Preface

Connectivity is one of the most basic concepts of graph-theoretical subjects, both in a combinatorial sense and in an algorithmic sense. As we know, the classical connectivity has two equivalent definitions, one is the "cut" version; the other is the "path" version. The generalized connectivity of a graph G is a natural generalization of the "path" version definition. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is such a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$, a minimal Steiner tree connecting S is just a path connecting the two vertices of S. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint Steiner trees connecting S in G, that is, we search for the largest number of edge-disjoint Steiner trees which contain S and are vertex disjoint with the exception of the vertices in S. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* (or *k-tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k-subsets of $V(G)$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is just the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$, and $\kappa_k(G) = 0$ when G is disconnected. This concept was first mentioned in a paper by Hager published in 1985 and studied by himself in his other unpublished paper. So, it should have appeared before 1985.

There are many other kinds of generalizations of the classical connectivity, such as pendant tree-connectivity, path-connectivity, k-connectivity, restricted connectivity, super connectivity, etc. But, they are different from the generalized k-connectivity we deal with in this book.

As a natural counterpart of the generalized connectivity, we introduced the concept of generalized edge-connectivity $\lambda_k(G)$, which is defined similarly to $\kappa_k(G)$ only with the change of internally disjoint Steiner trees into edge-disjoint Steiner trees, i.e., only with $E(T) \cap E(T') = \varnothing$ but without $V(T) \cap V(T') = S$. The generalized edge-connectivity is related to two important problems. For a given graph G and $S \subseteq V(G)$, the problem of finding a maximum set of edge-disjoint Steiner trees connecting S in G is called the *Steiner tree packing problem*. The problem for $S = V(G)$ is called the *spanning-tree packing problem*. Note that spanning tree packing problem is a specialization of the Steiner tree packing problem (for $k = n$, each Steiner tree connecting S is a spanning tree of G). For any graph G of order n, the *spanning tree packing number* or *STP number* is the maximum number of edge-disjoint spanning trees contained in G. From the definitions of $\kappa_k(G)$ and $\lambda_k(G)$, $\kappa_n(G) = \lambda_n(G)$ is exactly the spanning-tree packing number of G (for $k = n$, internally disjoint Steiner trees connecting S are edge-disjoint spanning trees).

Both extremal cases of k, i.e., $k = 2$ and n, for the two parameters are fundamental theorems in graph theory. The extremal case $k = 2$ of the problem means when we have two terminals. In this case internally (edge-)disjoint trees are just internally (edge-)disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extremal case $k = n$ means when all the vertices are terminals. In this case, internally disjoint Steiner trees are just edge-disjoint spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem.

The generalized edge-connectivity and the Steiner tree packing problem have applications in *VLSI* circuit design. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. A Steiner tree is also used in computer communication networks and optical wireless communication networks. Another application, which is our primary focus, arises in the Internet domain. Imagine that a given graph G represents a network. We choose arbitrary k vertices as nodes. Suppose one of the nodes in G is a *broadcaster*, and all other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. So, in essence, we need to find the maximum number of Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda(S)$, where S is the set of the k nodes. Clearly, it is a Steiner tree packing problem. Furthermore, if we want to know whether for any k nodes the network G has the above properties, then we need to compute $\lambda_k(G) = \min\{\lambda(S)\}$ in order to prescribe the reliability and the security of the network.

After the concept of generalized connectivity was proposed by Hager in 1985, the study on this graph parameter kept silent for about 25 years. Only in recent 5 years when a paper by Chartrand et al. in 2010 reintroduced it, this parameter then caught people's attention. As a result, the study on this parameter was stimulated, and there have been quite a lot of results published. The goal of this book is to bring together most of the results that deal with it. We begin with an introductory chapter. In Chap. 2, we summarize the results on the generalized (edge-)connectivity for some graph classes, giving the reader some intuitive idea about the parameter. In Chap. 3, we address the problem of algorithms and computational complexity for the generalized (edge-)connectivity. In general, it is *NP*-hard. Chapter 4 is then to report sharp bounds that have been obtained in recent years. Chapter 5 is to characterize the graphs with large generalized (edge-)connectivity, and Chap. 6 is to develop Nordhaus-Gaddum-type inequalities. Some results on graph operations are reported in Chap. 7. Extremal problems on the generalized local (edge-) connectivity are reported in Chap. 8. The last chapter, Chap. 9, covers results for random graphs. In each chapter, we list some conjectures, open problems, or questions at proper places. We hope that this can motivate more young graph theorists and graduate students to do further study in this subject. We do not give proofs for all results. Instead, we only select some of them for which we gave their proofs because we feel that these proofs employed some typical techniques, and these proof techniques are popular in the study of generalized (edge-)connectivity. New results are still appearing. There must be some or even many of them for which we have not realized their existence and therefore have not included them in this book.

The readers of the book are expected to have some background in graph theory and some related knowledge in combinatorics, probability, algorithms, and complexity analysis. All relevant notions from graph theory are properly defined in Chap. 1, but also elsewhere where needed.

The anticipated readers of the book are mathematicians and students of mathematics, whose fields of interest are *graph theory* and *combinatorial optimization*. Consequently, this book will be found suitable for such courses. The exposition of the details of the proofs of some main results will enable students to understand and eventually master a good part of graph theory and combinatorial optimization. People working on *communication networks* may also be interested in some aspects of the book. They will find it useful for designing networks that can efficiently, reliably, and safely transfer information.

The material presented in this book was used in graph theory seminars, held three times at Nankai University, in 2013, 2014, and 2015. We thank all the members of our group for help in the preparation of this book. Without their help, we would have not finished writing it in such a short period of time. We also thank the NSFC (National Natural Science Foundation of China) for financial support to our research project on generalized (edge-)connectivity. Last but not least, we are very grateful to the editor for algebraic combinatorics and graph theory of this new series of books of Springer Briefs, Professor Ping Zhang, for inviting us to write this book. Without her encouragement, this book may not exist.

Tianjin, China

Xueliang Li
Yaping Mao

Generalized Connectivity of Graphs
Li, X.; Mao, Y.
2016, X, 143 p. 28 illus., 6 illus. in color., Softcover
ISBN: 978-3-319-33827-9