
Contents

Table of Notation	xvii
Picture Gallery	xxiii
Preface to the Second Edition	xxv
Introduction	xxvii
I A Brief Introduction to Lattices	1
1 Basic Concepts	3
1.1 Ordering	3
1.1.1 Ordered sets	3
1.1.2 Diagrams	5
1.1.3 Constructions of ordered sets	5
1.1.4 Partitions	7
1.2 Lattices and semilattices	9
1.2.1 Lattices	9
1.2.2 Semilattices and closure systems	10
1.3 Some algebraic concepts	12
1.3.1 Homomorphisms	12
1.3.2 Sublattices	13
1.3.3 Congruences	15
2 Special Concepts	19
2.1 Elements and lattices	19
2.2 Direct and subdirect products	20
2.3 Terms and identities	23

2.4	Gluing	26
2.5	Modular and distributive lattices	30
2.5.1	The characterization theorems	30
2.5.2	Finite distributive lattices	31
2.5.3	Finite modular lattices	32
3	Congruences	35
3.1	Congruence spreading	35
3.2	Finite lattices and prime intervals	38
3.3	Congruence-preserving extensions and variants	41
4	Planar Semimodular Lattices	47
4.1	Basic concepts	47
4.2	SPS lattices	48
4.3	Forks	49
4.4	Rectangular lattices	51
II	Some Special Techniques	55
5	Chopped Lattices	57
5.1	Basic definitions	57
5.2	Compatible vectors of elements	59
5.3	Compatible vectors of congruences	60
5.4	From the chopped lattice to the ideal lattice	62
5.5	Sectional complementation	63
6	Boolean Triples	67
6.1	The general construction	67
6.2	The congruence-preserving extension property	70
6.3	The distributive case	72
6.4	Two interesting intervals	73
7	Cubic Extensions	81
7.1	The construction	81
7.2	The basic property	83
III	Congruence Lattices of Finite Lattices	87
8	The Dilworth Theorem	89
8.1	The representation theorem	89
8.2	<i>Proof-by-Picture</i>	90

8.3	Computing	92
8.4	Sectionally complemented lattices	94
8.5	Discussion	95
9	Minimal Representations	101
9.1	The results	101
9.2	<i>Proof-by-Picture</i> for minimal construction	102
9.3	The formal construction	104
9.4	<i>Proof-by-Picture</i> for minimality	105
9.5	Computing minimality	107
9.6	Discussion	108
10	Semimodular Lattices	113
10.1	The representation theorem	113
10.2	<i>Proof-by-Picture</i>	114
10.3	Construction and proof	115
10.4	Discussion	122
11	Rectangular Lattices	123
11.1	Results	123
11.2	<i>Proof-by-Picture</i>	124
11.3	Discussion	125
12	Modular Lattices	127
12.1	The representation theorem	127
12.2	<i>Proof-by-Picture</i>	128
12.3	Construction and proof	131
12.4	Discussion	136
13	Uniform Lattices	141
13.1	The representation theorem	141
13.2	<i>Proof-by-Picture</i>	142
13.3	The lattice $N(A, B)$	144
13.4	Formal proof	149
13.5	Discussion	150
IV	Congruence Lattices and Lattice Extensions	155
14	Sectionally Complemented Lattices	157
14.1	The extension theorem	157
14.2	<i>Proof-by-Picture</i>	158
14.3	Simple extensions	160
14.4	Formal proof	162
14.5	Discussion	164

15	Semimodular Lattices	165
15.1	The extension theorem	165
15.2	<i>Proof-by-Picture</i>	165
15.3	The conduit	168
15.4	The construction	170
15.5	Formal proof	171
15.6	Discussion	171
16	Isoform Lattices	173
16.1	The result	173
16.2	<i>Proof-by-Picture</i>	174
16.3	Formal construction	177
16.4	The congruences	183
16.5	The isoform property	184
16.6	Discussion	185
17	The Congruence Lattice and the Automorphism Group	189
17.1	Results	189
17.2	<i>Proof-by-Picture</i>	190
17.3	The representation theorems	194
17.4	Formal proofs	195
17.4.1	An automorphism-preserving simple extension	195
17.4.2	A congruence-preserving rigid extension	197
17.4.3	Proof of the independence theorems	197
17.5	Discussion	199
18	Magic Wands	201
18.1	Constructing congruence lattices	201
18.1.1	Bijjective maps	201
18.1.2	Surjective maps	202
18.2	<i>Proof-by-Picture</i> for bijective maps	203
18.3	Verification for bijective maps	206
18.4	2/3-boolean triples	209
18.5	<i>Proof-by-Picture</i> for surjective maps	215
18.6	Verification for surjective maps	217
18.7	Discussion	218
V	Congruence Lattices of Two Related Lattices	223
19	Sublattices	225
19.1	The results	225
19.2	<i>Proof-by-Picture</i>	227
19.3	Multi-coloring	227

19.4	Formal proof	230
19.5	Discussion	231
20	Ideals	237
20.1	The results	237
20.2	<i>Proof-by-Picture</i> for the main result	238
20.3	The main result	240
20.3.1	A formal proof	240
20.3.2	The final step	248
20.4	<i>Proof-by-Picture</i> for planar lattices	249
20.5	Discussion	251
21	Tensor Extensions	253
21.1	The problem	253
21.2	Three unary functions	254
21.3	Defining tensor extensions	256
21.4	Computing	258
21.4.1	Some special elements	258
21.4.2	An embedding	260
21.4.3	Distributive lattices	261
21.5	Congruences	262
21.5.1	Congruence spreading	262
21.5.2	Some structural observations	265
21.5.3	Lifting congruences	267
21.5.4	The main lemma	269
21.6	The congruence isomorphism	270
21.7	Discussion	271
VI	The Ordered Set of Principal Congruences	273
22	Representation Theorems	275
22.1	Representing the ordered set of principal congruences	275
22.2	<i>Proof-by-Picture</i>	275
22.3	An independence theorem	277
22.4	Discussion	280
23	Isotone Maps	283
23.1	Two isotone maps	283
23.1.1	Sublattices	283
23.1.2	Bounded homomorphisms	284
23.2	Sublattices, sketching the proof	285
23.3	Isotone surjective maps	285

23.4 Proving the Representation Theorem 286
23.5 Discussion 287

VII Congruence Structure 289

24 Prime Intervals and Congruences 291

24.1 Introduction 291
24.2 The Prime-projectivity Lemma 292
24.3 The Swing Lemma 294
24.4 Some consequences of the Swing Lemma 304
24.5 Fork congruences 307
24.6 Discussion 313

25 Some Applications of the Swing Lemma 315

25.1 The Trajectory Theorem for SPS Lattices 315
25.2 The Two-cover Theorem 317
25.3 A counterexample 318
25.4 Discussion 321

Bibliography 323

Index 337



<http://www.springer.com/978-3-319-38796-3>

The Congruences of a Finite Lattice

A "Proof-by-Picture" Approach

Grätzer, G.

2016, XXXIV, 346 p. 159 illus., Hardcover

ISBN: 978-3-319-38796-3

A product of Birkhäuser Basel