Chapter

2

Special Concepts

In this chapter we introduce special elements, constructions, and classes of lattices that play an important role in the representation of finite distributive lattices as congruence lattices of finite lattices.

2.1. Elements and lattices

In a nontrivial finite lattice L, an element a is *join-reducible* if $a = 0$ or if $a = b \vee c$ for some $b < a$ and $c < a$; otherwise, it is *join-irreducible.* Let $\mathrm{J}(L)$ denote the set of all join-irreducible elements of L, regarded as an ordered set under the ordering of L. By definition, $0 \notin J(L)$. For $a \in L$, set

$$\mathrm{J}(a) = \{\, x \mid x \leq a,\ x \in \mathrm{J}(L)\,\} = \mathrm{id}(a) \cap \mathrm{J}(L),$$

that is, $\mathrm{J}(a)$ is $\downarrow a$ formed in $\mathrm{J}(L)$. Note that, by definition, 0 *is not* a join-irreducible element; and similarly, 1 is not a meet-irreducible element.

In a finite lattice, every element is a join of join-irreducible elements (indeed, $a = \bigvee \mathrm{J}(a)$), and similarly for meets.

Dually, we define *meet-reducible* and *meet-irreducible* elements.

An element a is an *atom* if $0 \prec a$ and a *dual atom* if $a \prec 1$. Atoms are join-irreducible.

A lattice L is *atomistic* if every element is a finite join of atoms.

In a bounded lattice L, the element a is a *complement* of the element b iff $a \wedge b = 0$ and $a \vee b = 1$. A *complemented lattice* is a bounded lattice in

G. Grätzer, *The Congruences of a Finite Lattice*,
DOI 10.1007/978-3-319-38798-7_2

which every element has a complement; the lattices of Figure 1.5 are complemented and so are all but one of the lattices of Figure 1.6. The lattice B_n is complemented.

Let $a \in [b, c]$; the element x is a *relative complement of* a *in* $[b, c]$ iff $a \wedge x = b$, and $a \vee x = c$. A *relatively complemented lattice* is a lattice in which every element has a relative complement in any interval containing it. The lattice N_5 of Figure 2.1 is complemented but not relatively complemented.

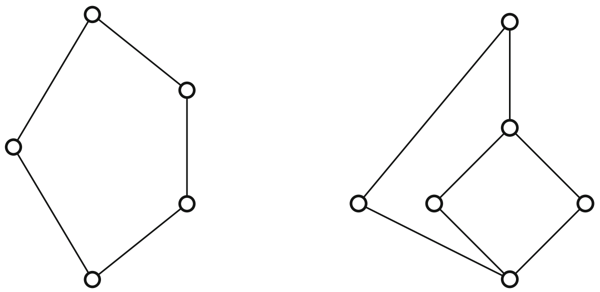

Figure 2.1: The lattices N_5 and N_6.

In a lattice L with zero, let $a \leq b$. A complement of a in $[0, b]$ is called a *sectional complement* of a in b. A lattice L with zero is called *sectionally complemented* if a has a sectional complement in b for all $a \leq b$ in L. The lattice N_6 of Figure 2.1 is sectionally complemented but not relatively complemented.

2.2. Direct and subdirect products

Let L and K be lattices and form the *direct product* $L \times K$ as in Section 1.1.3. Then $L \times K$ is a lattice and $\vee$ and $\wedge$ are computed "componentwise":

$$(a, b) \vee (c, d) = (a \vee c, b \vee d),$$
$$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d).$$

See Figure 2.2 for the example $\mathsf{C}_2 \times \mathsf{N}_5$.

Obviously, B_n is isomorphic to a direct product of n copies of B_1.

There are two *projection maps* (homomorphisms) associated with this construction:

$$\pi_L \colon L \times K \to L \quad \text{and} \quad \pi_K \colon L \times K \to K,$$

defined by $\pi_L \colon (x, y) \mapsto x$ and by $\pi_K \colon (x, y) \mapsto y$, respectively.

Similarly, we can form the *direct product* $L_1 \times \cdots \times L_n$ with elements $(x_1, \ldots, x_n)$, where $x_i \in L_i$ for $i \leq n$; we denote the projection map

$$(x_1, \ldots, x_i, \ldots, x_n) \mapsto x_i$$

by π_i. If $L_i = L$, for all $i \leq n$, we get the *direct power* L^n. By identifying $x \in L_i$ with $(0, \ldots, 0, x, 0, \ldots, 0)$ (x is the i-th coordinate), we regard L_i as an ideal of $L_1 \times \cdots \times L_n$ for $i \leq n$. The black-filled elements in Figure 2.2 show how we consider C_2 and N_5 ideals of $\mathsf{C}_2 \times \mathsf{N}_5$.

A very important property of direct products is:

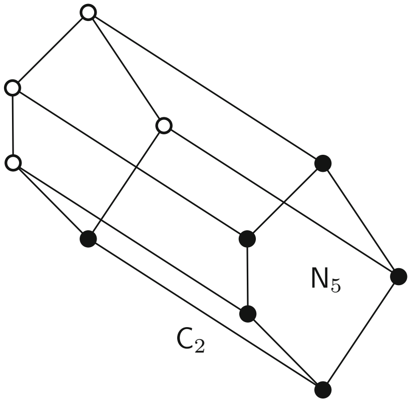

Figure 2.2: $\mathsf{C}_2 \times \mathsf{N}_5$, a direct product of two lattices.

Theorem 2.1. *Let L and K be lattices, let $\boldsymbol{\alpha}$ be a congruence relation of L, and let $\boldsymbol{\beta}$ be a congruence relation of K. Define the relation $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ on $L \times K$ by*

$$(a_1, b_1) \equiv (a_2, b_2)\,(\boldsymbol{\alpha} \times \boldsymbol{\beta}) \quad \textit{iff} \quad a_1 \equiv a_2 \pmod{\boldsymbol{\alpha}} \textit{ and } b_1 \equiv b_2 \pmod{\boldsymbol{\beta}}.$$

Then $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is a congruence relation on $L \times K$. Conversely, every congruence relation of $L \times K$ is of this form.

A more general construction is subdirect products. If $L \leq K_1 \times \cdots \times K_n$ and the projection maps π_i are onto maps, for $i \leq n$, then we call L a *subdirect product* of $K_1, \ldots, K_n$.

Trivial examples: L is a subdirect product of L and L if we identify $x \in L$ with $(x, x) \in L^2$ (*diagonal embedding*). In this example, the projection map is an isomorphism. To exclude such trivial cases, let us call a representation of L as a subdirect product of $K_1, \ldots, K_n$ *trivial* if one of the projection maps $\pi_1, \ldots, \pi_n$ is an isomorphism.

A lattice L is called *subdirectly irreducible* iff all representations of L as a subdirect product are trivial. Let L be a subdirect product of K_1 and K_2; then $\ker(\pi_1) \wedge \ker(\pi_2) = \mathbf{0}$. This subdirect product is trivial iff $\ker(\pi_1) = \mathbf{0}$ or $\ker(\pi_2) = \mathbf{0}$. Conversely, if $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 = \mathbf{0}$ in $\operatorname{Con} K$, then K is a subdirect product of $K/\boldsymbol{\alpha}_1$ and $K/\boldsymbol{\alpha}_2$, and this representation is trivial iff $\boldsymbol{\alpha}_1 = \mathbf{0}$ or $\boldsymbol{\alpha}_2 = \mathbf{0}$.

Every simple lattice is subdirectly irreducible. The lattice N_5 is subdirectly irreducible but not simple.

There is a natural correspondence between subdirect representations of a lattice L and sets of congruences $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_n\}$ satisfying $\boldsymbol{\gamma}_1 \wedge \cdots \wedge \boldsymbol{\gamma}_n = \mathbf{0}$. This representation is nontrivial iff $\boldsymbol{\gamma}_i \neq \mathbf{0}$ for all $i \leq n$. In this subdirect representation, the factors (the lattices $L/\boldsymbol{\gamma}_i$) are subdirectly irreducible iff the congruences $\boldsymbol{\gamma}_i$ are meet-irreducible, for all $i \leq n$, by the Second Isomorphism Theorem (Theorem 1.5).

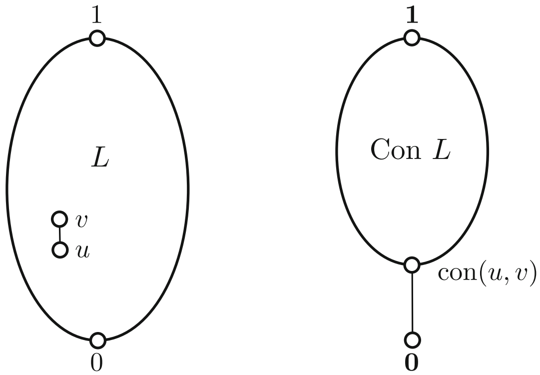

Figure 2.3: A subdirectly irreducible lattice and its congruence lattice.

For a finite lattice L, the lattice $\operatorname{Con} L$ is finite, so we can represent $\mathbf{0}$ as a meet of meet-irreducible congruences, and we obtain:

Theorem 2.2. *Every finite lattice L is a subdirect product of subdirectly irreducible lattices.*

This result (*Birkhoff's Subdirect Representation Theorem*) is true for any algebra in any variety (a class of algebras defined by identities, such as the class of all lattices or the class of all groups).

Finite subdirectly irreducible lattices are easy to recognize. If L is such a lattice, then the meet $\boldsymbol{\alpha}$ of all the $> \mathbf{0}$ elements is $> \mathbf{0}$. Obviously, $\boldsymbol{\alpha}$ is an atom, the unique atom of $\operatorname{Con} L$. Conversely, if $\operatorname{Con} L$ has a unique atom, then all $> \mathbf{0}$ congruences are $\geq \boldsymbol{\alpha}$, so their meet cannot be $\mathbf{0}$. We call $\boldsymbol{\alpha}$ the *base congruence* of L (called *monolith* in many publications).

If $u \neq v$ and $u \equiv v \pmod{\boldsymbol{\alpha}}$, then $\boldsymbol{\alpha} = \operatorname{con}(u, v)$. So

$$\operatorname{Con} L = \{\mathbf{0}\} \cup [\operatorname{con}(u, v), \mathbf{1}],$$

as illustrated in Figure 2.3.

Let L be a finite subdirectly irreducible lattice with $\operatorname{con}(u, v)$ the base congruence, where $u \prec v \in L$. By inserting two elements as shown in Figure 2.4, we embed L into a simple lattice.

Lemma 2.3. *Every finite subdirectly irreducible lattice can be embedded into a simple lattice with at most two extra elements.*

Note that every finite lattice can be embedded into a finite simple lattice; in general, we need more than two elements. For a stronger statement, see Lemma 14.3.

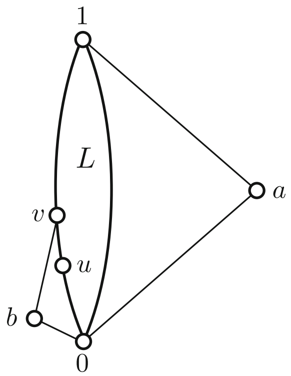

Figure 2.4: Embedding into a simple lattice.

2.3. Terms and identities

From the variables $x_1, \ldots, x_n$, we can form (n*-ary*) *terms* in the usual manner using $\vee, \wedge$, and parentheses. Examples of terms are: x_1, x_3, $x_1 \vee x_1$, $(x_1 \wedge x_2) \vee (x_3 \wedge x_1)$, $(x_3 \wedge x_1) \vee ((x_3 \vee x_2) \wedge (x_1 \vee x_2))$.

An n-ary term p defines a function in n variables (a *term function*, or simply, a *term*) on a lattice L. For example, if

$$p = (x_1 \wedge x_3) \vee (x_3 \vee x_2)$$

and $a, b, c \in L$, then

$$p(a, b, c) = (a \wedge c) \vee (c \vee b) = b \vee c.$$

If p is a *unary* ($n = 1$) lattice term, then $p(a) = a$ for any $a \in L$. If p is *binary*, then $p(a, b) = a$, or $p(a, b) = b$, or $p(a, b) = a \wedge b$, or $p(a, b) = a \vee b$ for all a, $b \in L$.

If $p = p(x_1, \ldots, x_n)$ is an n-ary term and L is a lattice, then by substituting some variables by elements of L, we get a function on L of n-variables. Such functions are called *term functions*. Unary term functions of the form

$$p(x) = p(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, x_n),$$

where $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in L$, play the most important role, see Section 3.1.

A term (function), in fact, any term function, p is *isotone*; that is, if $a_1 \leq b_1, \ldots, a_n \leq b_n$, then $p(a_1, \ldots, a_n) \leq p(b_1, \ldots, b_n)$. Furthermore,

$$a_1 \wedge \cdots \wedge a_n \leq p(a_1, \ldots, a_n) \leq a_1 \vee \cdots \vee a_n.$$

Note that many publications in Lattice Theory and Universal Algebra use *polynomials* and *polynomial functions* for terms and term functions.

Terms have many uses. We briefly discuss three.

(i) The sublattice generated by a set

Lemma 2.4. *Let L be a lattice and let H be a nonempty subset of L. Then $a \in \operatorname{sub}(H)$ (the sublattice generated by H) iff $a = p(h_1, \ldots, h_n)$ for some integer $n \geq 1$, for some n-ary term p, and for some $h_1, \ldots, h_n \in H$.*

(ii) Identities

A *lattice identity* (resp., *lattice inequality*)—also called *equation*—is an expression of the form $p = q$ (resp., $p \leq q$), where p and q are terms. An *identity $p = q$* (resp., $p \leq q$) *holds in the lattice L* iff $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_n)$ (resp., $p(a_1, \ldots, a_n) \leq q(a_1, \ldots, a_n)$) holds for all $a_1, \ldots, a_n \in L$. The identity $p = q$ is equivalent to the two inequalities $p \leq q$ and $q \leq p$; the inequality $p \leq q$ is equivalent to the identity $p \vee q = q$.

The most important properties of identities are given by

Lemma 2.5. *Identities are preserved under the formation of sublattices, homomorphic images, direct products, and ideal lattices.*

A lattice L is called *distributive* if the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

hold in L. In fact, it is enough to assume one of these identities, because the two identities are equivalent.

As we have just noted, the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ is equivalent to the two inequalities:

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z),$$
$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

However, the second inequality holds in any lattice. So a lattice is distributive iff the inequality $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ holds. By duality, we get a similar statement about the second identity defining distributivity.

The class of all distributive lattices will be denoted by **D**. A *boolean lattice* is a distributive complemented lattice. A finite boolean lattice is isomorphic to some B_n.

A lattice is called *modular* if the identity

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z))$$

holds. Note that this identity is equivalent to the following implication:

$$x \geq z \text{ implies that } (x \wedge y) \vee z = x \wedge (y \vee z).$$

The class of all modular lattices will be denoted by **M**.

Every distributive lattice is modular. The lattice M_3 is modular but not distributive. All the lattices of Figures 1.5 and 1.6 are modular except for Part $\{1, 2, 3, 4\}$ and N_5.

A class of lattices **V** is called a *variety* if it is defined by a set of identities. The classes **D** and **M** are examples of varieties, and so are **L**, the variety of all lattices and **T**, the (trivial) variety of one-element lattices.

(iii) Free lattices

Starting with a set H, we can form the set of all terms over H, collapsing two terms if their equality follows from the lattice axioms. We thus form the "free-est" lattice over H. For instance, if we start with $H = \{a, b\}$, then we obtain the four-element lattice, $\mathsf{F}(2)$, of Figure 2.5.

We obtain more interesting examples if we start with an ordered set P, and require that the ordering in P be preserved. For instance, if we start with $P = \{a, b, c\}$ with $a < b$, then we get the corresponding nine-element "free" lattice, $\mathsf{F}(P)$, of Figure 2.5.

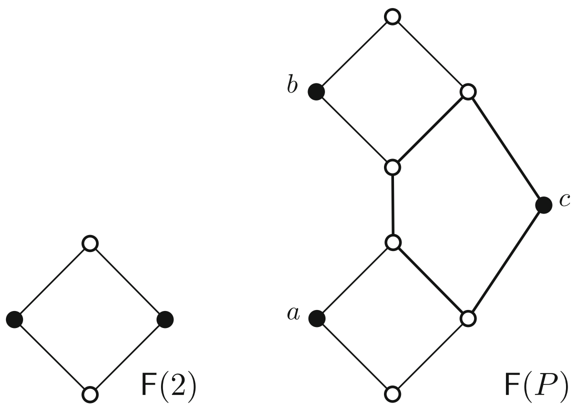

Figure 2.5: Two free lattices.

Sometimes, we need free lattices with respect to some special conditions. The following result illustrates this.

Lemma 2.6. *Let $x, y,$ and z be elements of a lattice L and let $x \vee y$, $y \vee z$, $z \vee x$ be pairwise incomparable. Then* $\mathrm{sub}(\{x \vee y, y \vee z, z \vee x\}) \cong \mathsf{B}_3$.

Lemma 2.6 is illustrated by Figure 2.6.

We will also need "free distributive lattices", obtained by collapsing two terms if their equality follows from the lattice axioms and the distributive identities. Starting with a three-element set $H = \{x, y, z\}$, we then obtain the lattice, $\mathrm{Free}_{\mathbf{D}}(3)$, of Figure 2.7.

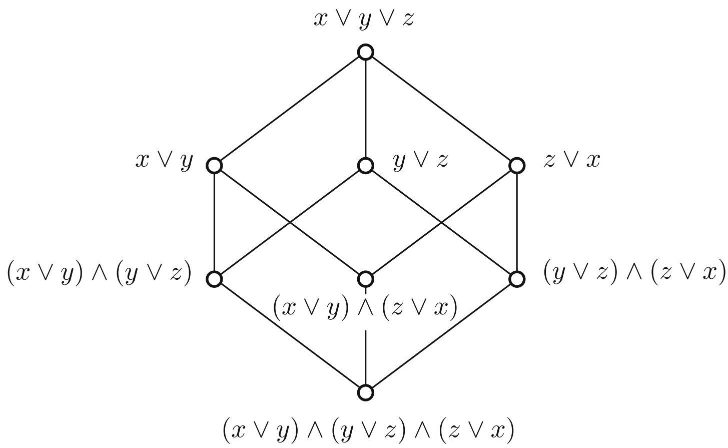

Figure 2.6: A free lattice with special relations.

Similarly, we can define "free modular lattices." Starting with a three-element set $H = \{x, y, z\}$, we then obtain the lattice, $\mathrm{Free}_{\mathbf{M}}(3)$, of Figure 2.8.

An equivalent definition of freeness is the following:

Let H be a set and let $\mathbf{K}$ be a variety of lattices. A lattice $\mathrm{Free}_{\mathbf{K}}(H)$ is called a *free lattice* over $\mathbf{K}$ *generated by* H iff the following three conditions are satisfied:

(i) $\mathrm{Free}_{\mathbf{K}}(H) \in \mathbf{K}$.

(ii) $\mathrm{Free}_{\mathbf{K}}(H)$ is generated by H.

(iii) Let $L \in \mathbf{K}$ and let $\psi \colon H \to L$ be a map; then there exists a (lattice) homomorphism $\varphi \colon \mathrm{Free}_{\mathbf{K}}(H) \to L$ extending ψ (that is, satisfying $\varphi a = \psi a$ for all $a \in H$).

2.4. Gluing

In Section 1.1.3 glued sums of ordered sets, $P \dotplus Q$, applied to an ordered set P with largest element 1_P and an ordered set Q with smallest element 0_Q

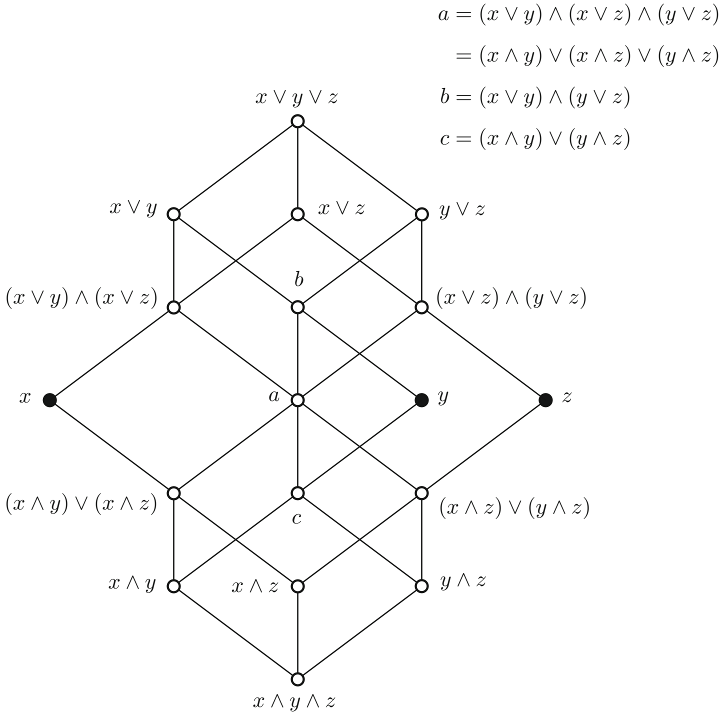

Figure 2.7: The free distributive lattice on three generators, $\mathrm{Free}_{\mathbf{D}}(3)$.

were introduced. This applies to any two lattices K with a unit and L with a zero. A natural generalization of this construction is gluing.

Let K and L be lattices, let F be a filter of K, and let I be an ideal of L. If F is isomorphic to I (with φ the isomorphism), then we can form the lattice G, the *gluing* of K and L over F and I (with respect to φ), defined as follows:

We form the disjoint union $K \cup L$, and identify $a \in F$ with $\varphi a \in I$, for all $a \in F$, to obtain the set G. We order G as follows (see Figure 2.9):

$$a \leq b \quad \text{iff} \quad \begin{cases} a \leq_K b & \text{if } a, b \in K; \\ a \leq_L b & \text{if } a, b \in L; \\ a \leq_K x \text{ and } \varphi x \leq_L b & \text{if } a \in K \text{ and } b \in L \\ & \qquad \text{for some } x \in F. \end{cases}$$

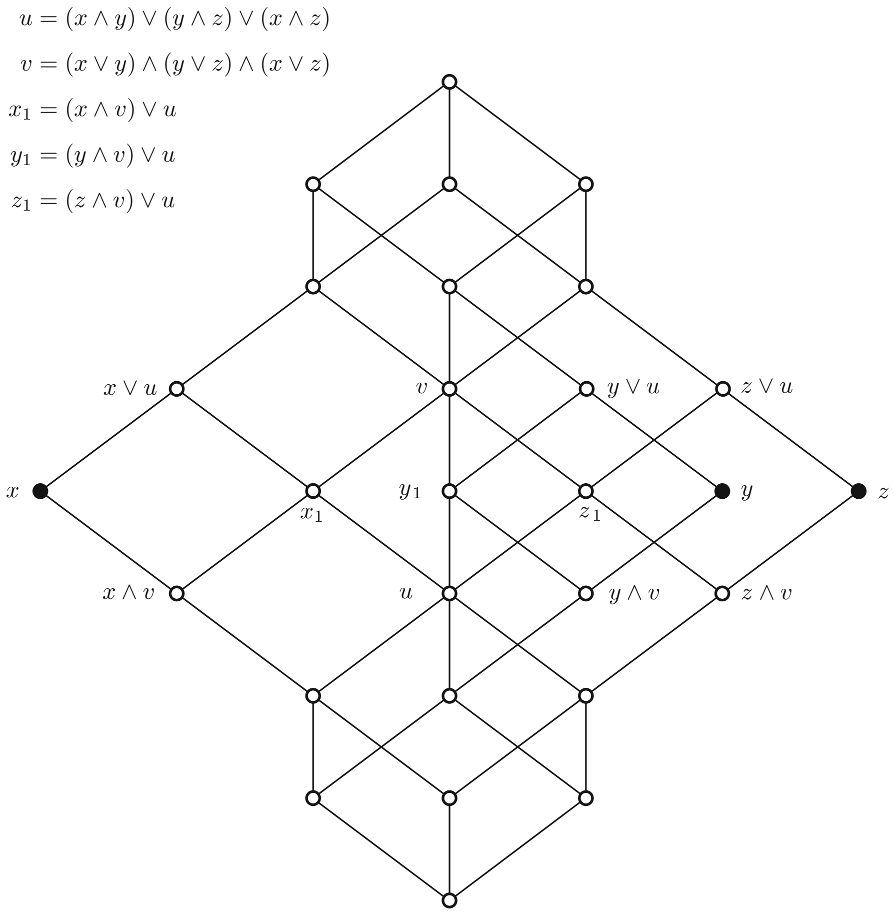

Figure 2.8: The free modular lattice on three generators, $\mathrm{Free}_{\mathbf{M}}(3)$.

Lemma 2.7. *G is an ordered set, in fact, G is a lattice. The join in G is described by*

$$a \vee_G b = \begin{cases} a \vee_K b & \textit{if } a, b \in K; \\ a \vee_L b & \textit{if } a, b \in L; \\ \varphi(a \vee_K x) \vee_L b & \textit{if } a \in K \textit{ and } b \in L \textit{ for any } b \geq x \in F, \end{cases}$$

and dually for the meet. If L has a zero, 0_L, then the last clause for the join may be rephrased:

$$a \vee_G b = \varphi(a \vee_K 0_L) \vee_L b \quad \textit{if } a \in K \textit{ and } b \in L.$$

G contains K and L as sublattices; in fact, K is an ideal and L is a filter of G.

An example of gluing is shown in Figure 2.10. There are more sophisticated examples in this book; see, for instance, Chapters 12 and 18.

Lemma 2.8. *Let K, L, F, I, and G be given as above. Let A be a lattice containing K and L as sublattices so that $K \cap L = I = F$. Then $K \cup L$ is a sublattice of A and it is isomorphic to G.*

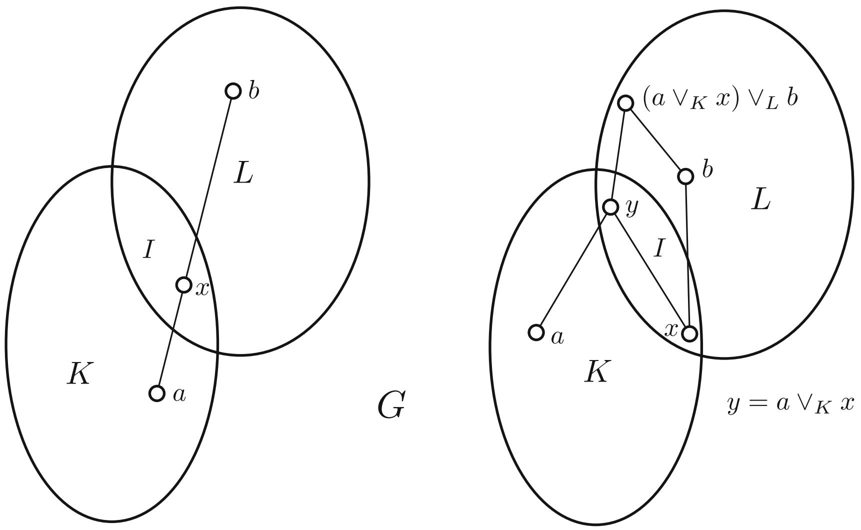

Figure 2.9: Defining gluing.

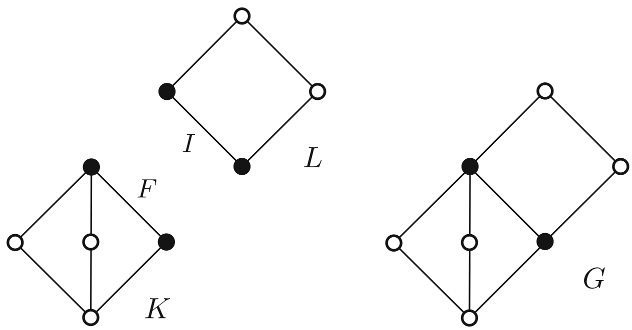

Figure 2.10: An easy gluing example.

Now if $\boldsymbol{\alpha}_K$ is a binary relation on K and $\boldsymbol{\alpha}_L$ is a binary relation on L, we define the *reflexive product* $\boldsymbol{\alpha}_K \overset{\mathrm{r}}{\circ} \boldsymbol{\alpha}_L$ as $\boldsymbol{\alpha}_K \cup \boldsymbol{\alpha}_L \cup (\boldsymbol{\alpha}_K \circ \boldsymbol{\alpha}_L)$.

We can easily describe the congruences of G.

Lemma 2.9. *A congruence* $\boldsymbol{\alpha}$ *of* G *can be uniquely written in the form*

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_K \overset{r}{\circ} \boldsymbol{\alpha}_L,$$

where $\boldsymbol{\alpha}_K$ *is a congruence of* K *and* $\boldsymbol{\alpha}_L$ *is a congruence of* L *satisfying the condition that* $\boldsymbol{\alpha}_K$ *restricted to* F *equals* $\boldsymbol{\alpha}_L$ *restricted to* I *(under the identification of elements by* φ*).*

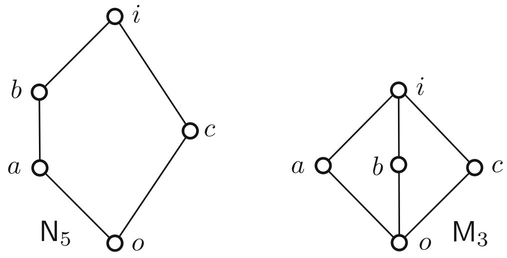

Figure 2.11: The two characteristic nondistributive lattices.

Conversely, if $\boldsymbol{\alpha}_K$ *is a congruence of* K *and* $\boldsymbol{\alpha}_L$ *is a congruence of* L *satisfying the condition that* $\boldsymbol{\alpha}_K$ *restricted to* F *equals* $\boldsymbol{\alpha}_L$ *restricted to* I*, then* $\boldsymbol{\alpha} = \boldsymbol{\alpha}_K \overset{r}{\circ} \boldsymbol{\alpha}_L$ *is a congruence of* G*.*

Let A and B be lattices, F_A a filter of A, I_B an ideal of B, and F_B a filter of B. Let us assume that F_A, I_B, and F_B are isomorphic. We now define what it means to obtain C *by gluing* B *to* A k*-times*. For $k = 1$, let C be the gluing of A and B over F_A and I_B with the filter F_B regarded as a filter F_C of C. Now if C_{k-1} with the filter $F_{C_{k-1}}$ is the gluing of B to A $k-1$-times, then we glue C_{k-1} and B over $F_{C_{k-1}}$ and I_B to obtain C the gluing of B to A k-times with the filter F_B regarded as a filter F_C of C.

2.5. Modular and distributive lattices

2.5.1 The characterization theorems

The two typical examples of nondistributive lattices are N_5 and M_3, whose diagrams are given (again) in Figure 2.11. The following characterization theorem follows immediately by inspecting the diagrams of the free lattices in Figures 2.5 and 2.8.

Theorem 2.10.

(i) *A lattice* L *is modular iff it does not contain* N_5 *as a sublattice.*

(ii) *A modular lattice* L *is distributive iff it does not contain* M_3 *as a sublattice.*

(iii) *A lattice L is distributive iff L contains neither N_5 nor M_3 as a sublattice.*

Theorem 2.11. *Let L be a modular lattice, let $a \in L$, and let U and V be sublattices with the property $u \wedge v = a$ for all $u \in U$ and $v \in V$. Then* $\mathrm{sub}(U \cup V)$ *is isomorphic to $U \times V$ under the isomorphism ($u \in U$ and $v \in V$)*

$$u \vee v \mapsto (u, v).$$

Conversely, a lattice L satisfying this property is modular.

Corollary 2.12. *Let L be a modular lattice and let $a, b \in L$. Then*

$$\mathrm{sub}([a \wedge b, a] \cup [a \wedge b, b]),$$

that is, the sublattice of L generated by $[a \wedge b, a] \cup [a \wedge b, b]$, is isomorphic to the direct product

$$[a \wedge b, a] \times [a \wedge b, b].$$

In the distributive case, the sublattice generated by $[a \wedge b, a] \cup [a \wedge b, b]$ is the interval $[a \wedge b, a \vee b]$; this does not hold for modular lattices, as exemplified by M_3.

Let G be the gluing of the lattices K and L over F and I, as in Section 2.4.

Lemma 2.13. *If K and L are modular, so is the gluing G of K and L. If K and L are distributive, so is G.*

The distributive identity easily implies that every n-ary term equals one we get by joining meets of variables. So we get:

Lemma 2.14. *A finitely generated distributive lattice is finite.*

2.5.2 Finite distributive lattices

For a nontrivial finite distributive lattice D, the ordered set $\mathrm{J}(D)$ is "equivalent" to the lattice D in the following sense:

Theorem 2.15. *Let D be a nontrivial finite distributive lattice. Then the map*

$$\varphi \colon a \mapsto \mathrm{J}(a)$$

is an isomorphism between D and $\mathrm{Down}(\mathrm{J}(D))$.

Corollary 2.16. *The correspondence $D \mapsto \mathrm{J}(D)$ makes the class of all nontrivial finite distributive lattices correspond to the class of all finite ordered sets; isomorphic lattices correspond to isomorphic ordered sets, and vice versa.*

In particular, $D \cong \mathrm{Down}(\mathrm{J}(D))$ and $P \cong \mathrm{J}(\mathrm{Down}\, P)$.

Let D and E be nontrivial finite distributive lattices, and let $\varphi\colon D \to E$ be a bounded homomorphism. Then with every $x \in \mathrm{J}(E)$, we can associate the smallest $y \in D$ with $\varphi y \geq x$. It turns out that $y \in \mathrm{J}(D)$, so we obtain an isotone map $\mathrm{J}(\varphi)\colon \mathrm{J}(E) \to \mathrm{J}(D)$.

Let P and Q be ordered sets, and let $\psi\colon P \to Q$ be a isotone map. Then with every $I \in \mathrm{Down}\, Q$, we can associate $\psi^{-1}I$. It turns out that $\psi^{-1} \in \mathrm{Down}\, P$, so we obtain the isotone map $\mathrm{Down}(\psi)\colon \mathrm{Down}\, Q \to \mathrm{Down}\, P$.

Theorem 2.17. *Let D and E be nontrivial finite distributive lattices, and let $\varphi\colon D \to E$ be a bounded homomorphism. Let φ_D and φ_E be the isomorphisms between D and $\mathrm{Down}(\mathrm{J}(D))$ and between E and $\mathrm{Down}(\mathrm{J}(E))$, respectively. Then the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\ \varphi_D\ } & \mathrm{Down}(\mathrm{J}(D)) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Down}(\mathrm{J}(\varphi))} \\ E & \xrightarrow{\ \varphi_E\ } & \mathrm{Down}(\mathrm{J}(E)) \end{array}$$

commutes, that is, $\mathrm{Down}(\mathrm{J}(\varphi))\varphi_D = \varphi_E\varphi$.

Let U be a finite order. If U is an antichain, then $\mathrm{Pow}\, U \cong \mathrm{Down}\, U$, the finite boolean lattice of the power set of H. Since $\mathrm{Down}(\mathrm{J}(D))$ is a sublattice of the finite boolean lattice of all subsets of $\mathrm{J}(D)$, we get

Corollary 2.18. *Every finite distributive lattice D can be embedded into a finite boolean lattice. If D is nontrivial, then it can be embedded into B_n, where $n = |\mathrm{J}(D)|$*

In Theorem 2.15, instead of $\mathrm{J}(D)$ and down sets, we could work with the dual concepts: $\mathrm{M}(D)$ (the ordered set of meet-irreducible elements of D) and up-sets. However, surprisingly, $\mathrm{J}(D)$ and $\mathrm{M}(D)$ are isomorphic. To see this, for $a \in \mathrm{M}(D)$, let $a^\dagger$ denote the smallest element x of D not below a (that is, with $x \nleq a$). By distributivity, it is easy to see that $a^\dagger \in \mathrm{J}(D)$; in fact, we have the following.

Theorem 2.19. *Let D be a nontrivial finite distributive lattice. Then the map*

$$\varphi\colon a \mapsto a^\dagger$$

is an isomorphism between the ordered sets $\mathrm{M}(D)$ and $\mathrm{J}(D)$.

2.5.3 Finite modular lattices

Take a look at the two positions of the pair of intervals $[a, b]$ and $[c, d]$ in Figure 2.12. In either case, we will write $[a, b] \sim [c, d]$, and say that $[a, b]$ is *perspective* to $[c, d]$. If we want to show whether the perspectivity is "up" or "down", we will write $[a,b] \overset{\text{up}}{\sim} [c,d]$ in the first case and $[a,b] \overset{\text{dn}}{\sim} [c,d]$ in the second case.

If for some natural number n and intervals $[e_i, f_i]$, for $0 \leq i \leq n$,

$$[a,b] = [e_0, f_0] \sim [e_1, f_1] \sim \cdots \sim [e_n, f_n] = [c,d],$$

then we say that $[a,b]$ is *projective* to $[c,d]$ and write $[a,b] \approx [c,d]$.

One of the most important properties of a modular lattice is stated in the following result:

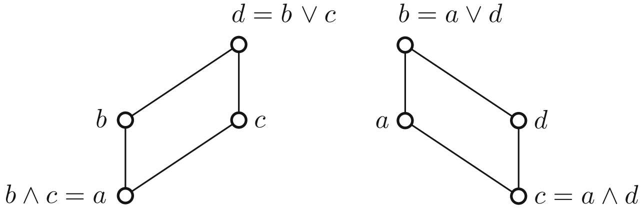

Figure 2.12: $[a,b] \overset{\text{up}}{\sim} [c,d]$ and $[a,b] \overset{\text{dn}}{\sim} [c,d]$.

Theorem 2.20 (Isomorphism Theorem for Modular Lattices)**.** *Let L be a modular lattice and let $[a,b] \overset{up}{\sim} [c,d]$ in L. Then*

$$\varphi_c \colon x \mapsto x \vee c, \quad x \in [a,b],$$

is an isomorphism of $[a,b]$ and $[c,d]$. The inverse isomorphism is

$$\psi_b \colon y \mapsto y \wedge b, \quad y \in [c,d].$$

(See Figure 2.13.)

Corollary 2.21. *In a modular lattice, projective intervals are isomorphic.*

Corollary 2.22. *In a modular lattice if a prime interval $\mathfrak{p}$ is projective to an interval $\mathfrak{q}$, then $\mathfrak{q}$ is also prime.*

Let us call the finite lattice L *semimodular* or *upper semimodular* if for $a, b, c \in L$, the covering $a \prec b$ implies that $a \vee c \prec b \vee c$ or $a \vee c = b \vee c$. The dual of an upper semimodular lattice is a *lower semimodular* lattice.

Lemma 2.23. *A modular lattice is both upper and lower semimodular. For a* finite *lattice, the converse also holds: a finite upper and lower semimodular lattice is modular, and conversely.*

The lattice S_8, in Figure 2.14, is an example of a semimodular lattice that is not modular. See Section 10.2 for an interesting use of this lattice.

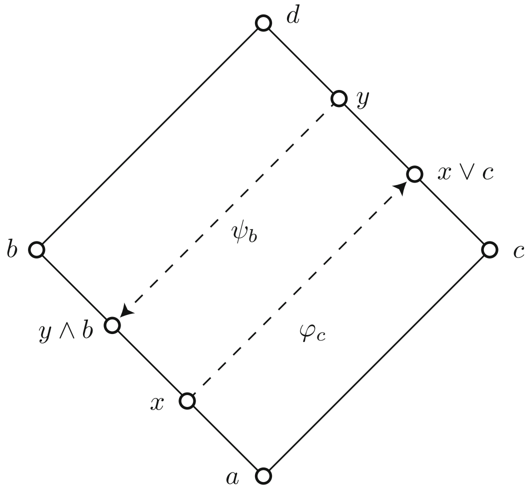

Figure 2.13: The isomorphisms φ_c and ψ_b.

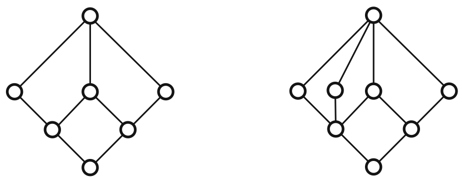

Figure 2.14: The lattices S_7 and S_8.

The following is even more trivial than Lemma 2.13:

Lemma 2.24. *If K and L are finite semimodular lattices, so is the gluing G of K and L.*

A large class of semimodular lattices is provided by

Lemma 2.25. *Let A be a nonempty set. Then* Part A *is semimodular; it is not modular unless $|A| \leq 3$.*

http://www.springer.com/978-3-319-38796-3

The Congruences of a Finite Lattice
A "Proof-by-Picture" Approach
Grätzer, G.
2016, XXXIV, 346 p. 159 illus., Hardcover
ISBN: 978-3-319-38796-3
A product of Birkhäuser Basel