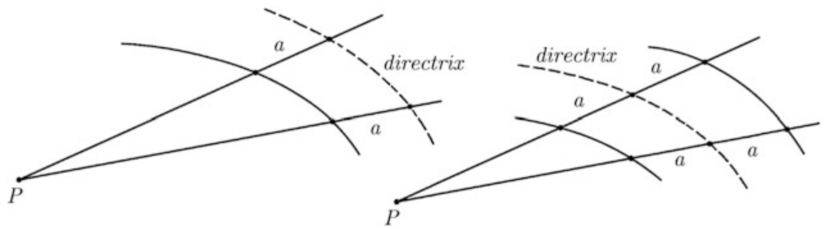



**Fig. 2.21** $D_1$ and $D_2$ are branches of a conchoid generated by $K$ (drawing by Paul Tytgat)

determine a point $B$ for which $\|AB\| = a$. Attach a pointer to $B$ and a pencil to $A$. Let the pointer $B$ follow the directrix $l$ and the pencil, at $A$, will draw the desired curve.

Of course, there are two points at a distance $a$ from a given point. So again take the ruler and from the end $A_1$ mark off a point $B$ at a distance $a$. Then from $B$, mark off another point $A_2$ at a distance $a$ from $B$. Put the slot of the ruler over the cursor at $P$ and attach a cursor to $B$. Attach pencils to both $A_1$ and $A_2$. Let $B$ follow the directrix $l$. The pencils now draw two branches of a curve, which we call a *conchoid* (see ▶ Sect. 5.1).

For directrices with polar equation $r = f(\theta)$, and the pole of the neusis in the origin, the associated conchoids have a polar equation $r = f(\theta) \pm a$.

Consider the following situation: the pole $P$ for a neusis is given, a line segment has length $a$, the directrix is a circle with radius $b$, the captrix is a certain curve (not shown in Fig. 2.22). One of the end points $A$ of the line segment is on the circle. Which curve will the other endpoint $B$ on $[AP$ describe? In other words, what is its locus when $A$ moves along the circle? The intersection of this line with the captrix will solve the problem. In Fig. 2.22 we can see examples of the locus of $B$ with $A$. In the figure on the left, the pole of the neusis $P$ and the centre of the circle $O$ coincide. The locus is a circle centered at $P$ and with radius $R = b - a$. In the figure on the right, the pole and the centre do not coincide, the locus now is a closed curve.

**Exercise 27**

Create an IGS file in which the origin is the pole $P$ of a neusis. Draw a point $A$ on the $x$-axis, which is the centre of a circle with radius 5. Select a point $B$ on the circle. Determine points $D$ and $E$ on $PB$ for which $\|DB\| = \|BE\| = 3$. Determine the loci of $D$ with $B$ and of $E$ with $B$ respectively. Move $A$ along the $x$-axis. What do you notice?

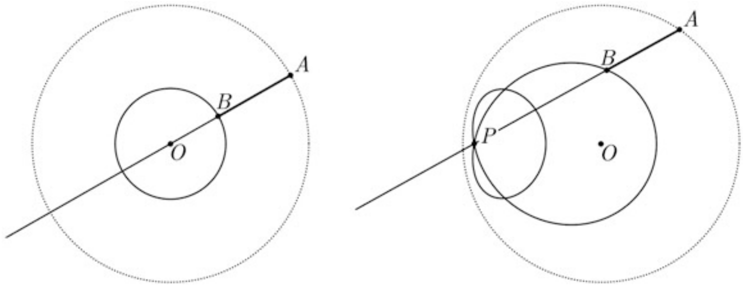



Fig. 2.22 The locus of $B$ with $A$ generated by a neusis in which the directrix is a circle. *Left* if the pole coincides with the centre of the circle, *right* if the pole and the centre do not coincide

**Exercise 28**

Create an IGS file in which the origin is the pole $P$ of a neusis and in which the directrix is a circle with radius 5 through the origin and with its centre on the $x$-axis. Define a slider with variable $a$. Let $a$ vary between 0 and three times the radius. Select a point $B$ on the circle. Draw the straight line $PB$. Determine points $D$ and $E$ on $PB$ for which $\|DB\| = \|BE\| = a$. Determine the loci of $D$ with $B$ and $E$ with $B$ respectively. Which kinds of curves are they?

**Exercise 29**

Determine the polar equations of the loci you found in the previous exercise. Hint: look at the triangle with $[BP]$ as one edge and the diameter through $P$ as another. Which kind of triangle is this? Can you determine $\|PB\|$? Remember for the locus of $D$ with $B$ $r = \|PB\| - \|DB\|$.

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Exploring Classical Greek Construction Problems with
Interactive Geometry Software
Meskens, A.; Tytgat, P.
2017, XII, 185 p. 160 illus., 10 illus. in color., Softcover
ISBN: 978-3-319-42862-8
A product of Birkhäuser Basel