Chapter 2
Preliminaries

Abstract This chapter provides the technical and mathematical background for the fuzzy model-based control which offers the equations of the fuzzy model and closed-loop systems, definition of variables, published stability conditions in terms of linear matrix inequalities (LMIs) and sum of squares (SOS). Numerical examples are given to demonstrate the motivation using polynomial fuzzy model over T-S fuzzy model. State-feedback fuzzy controller and polynomial fuzzy controller are introduced to close the feedback loop. Three main types of control design including perfectly, partially and imperfectly matched premises are discussed and compared. LMI/SOS-based stability conditions in the literature are reviewed, which will be used in other chapters for comparison purposes.

2.1 Introduction

Control of nonlinear systems is challenging because of the complexity of the system nonlinearities and control theories. Although linear control theory can be applied in some cases, it may not offer an acceptable performance when the system is working in a large operating domain. As most of real-world applications are nonlinear in nature, it is important to develop a nonlinear control approach to deal with the nonlinear plants with simple and easy-to-understand control theories and methodologies.

FMB control approach [1, 2] offers a systematic and efficient way to facilitate the system analysis and control design of nonlinear plants. T-S fuzzy model [3, 4] plays an important role in the FMB control, which offers a general framework to represent the nonlinear plants as an average weighted sum of some linear sub-systems. As the linear and nonlinear elements of the nonlinear plant are extracted as the linear sub-systems and membership functions, respectively, linear control theory can be applied to investigate the system stability and design the fuzzy controller by considering the linear part of the system. A fuzzy controller of state-feedback form [5, 6] was proposed to close the feedback loop to form an FMB control system. Throughout this book, we only focus on the state-feedback fuzzy controller. Stability conditions in terms of LMIs [7–14] or SOS [15–21] were obtained using Lyapunov-based approach to guarantee the system stability and facilitate the control synthesis.

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A feasible solution to the LMI- or SOS-based stability conditions can be found numerically using convex programming techniques.

In most of the existing work, stability analysis is carried out based on the MFI technique and the control design is based on the PDC design concept [5, 6]. Consequently, it suffers from (1) conservative stability conditions because the membership functions are ignored in the stability analysis; (2) restrictive control design because the fuzzy controller shares the same number of rules and premise membership functions as those of the T-S fuzzy model. The proposed approaches in this book will alleviate the shortcomings by using the MFD techniques for stability analysis and/or the fuzzy controllers with partially/imperfectly matched premises (where the number of rules and/or premise membership functions of the controller is/are not necessary to be the same as those of the fuzzy model) [19, 20, 22–33].

In this chapter, we will revise the basic and essential concepts, analysis techniques and mathematical tools, which will support the stability analysis of the FMB control systems. In Sect. 2.2, the notations used in this book are introduced. In Sects. 2.3 to 2.5, various types of fuzzy models, fuzzy controllers and FMB control systems are presented. Some examples of fuzzy modeling of nonlinear plants are given to demonstrate the model construction using the sector nonlinearity concept and their properties. In Sect. 2.6, some existing LMI/SOS-based stability analysis results are presented, which will be referred in other chapters for comparison purposes. In Sect. 2.7, a conclusion is drawn.

2.2 Notation

Throughout this book, the following notations are adopted [34]. The monomial in $\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ is defined as $x_1^{d_1}(t) \ldots x_n^{d_n}(t)$, where d_i, $i = 1, \ldots, n$, are non-negative integers. The degree of a monomial is defined as $d = \sum_{i=1}^{n} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be written as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a non-zero positive integer. Hence, it can be seen that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ if it is an SOS. The expressions of $\mathbf{M} > 0$, $\mathbf{M} \geq 0$, $\mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative, semi-negative definite matrices $\mathbf{M}$, respectively. It is stated in [35] that the polynomial $\mathbf{p}(\mathbf{x}(t))$ being an SOS can be represented in the form of $\hat{\mathbf{x}}(t)^T \mathbf{Q} \hat{\mathbf{x}}(t)$, where $\hat{\mathbf{x}}(t)$ is a vector of monomials in $\mathbf{x}$ and $\mathbf{Q}$ is a positive semi-definite matrix. The problem of finding a $\mathbf{Q}$ can be formulated as a semi-definite program (SDP). SOSTOOLS [36] is a third-party Matlab toolbox for solving SOS programs and its technical details can be found in [37].

2.3 Fuzzy Models

T-S fuzzy model is a powerful mathematical tool for the modeling of nonlinear plants. It offers a general representation for nonlinear plants in a favorable form to facilitate the stability analysis and control synthesis using LMI/SOS-based analysis approach. In this section, various types of fuzzy models such as the traditional T-S fuzzy model [3–6] and polynomial fuzzy model [15–17] are introduced to support the work in the subsequent chapters.

2.3.1 *T-S Fuzzy Model*

The dynamics of the nonlinear plant is described by p rules of which the premise membership functions divide the operating domain into a number of operating subdomains and each consequent is a local linear state-space model. The ith rule is shown below:

$$\begin{aligned}&\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i\\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t), i = 1, \ldots, p,\end{aligned} \tag{2.1}$$

where M_α^i is a fuzzy set of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, \ldots, \Psi$; $i = 1, \ldots, p$; Ψ is a positive integer; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known system and input matrices, respectively; $\mathbf{u}(t) \in \Re^m$ is the input vector. The dynamics of the nonlinear plant can be represented as below:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)), \tag{2.2}$$

where

$$w_i(\mathbf{x}(t)) \geq 0 \;\forall\, i, \sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \tag{2.3}$$

$$w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^{p}\prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))} \;\forall\, i, \tag{2.4}$$

$w_i(\mathbf{x}(t))$, $i = 1, \ldots, p$, is the normalized membership grade; $\mu_{M_l^i}(f_l(\mathbf{x}(t)))$, $l = 1, \ldots, \Psi$, is the membership function corresponding to the fuzzy set M_l^i.

There are in general two methods to obtain the T-S fuzzy model for a nonlinear plant.

1. By collecting the input-output data pairs, system identification algorithms [3, 4, 38] can be employed to construct the T-S fuzzy model.
2. Based on the mathematical model of the nonlinear plant, a T-S fuzzy model can be constructed using the sector nonlinearity concept [5, 39].

Example 2.1 A simple example is given to demonstrate the construction of T-S fuzzy model using the sector nonlinearity concept. Consider the following nonlinear plant:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))u(t), \tag{2.5}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 \\ x_1(t)^3 & 1 \end{bmatrix}$ and $\mathbf{B}(\mathbf{x}(t)) = \begin{bmatrix} 1 \\ \sin(x_2(t)) \end{bmatrix}$.

It is assumed that the nonlinear plant is operating in $x_1(t) \in \big[-2, 2\big]$ whereas there is no limitation on $x_2(t)$. As the nonlinear term $x_1(t)^3 \in \big[-8, 8\big]$, it can represent $x_1(t)^3$ as a linear combination of its lower and upper bounds, i.e., $x_1(t)^3 = -8 \times \omega_{11}(x_1(t)) + 8 \times \omega_{12}(x_1(t))$, where $\omega_{11}(x_1(t)) \geq 0$, $\omega_{12}(x_1(t)) \geq 0$ and $\omega_{11}(x_1(t)) + \omega_{12}(x_1(t)) = 1$. Thus, we have $x_1(t)^3 = -8 \times \omega_{11}(x_1(t)) + 8 \times (1 - \omega_{11}(x_1(t)))$ which leads to $\omega_{11}(x_1(t)) = \frac{x_1(t)^3 - 8}{-16}$ and $\omega_{12}(x_1(t)) = 1 - \omega_{11}(x_1(t)) = \frac{-x_1(t)^3 - 8}{-16}$.

Similarly, considering the nonlinear term $\sin(x_2(t)) \in \big[-1, 1\big]$, it can represent the term as $\sin(x_2(t)) = -1 \times \omega_{21}(x_2(t)) + 1 \times \omega_{22}(x_2(t))$, where $\omega_{21}(x_2(t)) \geq 0$, $\omega_{22}(x_2(t)) \geq 0$, $\omega_{21}(x_2(t)) + \omega_{22}(x_2(t)) = 1$, $\omega_{21}(x_2(t)) = \frac{\sin(x_2(t)) - 1}{-2}$ and $\omega_{22}(x_2(t)) = 1 - \omega_{21}(x_2(t)) = \frac{-\sin(x_2(t)) - 1}{-2}$.

Consequently, the nonlinear plant (2.5) can be described by a 4-rule T-S fuzzy model with the rules given below:

$$\begin{aligned} &\text{Rule } i\text{: IF } x_1(t) \text{ is } M_1^i \text{ AND } x_2(t) \text{ is } M_2^i \\ &\qquad\quad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t),\ i = 1, 2, 3, 4, \end{aligned} \tag{2.6}$$

where

$$\mu_{M_1^1}(x_1(t)) = \mu_{M_1^2}(x_1(t)) = \omega_{11}(x_1(t)),$$

$$\mu_{M_1^3}(x_1(t)) = \mu_{M_1^4}(x_1(t)) = \omega_{12}(x_1(t)),$$

$$\mu_{M_2^1}(x_2(t)) = \mu_{M_2^3}(x_2(t)) = \omega_{21}(x_2(t)),$$

$$\mu_{M_2^2}(x_2(t)) = \mu_{M_2^4}(x_2(t)) = \omega_{22}(x_2(t)),$$

$$\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ -8 & 1 \end{bmatrix},$$

$$\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ 8 & 1 \end{bmatrix},$$

$$\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The nonlinear plant (2.5) can be represented as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)). \tag{2.7}$$

where $w_i(\mathbf{x}(t))$ is defined in (2.7).

2.3.2 Polynomial Fuzzy Model

The polynomial fuzzy model is an extension of the traditional T-S fuzzy model in Sect. 2.3.1. Referring to Example 2.1, the assumption that the system states $x_1(t) \in \left[-2,\ 2\right]$ is made for the construction of the T-S fuzzy model. As a result, the T-S fuzzy model (2.7) is a local nonlinear model but not a global one. The polynomial fuzzy model is able to relax the limitation by allowing polynomials in the system matrices $\mathbf{A}_i$ and input matrices $\mathbf{B}_i$ such that the T-S fuzzy model can represent a wider range of nonlinear plant.

The nonlinear plant is described by a polynomial fuzzy model with p rules of the following format:

$$\begin{aligned}&\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i\\ &\qquad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t),\ i = 1, \ldots, p,\end{aligned} \tag{2.8}$$

where $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are the known polynomial system and input matrices, respectively; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}(t)$. It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{0}$; the variables of the rest are defined in Sect. 2.3.1.

The dynamics of the nonlinear plant is represented by the following polynomial fuzzy model.

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)), \tag{2.9}$$

where $w_i(\mathbf{x}(t))$ satisfies the conditions in (2.3) and (2.4).

Remark 2.1 An approach to construct the polynomial fuzzy model using Taylor series expansion was given in [17] based on the sector nonlinearity concept.

Remark 2.2 The T-S polynomial fuzzy model (2.9) is reduced to the traditional T-S fuzzy model (2.2) when $\mathbf{A}_i(\mathbf{x}(t))$ and $\mathbf{B}_i(\mathbf{x}(t))$ are constant matrices for all i and $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t)$.

Remark 2.3 The representation of the polynomial fuzzy model using the monomial vector $\hat{\mathbf{x}}(\mathbf{x}(t))$ is not unique [34]. The purpose of using $\hat{\mathbf{x}}(\mathbf{x}(t))$ is for the consideration of a general form. Its advantages are still not yet understood. It can be seen that the T-S polynomial fuzzy model, $\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\big(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{J}_i(\mathbf{x}(t))\big)\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\big)$, is valid and equivalent to (2.9) when the matrices $\mathbf{J}_i(\mathbf{x}(t))$ are chosen such that $\sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{J}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$. The success of finding a feasible solution to the stability conditions obtained from the polynomial fuzzy model depends on the chosen form.

Example 2.2 We consider the nonlinear plant in Example 2.1 to demonstrate the merits of using the polynomial fuzzy model. As the term $x_1(t)^3$ is a polynomial, it is not necessary to represent the term using the membership functions. Consequently, only the sinusoidal term $\sin(x_2(t))$ is needed to be considered. The nonlinear plant can be represented by a 2-rule polynomial fuzzy model with the rules shown below:

$$\begin{aligned}&\text{Rule } i\text{: IF } x_2(t) \text{ is } M_1^i\\&\qquad\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))u(t),\ i = 1, 2,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}\hat{\mathbf{x}}(\mathbf{x}(t)) &= \mathbf{x}(t),\\ \mu_{M_1^1}(x_2(t)) &= \omega_{21}(x_2(t)),\\ \mu_{M_1^2}(x_2(t)) &= \omega_{22}(x_2(t)),\end{aligned}$$

$$\mathbf{A}_1(\mathbf{x}(t)) = \mathbf{A}_2(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 \\ x_1(t)^3 & 1 \end{bmatrix},$$

$$\mathbf{B}_1(\mathbf{x}(t)) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\mathbf{B}_2(\mathbf{x}(t)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The nonlinear plant (2.5) can be represented by the polynomial fuzzy model shown as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))u(t)). \tag{2.11}$$

where $w_i(\mathbf{x}(t))$ is defined in (2.11).

Compared with the T-S fuzzy model (2.5), the polynomial fuzzy model (2.11) demonstrates the following advantages:

- The number of rules is comparatively less.
- The assumption that $x_1(t)$ works in a local operating domain is not necessary.
- The polynomial fuzzy model is a global nonlinear model rather than a local one. As a result, the system analysis result is global.

2.4 State-Feedback Fuzzy Controller

In this book, we will focus on the state-feedback fuzzy controller (hereafter fuzzy controller), which is the most popular type of fuzzy controllers in the literature, for the control process. Two types of fuzzy controllers reported in the literature are introduced in this section. The first type of fuzzy controllers is simply the traditional state-feedback fuzzy controller [5, 6] which is represented as an average weighted sum of some linear state-feedback sub-controllers. The second type is the polynomial fuzzy controller [15–17, 20, 27] which is an extension of the traditional ones. Instead of using linear state-feedback sub-controllers in the consequent of the fuzzy rules, polynomial state-feedback sub-controllers are used.

2.4.1 Fuzzy Controller

A fuzzy controller [5, 6] is described by c rules of the following format:

$$\begin{aligned}\text{Rule } j\text{: IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \cdots \text{ AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j\\ \text{THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t),\ j = 1, \ldots, c,\end{aligned} \tag{2.12}$$

where N_β^j is a fuzzy set of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, \ldots, \Omega$; $j = 1, \ldots, c$; Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$, $j = 1, \ldots, c$, is the constant feedback gain to be determined.

The fuzzy controller is defined as follows:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t), \tag{2.13}$$

where

$$m_j(\mathbf{x}(t)) \geq 0 \;\forall\; j, \sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1, \tag{2.14}$$

$$m_j(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^{c} \prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(t)))} \;\forall\; j, \tag{2.15}$$

$m_j(\mathbf{x}(t))$, $j = 1, \ldots, c$, is the normalized membership grade; $\mu_{N_l^j}(g_l(\mathbf{x}(t)))$, $l = 1, \ldots, \Omega$, is the membership function corresponding to the fuzzy set N_l^j.

2.4.2 Polynomial Fuzzy Controller

A polynomial fuzzy controller [20, 27] is described by c rules of the following format:

$$\begin{aligned} &\text{Rule } j\text{: IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \cdots \text{ AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j \\ &\quad \text{THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \; j = 1, \ldots, c, \end{aligned} \tag{2.16}$$

where $\mathbf{G}_j(\mathbf{x}(t)) \in \Re^{m \times N}$, $j = 1, \ldots, c$, are the polynomial feedback gains to be determined; the variables of the rest are defined in Sect. 2.4.1.

The polynomial fuzzy controller is defined as,

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \tag{2.17}$$

where $m_j(\mathbf{x}(t))$ satisfies the conditions in (2.14) and (2.15).

Remark 2.4 The polynomial fuzzy controller (2.17) is reduced to the traditional fuzzy controller (2.13) when the feedback gains $\mathbf{G}_j(\mathbf{x}(t))$ become constant matrices, i.e., $\mathbf{G}_j$ for all j.

2.5 Various Types of FMB Control Systems

An FMB control system is formed by a nonlinear plant represented by the fuzzy model and a fuzzy controller connected in a closed loop as shown in Fig. 1.2.

In this section, various types of FMB control systems found in the literature are presented. From (2.3) to (2.14), the following property is used to obtain the FMB control systems.

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = \sum_{j=1}^{c} m_j(\mathbf{x}(t)) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) = 1 \tag{2.18}$$

2.5.1 FMB Control System

With the consideration of the T-S fuzzy model (2.2) and the fuzzy controller (2.13), using the property of the membership functions in (2.18), the FMB control system is obtained as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t)) \left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \right) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{x}(t).\end{aligned} \tag{2.19}$$

2.5.2 PFMB Control System

From the polynomial fuzzy model (2.9) and the polynomial fuzzy controller (2.17), using the property of the membership functions in (2.18), the PFMB control system is obtained as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t)) \left(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \right) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t)))\hat{\mathbf{x}}(\mathbf{x}(t)).\end{aligned} \tag{2.20}$$

2.6 LMI/SOS-Based Stability Conditions

Referring to the FMB/PFMB control systems in Sect. 2.5, the control objective is to determine the feedback gains such that the FMB/PFMB control system is asymptotically stable.

Definition 2.1 [40] The equilibrium point $\mathbf{x}(t) = \mathbf{0}$ of the dynamic system is asymptotically stable if it is stable and there exists δ such that $||\mathbf{x}(0)|| < \delta \Rightarrow \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

In this book, we will focus on the Lyapunov stability theory, which is summarized in the following, for the stability analysis of the FMB control systems.

Theorem 2.1 *Lyapunov's direct method (also known as Lyapunov's second method) [40]: Let $\mathbf{x}(t) = \mathbf{0}$ be the equilibrium point for the nonlinear system and $V(\mathbf{x}(t))$ be a continuously differentiable function on a neighbourhood D of the equilibrium point.*

- *The equilibrium point is stable if $V(0) = 0$, $V(\mathbf{x}(t)) > 0$ in D for $\mathbf{x} \neq 0$ and $\dot{V}(\mathbf{x}(t)) \leq 0$ in D.*
- *The equilibrium point is asymptotically stable if $V(0) = 0$, $V(\mathbf{x}(t)) > 0$ in D for $\mathbf{x} \neq 0$ and $\dot{V}(\mathbf{x}(t)) < 0$ in D for $\mathbf{x} \neq 0$.*

In the following, some published LMI/SOS-based stability conditions are summarized. A feasible solution to the stability conditions can be found numerically using convex programming techniques. Throughout this book, Matlab LMI toolbox [41] and SOSTOOLS [36] are used to search for numerically a feasible solution of the LMI- and SOS-based stability conditions, respectively.

2.6.1 LMI-Based Stability Conditions for FMB Control Systems

The stability problem of the FMB control system (2.19) using LMI-based approach has been intensively investigated for the past decades. The most popular mathematical analysis tool is the Lyapunov stability theory which leads to extensive stability analysis results. In most of the work, Lyapunov function candidate in quadratic form was employed for the stability analysis. In this section, some LMI-based stability conditions obtained based on the quadratic Lyapunov function candidate are reviewed.

There are generally two cases considered in the stability analysis of the FMB control systems (2.19):

- Partially/Imperfectly matched premises: $c \neq p$ or $m_i(\mathbf{x}(t)) \neq w_i(\mathbf{x}(t))$ for any i.
- Perfectly matched premises (also know as PDC): $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i.

These two cases demonstrate different properties in stability analysis and control design. In general, the case of partially/imperfectly matched premises offers a higher design flexibility to the fuzzy controller as the number of rules and/or the premise membership functions can be chosen freely. As a result, when a smaller number of rules and/or simple membership functions are employed for the fuzzy controller, the implementation cost can be reduced. In terms of stability analysis, it tends to produce

Table 2.1 Properties of stability analysis under the cases of partially/imperfectly and perfectly matched premises

Properties	Partially/Imperfectly matched premises	Perfectly matched premises
Design flexibility	High	Low
Robustness	High	Low
Stability conditions	Conservative	Relaxed

comparatively conservative stability conditions because the premise membership functions of the T-S fuzzy model and fuzzy controller do not match. Under the MFI-based stability analysis, the membership functions are ignored. Thus, it offers an inherent robustness property to the FMB control system when the uncertainties of the nonlinear plant are embedded in the membership functions. Under the case of perfectly matched premises, the T-S fuzzy model and the fuzzy controller share the same number of rules and the same set of premise membership functions, the cross term of membership functions can be collected in the stability analysis, which potentially leads to relaxed stability conditions. However, the design flexibility and inherent robustness property are lost.

The properties of stability analysis and control design under the two cases are summarized in Table 2.1.

2.6.1.1 LMI-Based Stability Conditions Under Partially/Imperfectly Matched Premises

Considering the case of partially/imperfectly matched premises, basic LMI-based stability conditions for the FMB control system (2.19) are summarized as follows:

Theorem 2.2 ([5, 42]) *The FMB control system (2.19), formed by a nonlinear plant represented by the T-S fuzzy model (2.2) and the fuzzy controller (2.13) connected in a closed loop, is asymptotically stable if there exist matrices* $\mathbf{N}_j \in \Re^{m\times n}$, $j = 1, \ldots,$ c *and* $\mathbf{X} = \mathbf{X}^T \in \Re^{n\times n}$ *such that the following LMIs hold:*

$$\mathbf{X} > 0;$$

$$\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_j < 0 \;\forall\, i,\;\; j;$$

and the feedback gains are defined as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$ *for all* j.

Proof Consider the quadratic Lyapunov function candidate shown below:

$$V(\mathbf{x}(t)) = \mathbf{x}(t)^T\mathbf{P}\mathbf{x}(t), \tag{2.21}$$

where $0 < \mathbf{P} = \mathbf{P}^T \in \Re^{n\times n}$.

Define $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{z}(t) = \mathbf{X}^{-1}\mathbf{x}(t)$. From (2.19) to (2.21), we have

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \dot{\mathbf{x}}(t)^T\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^T\mathbf{P}\dot{\mathbf{x}}(t) \\
&= \left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{x}(t)\right)^T \mathbf{P}\mathbf{x}(t) \\
&\quad + \mathbf{x}(t)^T\mathbf{P}\left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{x}(t)\right) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) \\
&\quad \times \mathbf{x}(t)^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{G}_j^T\mathbf{B}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{P}\mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) \\
&\quad \times \mathbf{z}(t)^T\left(\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_j\right)\mathbf{z}(t).
\end{aligned} \tag{2.22}$$

By satisfying the LMI-based stability conditions in Theorem 2.2, $V(\mathbf{x}(t)) \geq 0$ (equality holds for $\mathbf{x}(t) = \mathbf{0}$) and $\dot{V}(\mathbf{x}(t)) \leq 0$ (equality holds for $\mathbf{x}(t) = \mathbf{0}$) can be achieved. The FMB control system (2.19) is asymptotically stable in the sense of Lyapunov.

2.6.1.2 LMI-Based Stability Conditions Under PDC Design

It can be seen from the proof of the LMI-based stability conditions above that the membership functions are ignored in the stability analysis, which is a source of conservativeness. Also, referring to Theorem 2.2, if there exists a feasible solution to the LMI-based stability conditions, the fuzzy controller can be reduced to a linear state-feedback controller by choosing, for example, $\mathbf{G}_j = \mathbf{G}_1$ for all j.

The PDC design concept is able to relax the conservativeness of the stability analysis result by considering the matched premise membership functions. Various LMI-based stability conditions at different levels of relaxation reported in the literature are given below.

Theorem 2.3 ([5, 6]) *The FMB control system (2.19), formed by a nonlinear plant represented by the T-S fuzzy model (2.2) and the fuzzy controller (2.13) under the PDC design, i.e., with $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i, connected in a closed loop, is asymptotically stable if there exist matrices $\mathbf{N}_j \in \Re^{m\times n}$, $j = 1, \ldots, p$, and $\mathbf{X} = \mathbf{X}^T \in \Re^{n\times n}$ such that the following LMIs hold:*

$$\mathbf{X} > 0;$$

$$\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_i^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_i < 0 \; \forall \; i;$$

$$\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_j + \mathbf{X}\mathbf{A}_j^T + \mathbf{A}_j\mathbf{X} + \mathbf{N}_i^T\mathbf{B}_j^T + \mathbf{B}_j\mathbf{N}_i \le 0 \; \forall \; j, i < j;$$

and the feedback gains are defined as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$ *for all* j.

Proof Denote $\mathbf{Q}_{ij} = \mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_j$. Choosing $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i, from (2.22), we have

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\mathbf{z}(t)^T\mathbf{Q}_{ij}\mathbf{z}(t) \\
&= \sum_{i=1}^{p} w_i(\mathbf{x}(t))^2\mathbf{z}(t)^T\mathbf{Q}_{ij}\mathbf{z}(t) \\
&\quad + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\mathbf{z}(t)^T(\mathbf{Q}_{ij} + \mathbf{Q}_{ji})\mathbf{z}(t).
\end{aligned} \tag{2.23}$$

By satisfying the LMI-based stability conditions in Theorem 2.3, $V(\mathbf{x}(t)) \ge 0$ (equality holds for $\mathbf{x}(t) = \mathbf{0}$) and $\dot{V}(\mathbf{x}(t)) \le 0$ (equality holds for $\mathbf{x}(t) = \mathbf{0}$) can be achieved. The FMB control system (2.19) with the PDC design is asymptotically stable in the sense of Lyapunov.

It can be seen from the above proof that $\mathbf{Q}_{ij}$ with the same cross term of membership functions $w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))$ can be collected. Thus, the stability analysis result can be relaxed by considering $\mathbf{Q}_{ij} + \mathbf{Q}_{ji} < 0$ for $i \le j$ rather than $\mathbf{Q}_{ij} < 0$ for all i and j.

Relaxation of stability analysis depends on the way of grouping the cross term of membership functions. In the following, the LMI-based stability conditions under the PDC design with different ways of grouping are given.

Theorem 2.4 ([8]) *The FMB control system (2.19), formed by a nonlinear plant represented by the T-S fuzzy model (2.2) and the fuzzy controller (2.13) under the PDC design, i.e., with* $c = p$ *and* $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ *for all* i *connected in a closed loop, is asymptotically stable if there exist matrices* $\mathbf{N}_j \in \Re^{m\times n}$, $\mathbf{X} = \mathbf{X}^T \in \Re^{n\times n}$ *and* $\mathbf{X}_{ij} = \mathbf{X}_{ji}^T \in \Re^{n\times n}$, $i, j = 1, \ldots, p$ *such that the following LMIs hold:*

$$\mathbf{X} > 0;$$

$$\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_i^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_i < \mathbf{X}_{ii} \; \forall \; i;$$

$$\begin{aligned}
&\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_j + \mathbf{X}\mathbf{A}_j^T + \mathbf{A}_j\mathbf{X} \\
&+ \mathbf{N}_i^T\mathbf{B}_j^T + \mathbf{B}_j\mathbf{N}_i \le \mathbf{X}_{ij} + \mathbf{X}_{ij}^T \; \forall \; j, i < j;
\end{aligned}$$

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_{11} & \cdots & \mathbf{X}_{1p} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{p1} & \cdots & \mathbf{X}_{pp} \end{bmatrix} < 0;$$

and the feedback gains are defined as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$ *for all* j.

Theorem 2.5 ([11]) *The FMB control system (2.19), formed by a nonlinear plant represented by the T-S fuzzy model (2.2) and the fuzzy controller (2.13) under the PDC design, i.e., with* $c = p$ *and* $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ *for all* i *connected in a closed loop, is asymptotically stable if there exist matrices* $\mathbf{N}_j \in \Re^{m\times n}$, j = *1, ...,* p, $\mathbf{X} = \mathbf{X}^T \in \Re^{n\times n}$, $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^T \in \Re^{n\times n}$, i = *1, ...,* p, $\mathbf{Y}_{iij} = \mathbf{Y}_{jii}^T \in \Re^{n\times n}$, $\mathbf{Y}_{iji} = \mathbf{Y}_{iji}^T \in \Re^{n\times n}$, i, j = *1, ...,* p; $i \neq j$, $\mathbf{Y}_{ijk} = \mathbf{Y}_{kji}^T \in \Re^{n\times n}$, $\mathbf{Y}_{ikj} = \mathbf{Y}_{jki}^T \in \Re^{n\times n}$ *and* $\mathbf{Y}_{jik} = \mathbf{Y}_{kij}^T \in \Re^{n\times n}$, i = *1, ..., p-2;* j = *i+1, ..., p-1;* k = *j+1, ...,* p *such that the following LMIs hold:*

$$\mathbf{X} > 0;$$

$$\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} + \mathbf{N}_i^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{N}_i < \mathbf{Y}_{iii}\ \forall\, i;$$

$$\begin{aligned} &2\mathbf{X}\mathbf{A}_i^T + \mathbf{X}\mathbf{A}_j^T + 2\mathbf{A}_i\mathbf{X} + \mathbf{A}_j\mathbf{X} + (\mathbf{N}_i + \mathbf{N}_j)^T\mathbf{B}_i^T + \mathbf{N}_i^T\mathbf{B}_j^T \\ &+ \mathbf{B}_i(\mathbf{N}_i + \mathbf{N}_j) + \mathbf{B}_j\mathbf{N}_i \le \mathbf{Y}_{iij} + \mathbf{Y}_{iji} + \mathbf{Y}_{iij}^T + \mathbf{Y}_{iji}^T\ \forall\, i,\ j;\ j \neq i; \end{aligned}$$

$$\begin{aligned} &2\mathbf{X}(\mathbf{A}_i + \mathbf{A}_j + \mathbf{A}_k)^T + (\mathbf{N}_j + \mathbf{N}_k)^T\mathbf{B}_i^T + (\mathbf{N}_i + \mathbf{N}_k)^T\mathbf{B}_j^T + (\mathbf{N}_i + \mathbf{N}_j)^T\mathbf{B}_k^T \\ &+ 2(\mathbf{A}_i + \mathbf{A}_j + \mathbf{A}_k)\mathbf{X} + \mathbf{B}_i(\mathbf{N}_j + \mathbf{N}_k) + \mathbf{B}_j(\mathbf{N}_i + \mathbf{N}_k) + \mathbf{B}_k(\mathbf{N}_i + \mathbf{N}_j) \\ &\le \mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{ijk}^T + \mathbf{Y}_{ikj}^T + \mathbf{Y}_{jik}^T, i = 1, \ldots, p-2; \\ &j = i+1, \cdots, p-1; i, k = j+1, \ldots, p; \end{aligned}$$

$$\tilde{\mathbf{Y}}_i = \begin{bmatrix} \mathbf{Y}_{1i1} & \cdots & \mathbf{Y}_{1ip} \\ \vdots & \ddots & \vdots \\ \mathbf{Y}_{pi1} & \cdots & \mathbf{Y}_{pip} \end{bmatrix} < 0\ \forall\, i,$$

where the feedback gains are defined as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$ *for all* j.

Remark 2.5 It should be noted that the stability conditions in Theorems 2.3 and 2.4 are particular cases of Theorem 2.5.

The LMI-based stability conditions under the PDC design can be generalized using the Pólya's permutation theorem [12] for grouping the terms $\mathbf{Q}_{ij}$ with the expansion of the degree of fuzzy summations by multiplying $\sum_{i_1=1}^{p} \ldots \sum_{i_k=1}^{p} w_{i_1}(\mathbf{x}(t)) \ldots w_{i_k}(\mathbf{x}(t)) = 1$ to $\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\mathbf{Q}_{ij}$.

Denote

$$I_q = \left\{\mathbf{i} = (i_1, \cdots, i_q) \in N^q | 1 \le i_j \le p\ \forall\, j = 1, \cdots, q\right\},$$

$$I_q^+ = \left\{\mathbf{i} \in I_q | i_k \le i_{k+1}, k = 1, \cdots, q-1\right\}$$

as a subset of I_q, $\sum_{\mathbf{i}\in I_q} \mathbf{w_i}(\mathbf{x}(t)) = \sum_{i_1=1}^{p} \dots \sum_{i_q=1}^{p} w_{i_1}(\mathbf{x}(t)) \dots w_{i_q}(\mathbf{x}(t))$, and the set of permutations as $P(\mathbf{i}) \subset I_q$, where $\mathbf{i} \in I_q$. The LMI-based stability conditions [12] for the FMB control system (2.19) obtained based on the Pólya's permutation theorem are given in the theorem below.

Theorem 2.6 ([12]) *The FMB control system (2.19), formed by a nonlinear plant represented by the T-S fuzzy model (2.2) and the fuzzy controller (2.13) under the PDC design, i.e., with $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i connected in a closed loop, is asymptotically stable if the following LMIs given in the following h steps are satisfied, where h = 0, 1, 2, ..., $h_{max} = floor\left(\frac{d-1}{2}\right)$ and $d \ge 2$. The dimension of the multi-indices in the iteration step h is denoted by $d_h = d - 2h$.*

1. *(Initialization) Choose the degree of the fuzzy summation as $d \ge 2$ and set $\mathbf{Q_i}^{[0]} = \mathbf{Q}_{i_1\cdots i_d}$ for $\mathbf{i} \in I_d$ and $h = 0$. Define matrices $\mathbf{Q_j}^{[0]} = \mathbf{Q_j}^{[0]^T} \in \Re^{n\times n}$ for $\mathbf{j} \in P(\mathbf{i})$, $\mathbf{i} \in I_{d_o}^+$.*
2. *(Recursive procedure) In the iterative step h, when $h < h_{max}$, the following inequality is included as the LMI condition:*

$$\sum_{\mathbf{j}\in P(\mathbf{i})} \mathbf{Q_j}^{[h]} < \frac{1}{2} \sum_{\mathbf{j}\in P(\mathbf{i})} (\mathbf{X_j}^{[h]} + \mathbf{X_j}^{[h]^T}) \ \forall \, \mathbf{i} \in I_{d_h}^+,$$

   and set

$$\mathbf{Q_k}^{[h+1]} = \begin{bmatrix} \mathbf{X}_{(\mathbf{k},1,1)}^{[h]} & \cdots & \mathbf{X}_{(\mathbf{k},1,p)}^{[h]} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{(\mathbf{k},p,1)}^{[h]} & \cdots & \mathbf{X}_{(\mathbf{k},p,p)}^{[h]} \end{bmatrix} \ \forall \, \mathbf{k} \in I_{d_h-2}^+,$$

   where $\mathbf{X}_{(\mathbf{k},i_{d-1},i_d)}^{[h]} = \mathbf{X}_{(\mathbf{k},i_d,i_{d-1})}^{[h]^T} \in \Re^{n\times n}$ when $i_{d-1} = i_d \ \forall \, \mathbf{k} \in I_{d_h-2}^+$, i_{d-1} = 1, ..., p. It should be noted that $d_{h+1} = d_h - 2$.
3. *Set $h = h + 1$. If $h < h_{max}$, go to step 2, otherwise, go to next step.*
4. *(Termination) When $d_{h_{max}}$ = 1, the stability conditions in Theorem 2.3 are included as the LMI conditions in this theorem. When $d_{h_{max}}$ = 2, the stability conditions in Theorem 2.4 are included as the LMI conditions in this theorem.*

Remark 2.6 The stability conditions in Theorems 2.4 and 2.5 are special cases of Theorem 2.6 with d = 2 and d = 3, respectively.

Remark 2.7 In view of the properties of the cases of partially/imperfectly and perfectly matched premises as shown in Table 2.1, it is suggested to design the fuzzy controller using Theorem 2.2 as the first trial to take advantage of the better design flexibility and robustness property. If the design fails, other relaxed stability conditions can then be applied.

Remark 2.8 It should be noted that the number of slack matrices introduced to the LMI-based stability conditions increases when the degree of fuzzy summation increases, which will increases the computational demand to search numerically for a feasible solution using convex programming techniques.

Example 2.3 In this simulation example, the LMI-based stability conditions from Theorems 2.2 to 2.6 are investigated. The stability region produced by each theorem is investigated by an FMB control system in the form of (2.19), which is formed by a 3-rule T-S fuzzy model in the form of (2.2) and a 3-rule fuzzy controller in the form of (2.13) connected in a closed loop. As the LMI-stability conditions in Theorems 2.2 to 2.6 are independent of the membership functions, the T-S fuzzy model and the fuzzy controller under the partially/imperfectly matched premises ($c \neq p$ and/or $m_i(\mathbf{x}(t)) \neq w_i(\mathbf{x}(t))$ for any i) and PDC design ($c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for $i = 1, 2, 3$) can take any shapes of membership functions satisfying the properties (2.3) and (2.14).

The 3-rule T-S fuzzy model is chosen with the following system and input matrices [11]:

$$\mathbf{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} -a & -4.33 \\ 0 & -0.05 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} -b + 6 \\ -1 \end{bmatrix},$$

where a and b are constant parameters.

With the Matlab LMI toolbox, the stability regions for $2 \leq a \leq 12$ and $2 \leq b \leq 12$ both at the interval of 1 given by the LMI-based stability conditions in Theorems 2.2 to 2.6 ($d = 4$ for Theorem 2.6) are shown in Fig. 2.1 indicated by different symbols. An empty point means that no feasible solution is found. It should be noted that no feasible solution is found for the LMI-based stability conditions in Theorems 2.2 and 2.3. It can be seen from Fig. 2.1 that the LMI-based stability conditions in Theorem 2.6 offer the largest size of stability region.

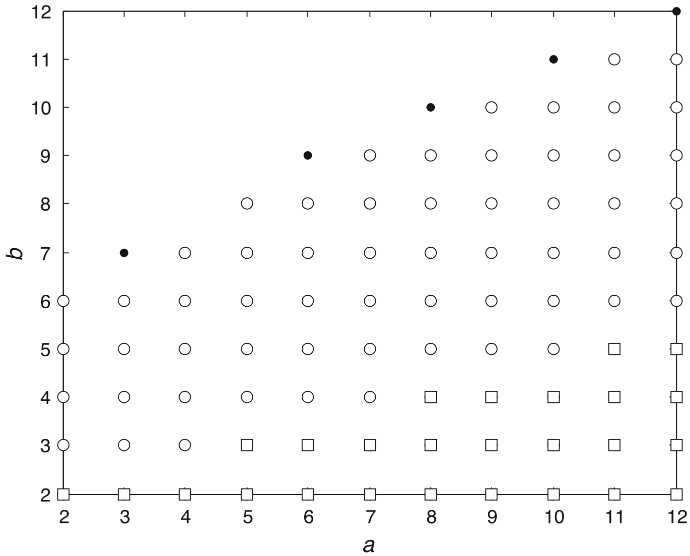

Fig. 2.1 Stability regions given by Theorems 2.4 ('□'), 2.5 ('□' and '○') and 2.6 ('□', '○' and '•')

2.6.2 SOS-Based Stability Conditions for FMB Control Systems

The stability of PFMB control system under the PDC design was investigated in [15] based on the Lyapunov stability theory using a polynomial Lyapunov function candidate. Stability conditions in terms of SOS were obtained in [15] and summarized in the following theorem.

Theorem 2.7 ([15]) *The PFMB control system (2.20), formed by a nonlinear plant represented by the polynomial fuzzy model (2.9) and the polynomial fuzzy controller (2.17) under the PDC design, i.e., with $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i connected in a closed loop, is asymptotically stable if there exist polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}(t)) = \mathbf{X}(\tilde{\mathbf{x}}(t))^T \in \Re^{N\times N}$ and $\mathbf{N}_j \in \Re^{m\times N}$ such that the following SOS hold:*

$$\mathbf{r}(t)^T \big(\mathbf{X}(\tilde{\mathbf{x}}(t)) - \varepsilon_1(\tilde{\mathbf{x}}(t))\mathbf{I}\big)\mathbf{r}(t) \text{ is SOS};$$

$$-\mathbf{r}(t)^T \big(\mathbf{Q}_{ij}(\mathbf{x}(t)) + \mathbf{Q}_{ji}(\mathbf{x}(t)) + \varepsilon_2(\mathbf{x}(t))\mathbf{I}\big)\mathbf{r}(t) \text{ is SOS } \forall\ i \le j,$$

where $\mathbf{Q}_{ij}(\mathbf{x}(t)) = \mathbf{X}(\tilde{\mathbf{x}}(t))\mathbf{A}_i(\mathbf{x}(t))^T\mathbf{T}(\mathbf{x}(t))^T + \mathbf{T}(\mathbf{x}(t))\mathbf{A}_i(\mathbf{x}(t))\mathbf{X}(\tilde{\mathbf{x}}(t)) + \mathbf{N}_j(\mathbf{x}(t))^T\mathbf{B}_i(\mathbf{x}(t))^T\mathbf{T}(\mathbf{x}(t))^T + \mathbf{T}(\mathbf{x}(t))\mathbf{B}_i(\mathbf{x}(t))\mathbf{N}_j(\mathbf{x}(t)) - \sum_{k\in\mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}}(t))}{\partial x_k(t)} \times \mathbf{A}_i^k(\mathbf{x}(t))\hat{\mathbf{x}}(t)$; $\mathbf{T}_{ij}(\mathbf{x}(t)) \in \Re^{N\times n}$ is a polynomial matrix with its (i, j)th entry given by $T_{ij}(\mathbf{x}(t)) = \frac{\partial \hat{x}_i(\mathbf{x}(t))}{\partial x_j(t)}$; $\mathbf{A}_i^k(\mathbf{x}(t))$ denotes the kth row of $\mathbf{A}_i(\mathbf{x}(t))$; $\mathbf{K} = \{k_1, \ldots, k_q\}$ denotes the row indices of $\mathbf{B}_i(\mathbf{x}(t))$ in which the corresponding row is equal to 0;

$\tilde{\mathbf{x}}(t)$ is a vector consists of any system states $x_{k_1}(t)$ to $x_{k_q}(t)$; $\hat{\mathbf{x}}(t)$ is a vector of monomials in $\mathbf{x}(t)$; $\mathbf{r}(t) \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}(t)$; $\varepsilon_1(\tilde{\mathbf{x}}(t)) > 0$ and $\varepsilon_2(\mathbf{x}(t)) > 0$ are predefined scalar polynomials; the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}(t)) = \mathbf{N}_j(\mathbf{x}(t))\mathbf{X}(\tilde{\mathbf{x}}(t))^{-1}$ for all j.

Remark 2.9 The SOS-based stability conditions contain the system states such that the Matlab LMI toolbox [41] cannot be employed to search numerically for a feasible solution. Instead, SOSTOOLS [36], which is a third party Matlab toolbox, is employed. The purpose of the predefined scalar polynomials $\varepsilon_1(\tilde{\mathbf{x}}(t))$ and $\varepsilon_2(\mathbf{x}(t))$ is to make sure that the SOS-based stability conditions in Theorem 2.7 are strictly positive definite.

2.7 Conclusion

In this chapter, the preliminaries of the FMB/PFMB control systems have been reviewed. Various fuzzy models, such as T-S fuzzy model and T-S polynomial fuzzy model have been introduced, which are used to represent the nonlinear plants to facilitate the system stability and control synthesis. Examples have been given to demonstrate the properties and construction of fuzzy models using the sector nonlinear concept. Corresponding to each type of fuzzy models and fuzzy controllers, traditional state-feedback fuzzy controller, polynomial fuzzy controller and their technical details have been presented. By connecting the fuzzy controller to the fuzzy model in a closed-loop, an FMB/PFMB control system is obtained. Various LMI/SOS-based stability conditions reported in the literature for the FMB/PFMB control systems have been reviewed, which will be employed in different chapters for comparison purposes. The materials in this chapter provides technical details of the FMB control systems to support the work in this book.

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