Chapter 2
The Brachistochrone Problem: Johann and Jakob Bernoulli

In this chapter we present the early solutions of the brachistochrone problem in the same mathematical words of their authors, if not simply in their words. We restate the problem:

> A New Problem
> that we invite the mathematicians to solve.
> Given points A and B in a vertical plane to find the path AMB down which a movable point M must, by virtue of its weight, proceed from A to B in the shortest possible time.[1]

2.1 Johann Bernoulli's 1697 Paper

We begin with the solution of Johann Bernoulli [20][2] that heavily relies on Fermat's minimum time principle, see Section 1.2.1 and, in particular, on its consequence Snell's law: If two media are separated by a horizontal plane, if v_1 and v_2 are the velocities of light respectively in the two media and if θ_1 and θ_2 are the angles formed by the incident and refracting rays with the vertical, then

[1]In Latin, *Opera Omnia*, t. I, p. 155 and p.166,

> Problema Novum,
> ad cujus Solutionem Mathematici invitantur.
> Datis in plano verticali duobus punctis A et B assignare mobili M viam AMB, per quam gravitate sua descendens et moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B.

[2]The translation of the original Latin sounds as: "The curvature of a ray in nonuniform media, and the solution of the proposed problem in Acta [Eruditorum] 1696, p. 269, to find the brachistochrone line, that is, the curve on which a heavy point falls from a given position to another given position in the shortest time, as well as on the construction of the synchrone or the wave of the rays".

P. Freguglia and M. Giaquinta, *The Early Period of the Calculus of Variations*, DOI 10.1007/978-3-319-38945-5_2

$$\frac{\sin\theta_2}{v_2} = \frac{\sin\theta_1}{v_1}.$$

Johann Bernoulli writes in [20]

[...] We have a just admiration for Huygens, because he was the first to discover that a heavy point of an ordinary *Cycloid* falls in the same time [*tautochronos*] whatever the position from which the motion begins[3]. But the reader will be greatly amazed, when I say that exactly this *Cycloid*, or *Tautochrone of Huygens*, is our required *Brachistochrone*. I reached this understanding in two ways, one indirect and one direct. When I pursued the first, I discovered a wondrous agreement between the curved path of a light ray in continuously varying medium and our *Brachistochrone*. I also found other rather mysterious things which might be useful in dioptric investigations. It is therefore true, as I claimed when I proposed the problem, that it is not just naked speculation, but also very useful for other branches of knowledge, namely, for dioptric. But in order to confirm my words by the deed, let me here give the first mode of proof![4]

Fermat, in a letter to De la Chambre[5], has shown that a light ray passing from a thin to a more dense medium, is bent toward the perpendicular in such a way that, under the supposition that the ray moves continuously from the light to the illuminated point, it follows the path that requires the shortest time. With the aid of these principles he showed that the sine of the angle of incidence and the sine of the angle of refraction are in inverse proportion to the densities of the media, hence directly as the velocities with which the light ray penetrates these media. Later Leibniz, in the Acta Eruditorum, 1682, pp. 185 and sequ.[6], and soon after the famous Huygens in his *Treatise on light*[7], p.40, have demonstrated this more comprehensively and, by most valid arguments, have established the physical, or better the metaphysical, principle

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[3]Johann refers to [127] Proposition XXV, [129] vol. 18; for a presentation see [203] [130] [110].

[4]We shall discuss the second direct method in Section 2.5. A direct discover of the agreement of the isochronous curve and the brachistochrone could go along the following lines. We may think of the idea of Huygens as of modifying the circle of the standard pendulum in such a way that the accelerating force becomes proportional to the arc length s. This way, the movement of the pendulum would be described by

$$\ddot{s} + ks = 0,$$

which has oscillations independent of the amplitude.
The accelerating force is $f = -dy/ds$, hence our requirement $f = -ks$ becomes

$$dy = ksds.$$

Integrating ($s = 0$ for $y = 0$)

$$y = \frac{k}{2}s^2 \text{ or } s = \sqrt{\frac{2y}{k}}.$$

Therefore, for our curve the height is proportional to the square of the arc length and, in terms of the variables x, y one finds

$$\sqrt{\frac{c-y}{y}}dy = dx$$

with $c = 1/(2k)$. A part from the shift in y, as we shall see, this is precisely the equation of the brachistochrone.

[5]See [96] vol. II.

[6]See [160].

[7]See [128].

which Fermat seems to have abandoned at the insistence of Clerselier, remaining satisfied with his geometric proof and giving up the rights all too lightly[8].

Johann Bernoulli then reverses the argument.

Now we shall consider a medium that is not homogeneously dense, but consists of purely parallel horizontally superimposed layers, each of which consists of a diaphanous matter of a certain density decreasing or increasing according to a certain law. It is then manifest that a ray which we consider as a particle will not be propagated in a straight line, but in a curved path. [. . .] We know that the sines of the angles of refraction at a separation points are to each other inversely as the densities of the media or directly as the velocities of the particles, so that the brachistochrone curve has the property that the sines of its angles of inclination with respect to the vertical are everywhere proportional to the velocities. But now we can see immediately that the brachistochrone is the curve that a light ray would follow on its way through a medium whose density is inversely proportional to the velocity that a heavy body acquires during its fall. Indeed, whether the increase of the velocity depends on the constitution of a more or less resisting medium, or whether we forget about the medium and suppose that the acceleration is generated by another cause according to the same law as that of gravity, in both cases the curve is traversed in the shortest time. Who prohibits us from replacing one with the other?

In this way we can solve the problem for an arbitrary law of acceleration, since it is reduced to the determination of the path of a light ray through a medium of arbitrarily varying density[9].

Johann Bernoulli then continues, with reference to Figure 2.1: Let FGD be the medium bounded by the horizontal line FG and let A be the luminous point. Let the curve AHE, with vertical axis AD, be given, the ordinates HC determining the density of the medium at altitude AC or the velocity of the light ray or particles at M. Set $x := AC$, $y := CM$, and $t := CH$; for the ordinate of the curve AHE that yields the velocity of the light at the point of altitude AC. Also, write for the *infinitesimals* Cc, mn and Mm respectively dx, dy, and dz. If θ_r is the angle of refraction at M, then

[8] The translation is taken from [187] p. 392-3.

[9] The translation is taken from [187] p. 393. The original passage in Latin is:

Si nunc concipiamus medium non uniformiter densum, sed velut per infinitas lamellas horizontaliter interjectas distinctum, quarum interstitia sint repleta materia diaphana raritatis certa ratione accrescentis vel decrescentis; manifestum est, radium, quem ut globulum consideramus, non emanaturum in linea recta, sed in curva quadam [...]. Constat quoque, cum sinus refractionum in singulis punctis sint respective ut raritates medii vel celeritates globuli, curvam habere eam proprietatem, ut sinus inclinationum suarum ad lineam verticalem sint ubique in eadem ratione celeritatum. Quibus praemissis nullo negotio perspicitur, curvam *brachystochronam* illam ipsam esse, quam formaret radius transiens per medium, cujus raritates essent in ratione velocitatum, quas grave verticaliter cadendo acquireret. Sive enim velocitatum incrementa dependeant a natura medii magis minusve resistentis, ut in radio; sive abstrahatur a medio, & ab alia causa acceleratio eadem tamen lege generari intelligatur,

ut in gravi: cum utroque in casu curva brevissimo tempore percurri supponatur, quid vetat, quo minus altera in alterius locum substitui possit?

Sic generaliter solvere licet problema nostrum, quamcunque statuamus accelerationis legem. Eo enim reductum est, ut quaeratur curvatura radii in medio secundum raritates, prout libuerit, variante.

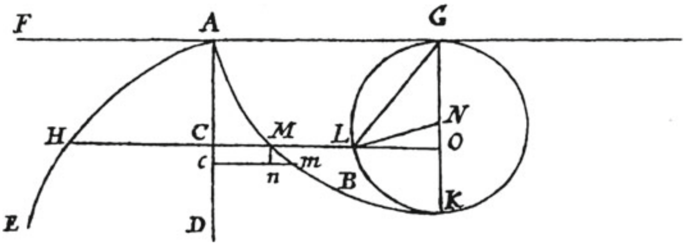

Fig. 2.1 Fig. I of Johann Bernoulli's paper.

$$\sin\theta_r = \frac{nm}{Mm} = \frac{dy}{dz} \tag{2.1}$$

and, according to Fermat's principle or Snell's law, for the path we wish to determine this quantity as to be proportional to the velocity t :

$$\frac{dy}{dz} = \frac{1}{a}t \tag{2.2}$$

for some positive constant a. This yields, since $dz^2 = dx^2 + dy^2$

$$dy^2 = \frac{t^2}{a^2}dz^2 = \frac{t^2}{a^2}(dx^2 + dy^2) = \frac{t^2}{a^2}dx^2 + \frac{t^2}{a^2}dy^2,$$

hence

$$dy^2 = \frac{t^2}{a^2 - t^2}dx^2$$

that is,

$$\frac{dy}{dx} = \frac{t}{\sqrt{a^2 - t^2}} \tag{2.3}$$

At this point Johann Bernoulli takes the special case of the brachistochrone problem where, according to *Galilei's law of falling bodies* the velocity t is proportional to the square root of the falling height, and sets[10] $t = \sqrt{ax}$. Replacing such a value of t in the equation (2.3) he then finds

$$dy = \sqrt{\frac{x}{a - x}}dx \tag{2.4}$$

[10]In principle, the choice of the factor $\sqrt{a}$ is arbitrary and, as we know, a more reasonable value is $t = \sqrt{2gx}$, where g is the gravity constant, but it is easily seen that this leads to the same equation (2.4), of course with a different constant as a, that we may again name a.

from which he concludes, as he states it, that the *brachistochrone* is the ordinary *cycloid*.

Of course, the claim rests on the integration of the differential equation (2.4). He rewrites (2.4) as

$$dy = dx\sqrt{\frac{x}{a-x}} = \frac{1}{2}\frac{adx}{\sqrt{ax-x^2}} - \frac{1}{2}\frac{a-2x}{\sqrt{ax-x^2}}dx \tag{2.5}$$

in such a way that the first term on the right-hand side be easily integrated as

$$\int \frac{1}{2}\frac{a-2x}{\sqrt{ax-x^2}}dx = \sqrt{ax-x^2}\,[+\text{const}]. \tag{2.6}$$

and observes that geometrically, with[11] reference to Figure 2.1,

$$\sqrt{ax-x^2} = LO \tag{2.7}$$

where LN is the radius of a circumference GLK with diameter $GK = a$ and, recall, $x = GO$[12].

Next, Johann Bernoulli claims[13] that the first term on the right-hand side of (2.5) is the differential of arc GL, i.e.,

$$\text{arc}\,GL = \int \frac{a}{2\sqrt{ax-x^2}}dx \tag{2.8}$$

[11] Thus

$$y = \sqrt{ax-x^2} + \frac{1}{2}\int \frac{a}{\sqrt{ax-x^2}}$$

that is the equation of the cycloid that Leibniz had found in 1686.

[12] This is easily seen analytically or as consequence of Euclid's theorem; in fact,

$$\begin{aligned}\sqrt{ax-x^2} &= \sqrt{GK\cdot GO - GO^2} = \sqrt{GO(GK-GO)} = \\ &= \sqrt{GO\cdot KO} = LO,\end{aligned}$$

since KLG is a right-angle triangle.

[13] As for instance,

$$\begin{aligned} d[\text{arc}\,GL] &= \sqrt{dLO^2 + dOG^2} = \\ &= \sqrt{[d(\sqrt{ax-x^2})]^2 + dx^2} = \sqrt{[\frac{adx-2xdx}{2\sqrt{ax-x^2}}]^2 + dx^2} = \\ &= \sqrt{\frac{a^2dx^2 + 4x^2dx^2 - 4axdx^2 + 4axdx^2 - 4x^2dx^2}{(2\sqrt[2]{ax-x^2})^2}} = \\ &= \frac{adx}{2\sqrt{ax-x^2}}.\end{aligned}$$

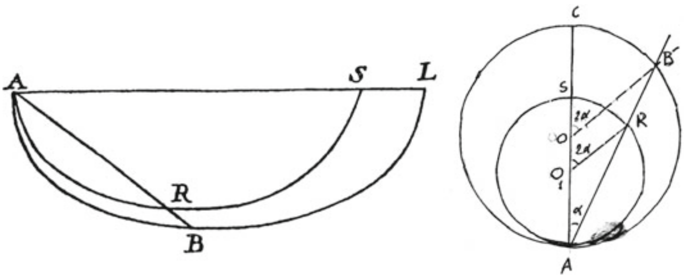

Fig. 2.2 Fig. II of Johann Bernoulli's paper and a complementary one.

and infers from (2.4), (2.5), (2.6), (2.7) and (2.8):

$$CM = y = \int dy = \text{arc}\, GL - LO. \tag{2.9}$$

As

$$\begin{aligned} MO = CO - CM &= CO - \text{arc}\, GL + LO = \\ &= \text{semicirc}\, GLK - \text{arc}\, GL + LO = \\ &= \text{arc}\, LK + LO \end{aligned}$$

and

$$MO = ML + LO,$$

he finally concludes:

$$\text{arc}\, LK = ML \tag{2.10}$$

By taking into account the definition of the cycloid, (2.10) readily yields that the curve *AMK* that solves the differential equation (2.4) is a cycloid[14].

So far Johann Bernoulli has proved that the quickest descent curve must be a cycloid[15]. He then proves that indeed there is a unique cycloid passing through

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[14]This is also seen analytically as in [117] p.41 (footnote 49): let $b = a/2$, arc $LK = b\varphi$ and arc $GL = b\pi - b\varphi = bt$. Then $\sin t = \sin\varphi = LO/b$ and $CO = AG = b\pi = y + ML + LO = y + \text{arc}\, LK + b\sin t$; hence $y = b\pi - \text{arc}\, LK - b\sin t = b\pi - b\varphi - b\sin t = b(t - \sin t)$. To find x, note that $\cos t = -\cos\varphi = -(b - OK)$ and that $x = AC = GK - OK = b(1 - \cos t)$. Note that b is the radius of the generating circle.
In the remainder of this section we shall see how, in the light of the common knowledge of the time about the geometrical properties of the cycloid, there are several ways of inferring from (2.1) that the solution is a cycloid. Such a geometrical understanding most likely was the way to the solution for the scholars of the time including Leibniz, Newton, Johann and Jakob Bernoulli. It is also likely that the mathematicians of the Continent stressed the analytical aspects in view of the generality of the approach and in contrast to the geometric character of the Newtonian calculus.

[15]For modern standard, if a quickest time descent curve *exists*.

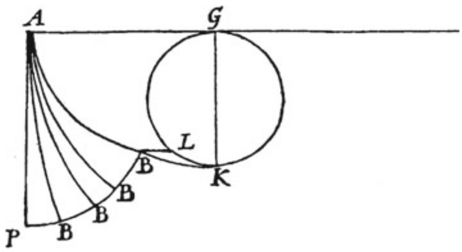

Fig. 2.3 Fig. III of Johann Bernoulli's paper.

the given points A and B and shows how it can be constructed. He starts from the observation that, given two cycloids ARS and ABL on the horizontal line AL and the straight line AB as in Figure 2.2, then

$$AR : AB = AS : AL, \tag{2.11}$$

consequently, given any cycloid ARS with basis AS and a point B, the cycloid from A with basis AL given by the proportion (2.11) is the unique cycloid starting in A with basis on the infinite straight line AL through B.[16] This is easily seen via Figure 2.2, on the right, as the angular velocities of the generating circles (with bases AS and AL) are equal.

In the last two paragraphs of his paper, Johann Bernoulli deals with the problem to finding the curve PB (see Figure 2.3) called by him *synchrone* that heavy bodies falling from A on cycloids reach in the same time. The problem is connected to the wave front problem of Huygens that in turn is related to the problem of cutting orthogonally a family of curves (in the specific, cycloids). Johann Bernoulli indeed shows that the synchrone PB cuts the family of cycloids orthogonally and indicates how to construct it[17]. Proofs are not given. We shall not deal with this topic, for a reconstruction of the proofs and a discussion of Johann Bernoulli's claims the reader may see [186] pp. 137-138 note 10, [117] pp.42-44, and the introduction of [118].

2.2 Jakob Bernoulli's Solution

Jakob Bernoulli begins his paper [9] saying that he would have paid no attention to his brother's problem were not for Leibniz's invitation and that then he had solved the problem in a few weeks, by October 1696, finding what he calls the *oligochrone*.

[16]Given a cycloid ARS on the horizontal line AS, the cycloid ABL defined as above is the quickest time descent curve from A to B. This is Newton's solution to the brachistochrone problem without any motivation of why the solution has to be a cycloid, see Section 2.4.

[17]This may be regarded as an anticipation of the transversality condition that will play an important role in the Nineteenth century.

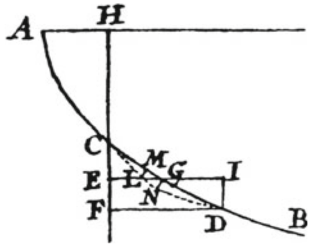

Fig. 2.4 Jakob Bernoulli's solution.

Jakob's approach is quite different from that of Johann, and apparently was very influential, in particular to Euler, in developing the early techniques of the calculus of variations.

First, Jakob Bernoulli notices or proves[18] that if C and D are two points of the least time descent curve from A to B, then arc CD is, among all arcs joining C to D (see Figure 2.4), that which a falling body travels through most rapidly. In fact, if the body travelled more quickly on an other subarc CED then $ACEDB$, instead of $ACDB$, had to be the least time descent curve.

Then he chooses C and D infinitely close on the minimizing curve and a point L on the straight line EI, E being the middle point between C and F, in such a way that GL be the differential of GE. In other words GL is an infinitesimal of higher order[19] with respect to GE; the lines AH, EI, and FD are parallel, and HF is orthogonal to them, compare Figure 2.4. Since the arc CGD is the (a) minimizing arc through C and D Jakob Bernoulli concludes

$$t_{CL} + t_{LD} = t_{CG} + t_{GD},$$

where we recall that, for instance, t_{CL} means the time to descent from C to G, and so

$$t_{CG} - t_{CL} = t_{LD} - t_{GD}. \tag{2.12}$$

According to the law of the inclined plane (considering CG as an inclined plane), see Section 1.2.5, he states

$$\begin{aligned} CE \; : \; CG &= t_{CE} \; : \; t_{CG}, \\ CE \; : \; CL &= t_{CE} \; : \; t_{CL}, \end{aligned}$$

hence (on account of a well-known Euclidean rule relative to proportions) he deduces:

$$CE \; : \; (CG - CL) = t_{CE} \; : \; (t_{CG} - t_{CL}). \tag{2.13}$$

[18]Previously we mentioned that this is often called *Euler's lemma*.

[19]This is essential for the following and involves the second differentials.

Let M be the orthogonal projection of L on CG. Since GL is an infinitesimal of higher order with respect to EG, he may assume[20] that $CG - CL = MG$. On the other hand, the similarity of the triangles MLG and CEG yields

$$EG : CG = MG : GL.$$

Multiplying by $t_{CE} : (t_{CG} - t_{CL})$ Jakob then concludes

$$CE : GL = EG \cdot t_{CE} : CG \cdot (t_{CG} - t_{CL}). \tag{2.14}$$

on account of (2.13).

Similarly, if N is the projection of G on LD, we have

$$EF : GL = GI \cdot t_{EF} : GD \cdot (t_{LD} - t_{GD}). \tag{2.15}$$

Comparing equations (2.14) and (2.15), he gets

$$EG \cdot t_{CE} : CG \cdot (t_{CG} - t_{CL}) = GI \cdot t_{EF} : GD \cdot (t_{LD} - t_{GD}) = CG : GD$$

and

$$EG \cdot t_{CE} : GI \cdot t_{EF} = CG \cdot (t_{CG} - t_{CL}) : GD \cdot (t_{LD} - t_{GD}) = CG : GD,$$

by taking into account (2.12). As, according to the gravity law (1.3), we have

$$EG \cdot t_{CE} : GI \cdot t_{EF} = \frac{EG}{\sqrt{HC}} : \frac{GI}{\sqrt{HE}},$$

Jakob Bernoulli concludes that

$$\frac{EG}{\sqrt{HC}} : \frac{GI}{\sqrt{HE}} = CG : GD, \tag{2.16}$$

that is, the element of line of the minimizing curve is directly proportional to the element of abscissa and inversely proportional to the square root of the ordinate (contrary to Leibniz, he uses our denomination of coordinates). He then claims that

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[20]In fact

$$\begin{aligned} CL = \sqrt{CM^2 + ML^2} = CM\sqrt{1 + \frac{ML^2}{CM^2}} \\ \backsimeq CM + \frac{1}{2}\frac{ML^2}{CM} \backsimeq CM. \end{aligned}$$

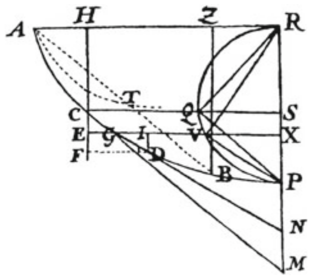

Fig. 2.5 Jakob Bernoulli characterization of cycloid.

such a property belongs to and characterizes the isochronous curve of Huygens, thus the minimizing curve is a cycloid.

He proves his claim geometrically, with reference to Figure 2.5. He notices that it is characteristic of cycloides that

$$GD : GI = GN : GX = VP : VX = VR : RX = \sqrt{RT}$$

and

$$EG : CG = CS : CM = QS : QP = RS : RQ = \sqrt{RS} : \sqrt{RP},$$

Hence

$$GD : CG = GI\sqrt{RP}\sqrt{HC} : EG\sqrt{HE}\sqrt{RP} = GI\sqrt{HC} : EG\sqrt{HE}.$$

In [8] he argues analytically. He sets (the naming of coordinates being the old one)

$$CG := ds = \sqrt{dx^2 + dy^2},\ HE := x,\ CE = dx,\ EG := dx,$$

and writes (2.16) as

$$ds = \frac{k}{\sqrt{x}}dy,$$

that is

$$\frac{dy}{dx} = \sqrt{\frac{x}{k^2 - x}},$$

and claims that the solution is a cycloid.

Finally, Jakob Bernoulli shows (exactly as Johann) that there is a cycloid starting from A and going through B.

The paper concludes with three problems on which Jakob challenges geometers, and in particular his brother Johann. The first problem asks to find among all cycloids through A (whit horizontal basis AH the one that intersecting a vertical line ZB is of least time descent between A and B. A problem that had been essentially already solved by Johann (the cycloid has to cut orthogonally ZB). The second problem asks to find the path of a falling body in a resistant medium. It shows that it was clear also to Jakob the relevance both optical and mechanical of the brachistochone problem. Finally the third problem concerns the isoperimetric problem on which we shall return later.

2.3 Leibniz's Solution

Leibniz's solution to the brachistochrone problem is contained in his reply letter to Johann Bernoulli[21] dated 16 June 1696. Denoting the altitude coordinate by x and the horizontal longitude by y, Leibniz claims that the curve of least time descent is the one for which the element of line is directly proportional to element of latitude and inversely proportional to the square root of the altitude, that is,

$$ds = \frac{k}{\sqrt{x}}dy,$$

or, since $ds^2 = dx^2 + dy^2$,

$$\frac{dy}{dx} = \sqrt{\frac{x}{2b - x}},$$

where $2b := k^2$.

The letter contained an addendum[22] in which he explained how he deduced the equation[23]. Following [117] we shall report on the main steps of Leibniz's argument.

With reference to Figure 2.6, on the left, Leibniz tries to find the point D on the horizontal line through the middle point E of AC and parallel to CB in such a way that the path ADB be of least time descent.

According to Galilei we have:

$$t_{AE} = \sqrt{\frac{AE}{AC}}\, t_{AC} \quad t_{EC} = \left(1 - \sqrt{\frac{AE}{AC}}\right) t_{AC},$$

hence

$$\begin{aligned} t_{AD} &= \frac{AD}{AE}\, t_{AE} = \frac{AD}{AE}\sqrt{\frac{AE}{AC}}\, t_{AC}, \\ t_{DB} &= \frac{DB}{EC}\, t_{EC} = \frac{DB}{EC}\left(1 - \sqrt{\frac{AE}{AC}}\right) t_{AC}. \end{aligned}$$

[21] [166], III/1, n. XXVIII and XXIX, pp. 277-290. See also [140] [139] [138].

[22] Gerhardt edited as *Beitrage* to the letter, [166], III/1 pp. 290-295.

[23] It is not proved that the addendum was sent together with the letter or later or ever. But, there are reasons to believe that it was written before the letter was sent, see [117].

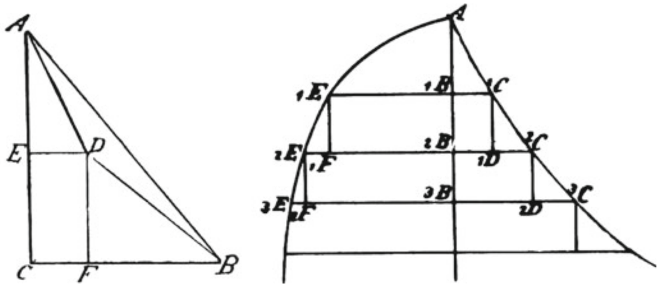

Fig. 2.6 Illustration of Leibniz's argument.

Consequently, time t_{ADB} to go from A to D and from D to B is

$$t_{ADB} = \Big[\frac{AD}{AE}\sqrt{\frac{AE}{AC}} + \frac{DB}{EC}\Big(1 - \sqrt{\frac{AE}{AC}}\Big)\Big]\, t_{AC}.$$

In this expression only the quantities AD and DB vary, when D varies; and, since

$$DB^2 = EC^2 + (CB - ED)^2 \quad \text{and} \quad AD^2 = AE^2 + ED^2,$$

we may write

$$t_{ADB} = \Big[\frac{\sqrt{AC^2 + ED^2}}{AE}\sqrt{\frac{AE}{AC}} + \frac{\sqrt{EC^2 + (CB - ED)^2}}{EC}\Big(1 - \sqrt{\frac{AE}{AC}}\Big)\Big] t_{AC},$$

in which the unique variable quantity is ED. Differentiating and setting $dt_{ADB} = 0$, we easily infer

$$\frac{ED}{\sqrt{AE^2 + ED^2}}\,\frac{1}{AD} t_{AD} - \frac{1}{DB}\,\frac{CB - ED}{\sqrt{EC^2 + (CB - ED)^2}}\, t_{DB} = 0,$$

that is, the condition on D

$$\frac{ED}{AD^2}\, t_{AD} = \frac{FB}{DB^2}\, t_{DB}.$$

Given this, Leibniz makes use of Figure 2.6, on the right, where AE is the parabola with vertex in A and axis AB, so that a body falls vertically from A to B in the time BE. If AC is the brachistochrone and B_1, B_2 and B_3 are equispaced, Leibniz finds then

$$t_{C_1C_2}\,\frac{D_1C_2}{(C_1C_2)^2} = t_{C_2C_3}\,\frac{D_2C_3}{(C_2C_3)^2},$$

and, according to Galilei law, (1.1),

$$t_{C_1C_2} = \frac{C_1C_2}{C_1D_1} t_{C_1D_1} \quad t_{C_2C_3} = \frac{C_2C_3}{C_2D_2} t_{C_2D_2}.$$

Leibniz may then conclude

$$t_{C_1D_1} \frac{D_1C_2}{C_1C_2} = t_{C_2D_2} \frac{D_2C_3}{C_2C_3},$$

that, since $C_1D_1 = C_2D_2$, becomes

$$\frac{D_1C_2}{D_2C_3} = \frac{F_2E_3}{F_1E_2} \frac{C_1C_2}{C_2C_3}.$$

As clearly $D_1C_2 \propto dy$, $F_1E_2 = t_{B_1B_2} \propto \sqrt{x}$ and $C_1C_2 \propto ds$ the initial equation $ds = k/\sqrt{x}dy$ in the letter to Bernoulli is proved.

In his paper in the May 1697 issue of *Acta Eruditorum*, Leibniz did not add any proof to his solution, but stressed that "calculus had given him the sought curve". He notices that the curve of quickest descent is characterized by the property that, see Figure 2.1, $\frac{1}{2}\, GK \times CM$ equals the area enclosed between arc GL and the cord GL and shows that such a property is characteristic of the upwards-facing cycloid generated by the circle of radius LN.

2.4 Newton's Solutions

Newton's solution of the brachistochrone problem was first published anonymously in the January issue 1697 of the *Philosophical Transactions* and later, still anonymously, in the May issue of *Acta Eruditorum*. It consists simply in showing (as we have already seen when discussing Johann Bernoulli's solution) how one can construct a cycloid going from point A to point B. However, in Newton's paper appears no reason as to why the solution should be a cycloid and there is anywhere (at least up to now) no record of the method followed by Newton to face Bernoulli's challenge is to be found.

Experts of the dualism Newton/Leibniz, United Kingdom/Continental scholars, geometrical/analytical methods, conjectured quite reasonably that Newton reasoned in geometrical terms. In support of this in [130] an exert of Newton taken from [176], IV, p. 409, is pointed out (and translated), where Newton already gives the name of *shortest* to the time to fall along an arc of cycloid:

> About the ratio of the time to slide along a straight line, down through the given points, to the shortest, of sliding by the force of gravity, from one of these points to the other along an arc of cycloid.

There Newton proves by means of Huygens's Proposition XXV, which we stated in Section 1.4, that, see Figure 2.7, the time for a heavy body to go through the straight

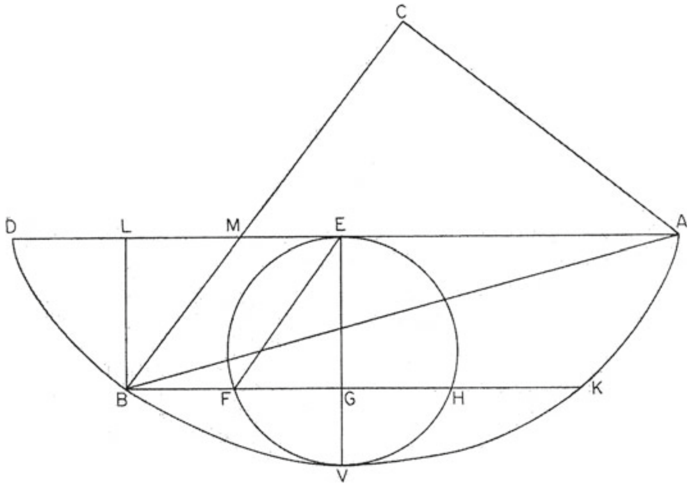

Fig. 2.7 Newton's theorem on cycloids.

line AB starting from rest, is to the time to go through arc AVB as the straight line AB is to the straight line AC.

It is hard to distinguish sharply between geometrical and analytical methods and each author may have followed mixtures of arguments. Nevertheless, geometric features must certainly have played by sure a somewhat prevalent role, not only in Newton but probably in all of the early scholars in the Continent, because after all their education was mostly geometrical and because of the nature of the problem. Scholars like Leibniz or the Bernoullis might have stressed the use of analytical methods partly because of contingent controversies partly because analytical methods were surely more suited to treat general problems with a sort of unique method, indeed via analysis more than synthesis. This, actually soon became one of the main aim of the topic.

The following two facts relative to brachistochrones and cycloids, pointed out in [130], may support the previous claims:

- In a brachistochrone the angles of inclination, measured from the vertical, vary everywhere in the same ratio as the speed:

$$\sin\theta = k\,v,$$

  where θ is the inclination angle, measured from the vertical, k a constant, and v the speed of the falling body, which varies as the square root of the height, according to Galilei;
- Figure 2.8 as a constant subfigure of all figures connected to the problem. If GO represents the fallen height along the diameter GK, and OL is horizontal, then from the similarity of the triangles LGK and LGO we can see that

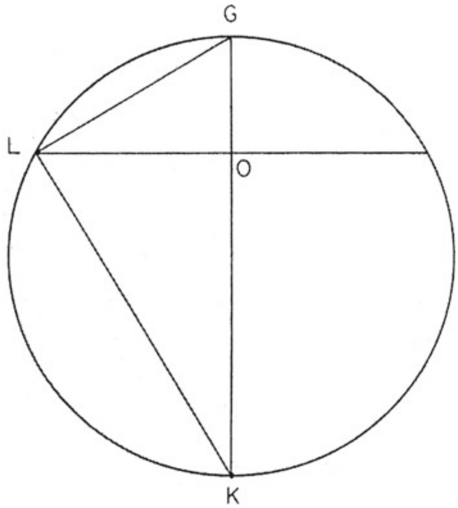

Fig. 2.8 A graphical solution.

$$LG^2 = GK\,GO,$$

that is, LG varies as the square root of segment GO.

Imposing that $\sin\theta$ is directly proportional to the segment LG means that the angle θ is identified with the angle LKG. Therefore, when the body has fallen the distance GO the inclination angle should be LKG, while the straight line LK should be parallel to the instantaneous tangent. On account of the drawing of tangents to a cycloid we may therefore conclude the figure to be a cycloid. It remains to give a criterion to choose the right cycloid, and this is exactly what Newton gave.

2.5 The *Addendum* in Johann Bernoulli's 1718 Paper

As Carathéodory noted in his thesis [45], the first satisfactory and completely rigorous solution of a variational problem was given in the paper [28] by Johann Bernoulli. The paper is mostly dedicated to the isoperimetric problem (we shall return on it in the next chapter), but it ends with few pages, a sort of *addendum*, headed "Problème. De la plus vite descente resolue d'une manière direct et extraordinaire" where he gives a simple proof that the cycloid is in fact the curve of quickest descent[24].

---

[24]It is the first time an extremal is shown to yield actually a minimum. Indeed, giving *sufficient conditions* so that such a claim holds was a central problem in the calculus of variations until the beginning of last century, and several approaches were developed: Jacobi's theory, the approaches of Weierstrass, Kneser, Hilbert, and Mayer, and the quickest and probably most elegant approach of Carathéodory, known as the *royal road to calculus of variations*, all via field theory. The interested reader is referred, for instance, to [33] [112] [117] [193].

The *addendum* presents also a first glimpse at what will be later *field theory* but, from this point of view, it was ignored until 1904. Thus it may be surprising to learn that Carathéodory was led to his approach to sufficient conditions by Johann Bernoulli's paper of 1718, the essence of which he had already described in the appendix to his thesis, see [51] I, pp. 69-78. In Vol. 2 of [51], pp. 97-98, Carathéodory wrote:

> In the ancient oriental courts there was often besides the official history written by an appointed historian, a secret history that was not less thrilling and interesting than the former. Something of this kind can be traced also in the history of the Calculus of Variations.
>
> It is a known fact that the whole of the work on Calculus of Variations during the eighteenth century deals only with necessary conditions for the existence of a minimum and that most of the methods employed during that time do not allow even to separate the cases in which the solution yields a maximum from those in which a minimum is attained. According to general belief Gauss[25] in 1823 was the first to give a method of calculation for the problem of geodesics which was equivalent to the sufficient conditions emphasized fifty years later by Weierstrass for more general cases.
>
> It is therefore important to know that the very first solution which Johann Bernoulli found for the problem of the quickest descent contains a demonstration of the fact that the minimum is really attained for the cycloid and it is more important still to learn from a letter which Bernulli addressed to Basnage, in 1697, that he himself was thoroughly aware of the advantages of is method [...]. But just as in the case of the problem of geodesics Bernoulli did not publish this most interesting result until 1718 and he did this on the very last pages of a rather tedious tract.
>
> Thus this method of Bernoulli, in which something of the field theory of Weierstrass appears for the first time [...] did not attract the attention even of his contemporaries and remained completely ignored for nearly two hundred years.
>
> These two pages of Bernoulli, which I discovered by chance more than thirty years ago [Thesis, 1904] have had a very decisive influence on the work which I myself did in the Calculus of Variations. I succeeded gradually in simplifying the exposition of this theory and came finally to a point where I found to my astonishment that the method to which I had been directed through long and hard work was contained, at least in principle in the *Traité de la lumière* of Christiaan Huygens.[26,27]

This is the beginning of the addendum of Johann Bernoulli's paper [28]:

> To bring this memoir to a conclusion I proceed to add my direct method for solving the famous problem of quickest descent, not having yet published this method although I had communicated it to several of my friends as early as 1697 when I published my other indirect one. The incomparable Mr. Leibniz to whom I had communicated both, as he has himself testified in the Leipzig Acts for the same year 1697, p. 204, found this direct method of

(Footnote 24 continued)
The modern reader might be surprised that the simple remark "extremals of a strictly convex functional are indeed minimizers" did not occur for so long. However, though already Archimedes investigated convex curves, in the Eighteenth and Nineteenth centuries the notion of convexity appeared only sporadically; the foundation of the geometry of convex bodies is due to Brunn and Minkowski around 1900.

[25] In [109] he deals with geodesic fields and transversal surfaces.

[26] Here he refers to *Huygens principle* on light rays and wave fronts.

[27] Quotation from [112] I, pp. 396-397.

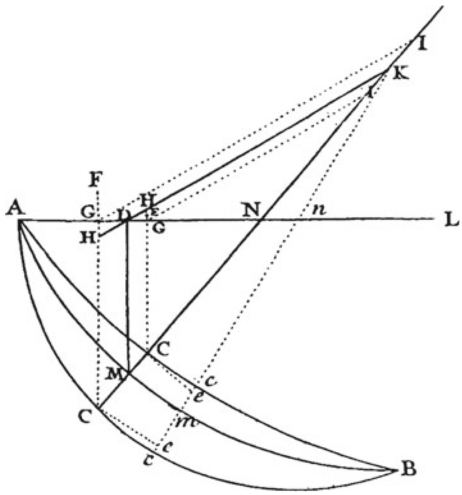

Fig. 2.9 Illustration from Johann Bernoulli's 1718 paper.

> such elegance that he counselled me not to publish it[28] for the reasons which then obtained but which do not any longer. I hope it will also please the reader as much that, although the analysis concerns only the radius of curvature of the osculating circle of the desired curve, what one finds however is the common cycloid having at whatever point such a radius of curvature or of the osculating circle; this method also provides me meanwhile with a synthetic demonstration which with extraordinary and agreeable case shows that this cycloid is effectively the desired curve of quickest descent.[29]

Johann Bernoulli considers, see Figure 2.9, a generic point M on the curve of quickest descent AMB and let K, on the normal line to AMB through M, to be the *centre of curvature*, consequently, MK to be the *radius of curvature* of AMB in M. He denotes by N the point where the line MK meets the horizontal line AL and sets

[28]See [30] 1, p. 348 and [166] III.1, p. 370. See also [22] in [118] p. 285.

[29]The following is the original passage in French:

> Pour mettre fin à ce Memoire l'y vas ajouter ma Methode directe de resoudre le fameux Problème de la plus vîte descente, n'ayant point encore publié cette Methode, quoi-que je l'aye communiquée à plusieurs de mes Amis dès 1697 que je publiai mon autre indirecte. L'incomparable M.Leibnitz, à qui je les avois communiquées toutes deux, comme il l'a temoigné lui-meme dans les Actes de Leipsick de cette meme année 1697 [G.W. Leibniz, *Communicatio suae pariter, duarumque alienarum [...] solutionum problematis curvae celerrimi descensus*, AE, Maji 1697 , pp.201-205], pag 204, trouva cette Methode directe d'une beauté si singuliere, qu'il me conseilla de ne la pas publier, pour de raisons qui étoient alors, & qui ne subsistent plus. J'espere qu'elle plaira aussi d'autant plus au Lecteur que, quoi-que l'Analyse n'en conduise qu'au rayon de la curvité ou du cercle osculateur de la Courbe cherchée, laquelle se trouve ainsi être la cycloïde ordinaire qu'on sçait avoir seule, en quelque point que ce soit, un tel rayon da sa curvité ou de son cercle osculateur; cette Methode me fournit cependant aussi une démonstration synthetique, qui avec une facilité surprenante & agreeable fait voir que cette cycloïde est effectivement la Courbe cherchée de la plus vîte descente.

$NK = a$, $MN = \xi$. Then he draws the vertical MD and considers a straight line Kcm that forms with KCM an infinitesimal angle $d\theta$ so that the arc of circles Ce and Mm, (with centre in K, C denoting the intersection of the line through M and K with any other curve ACB close to AMB) may be assumed to be straight, that is, regarded as inclined planes; he then sets $p := MD/MN$, and $q := Mm/MK = Ce/CK$.[30] Then $Mm = qx + qa$ and, since AMK is a curve of quickest descent, the time of descent of Mm, that according to Galilei is proportional to

$$\frac{q(x+a)}{(px)^{1/2}},$$

has to be minimum. This yields by differentiation that x must be equal to a, $x = a$. It follows that $MK = Mm/q = x + a = 2a = 2NK$, that is N divides MK in two equal parts. But, this is another characterization of the cycloid[31]

Let us now present Johann Bernoulli's proof that the cycloid actually provides the least time descent. The argument is completely geometric and uses again Figure 2.9.

Let CG and MD be orthogonal to AL and GI parallel to DK and let H be the intersection of the extensions of DK and CG; finally, let us choose F so that $MD/CH = CH/CF$.

For the time of fall along Mm, here we use $\propto$ as an abbreviation for 'proportional', we have

$$t_{Mm} \propto Mm/v_M \propto Mm/\sqrt{MD};$$

similarly $t_{Ce} \propto Ce/\sqrt{CG}$. By the choice of F we have $CH = (CF \times MD)^{1/2}$, while by similarity of triangles we have

$$\frac{Mm}{Ce} = \frac{MK}{CK} = \frac{MD}{CH} = \left(\frac{MD}{CF}\right)^{1/2}.$$

It follows that the ratio of the times of fall along Mm and along Ce is

$$\frac{Mm}{Ce} \times \left(\frac{CG}{MD}\right)^{1/2} = \left(\frac{MD}{CF}\right)^{1/2} \times \left(\frac{CG}{MD}\right)^{1/2} = \left(\frac{CG}{CF}\right)^{1/2}.$$

---

[30] We have replaced the original letters x, m, n with ξ, p, and q, respectively, to avoid possible confusions.

[31] Chosing horizontal coordinate x and downwards vertical coordinate y, the cycloid generated by a circle of radius r has parametric equations

$$\begin{cases} x = r(\varphi - \sin\varphi) \\ y = r(1 - \cos\varphi). \end{cases}$$

We can easily compute

$$\frac{dy}{dx} = \cot\frac{\varphi}{2}, \quad \frac{ds}{d\varphi} = 2a\sin\frac{\varphi}{2},$$

and show that $MK = y\,ds/dx$ where s is the arc length. The radius of curvature (as inverse of the curvature) is $ds/d\theta$ where θ is $\arctan dy/dx$. Since $\cot\varphi/2 = dy/dx = \tan\theta$ we see that $\theta = \pi/2 - \varphi/2$ and $ds/d\theta = 2ds/d\varphi$, i.e. $MK = 2NK$. Actually, this is characteristic of a cycloid.

Johann Bernoulli shows now that $CG/CF < 1$, that is, the time of fall along the arc of cycloid Mm is less then the time along Ce and, as the time of fall along Cc[32], is even greater, he may conclude that the time of fall along the arc of cycloid Mm is less than the time along any other arc Cc.

In order to prove that $CG/CF < 1$ he remarks that $MN = NK$, since AMB is a cycloid; by similarity of triangles

$$CN/MN = GN/DN = NI/NK,$$

yielding $CN = NI$. Hence, from

$$CN^2 + NK^2 > 2CN \times NK,$$

that implies $(CN + NK)^2 > 4CN \times NK = CI \times MK$, he deduces $CK^2 > CI \times MK$, that is $MK/CK < CK/CI$. Finally, he notices that $MK/CK = MD/CH = CH/CF$ and concludes that $CH/CF < CH/CG$ or $CG < CF$.

[32] Cc is the hypotenuse of the right triangle Cec.

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The Early Period of the Calculus of Variations
Freguglia, P.; Giaquinta, M.
2016, XII, 293 p. 59 illus., Hardcover
ISBN: 978-3-319-38944-8
A product of Birkhäuser Basel