Chapter 2
Results for Some Graph Classes

In this chapter, we study the generalized k-connectivity for some special graph classes. As one will see, even for special graphs, it is not easy to get the exact values of the generalized connectivity for them. The following two observations are easily seen.

Observation 2.1. *Let n, k be two integers with $2 \leq k \leq n$. If G is a connected graph of order n, then $\kappa_k(G) \leq \lambda_k(G) \leq \delta(G)$.*

Observation 2.2. *Let n, k be two integers with $2 \leq k \leq n$. If H is a spanning subgraph of a graph G with order n, then $\kappa_k(H) \leq \kappa_k(G)$ and $\lambda_k(H) \leq \lambda_k(G)$.*

2.1 Results for the Spanning-Tree Packing Number

At first we look at the case for $k = n$, i.e., the spanning-tree packing numbers of graphs. In his survey paper [114], Palmer listed the spanning-tree packing numbers for some graph classes. Since then, some new results have been derived. We now give an update for Palmer's table; see Table 2.1.

First of all, we give the definitions for the special graph classes. An *r-dimensional torus* is the Cartesian product of r cycles of sizes at least 3. An *r-dimensional torus-connected k-ary hypercube* is a special case of an r-dimensional torus, in which the r cycles are all of the same size k, where $k \geq 3$. Simply, we call a 2-dimensional torus a torus. A graph G is called *maximal planar* if there is an embedding of G such that each region is bounded by three edges. A maximal toroidal graph is defined in a similar way. A *toroidal graph* is defined as a graph that can be embedded on the torus but not on the plane. A graph G is called *maximal toroidal* if G is toroidal and, for some embedding, every region has three sides. Let q be a prime power congruent to 1 (mod 4). The *Paley graph* P_q has the q elements of the field F_q as its vertices and x are adjacent to y if $x - y$ is a square in F_q. These graphs are regular of degree

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Table 2.1 Results for STP number

Graphs or networks	STP number
K_n $(n \geq 1)$	$\lfloor n/2 \rfloor$
$K_{a,b}$ $(1 \leq a \leq b)$	$\lfloor ab/(a+b-1) \rfloor$
Complete multipartite graph (order n and size m)	$\lfloor m/(n-1) \rfloor$
k-tree (order n and size m)	$\lfloor m/(n-1) \rfloor$
Q_n $(n \geq 1)$	$\lfloor n/2 \rfloor$
Paley P_q	$(q-1)/4$
Octahedron	2
Icosahedron	2
Maximal planar	2
Connected cubic $(n \geq 6)$	1
Connected 4-regular $(n \geq 7)$	1 or 2
Maximal toroidal $(n \geq 7)$	3
Random r-regular	$\lfloor r/2 \rfloor$
Harary $H_{n,d}$	$\lfloor \lfloor 2d/n \rfloor /2 \rfloor$
Quasi-random	$(1+o(1))n/4$
r-d torus	r (maximum)
r-d torus-connected k-ary hypercube	r (maximum)
r-d hypercube	$\lfloor r/2 \rfloor$ (maximum)
r-d mesh (having $3m$ dimensions of size at least 3)	$2m + \lfloor (r-3m)/2 \rfloor$ (maximum)
r-d mesh-connected k-ary hypercube	$\lfloor 2r/3 \rfloor$ (maximum for $k=3$)
r-d generalized hypercube (m_j, size of the j-th clique ; f, number of even-size cliques)	$\sum_{j=1}^{n} \lfloor m_j/2 \rfloor - \lceil f/2 \rceil$ (maximum)
r-d mesh-connected tree (h, height of the trees)	$\lfloor 2r/3 \rfloor$ (maximum for $h=2$)
r-d hyper Petersen network	$\lfloor r/2 \rfloor$ (maximum)
$K_n \square C_m$	$\lfloor (nm(n+1))/2(nm-1) \rfloor$
$(K_a \circ bK_1) \square C_p$	$\lfloor (abp(ab-b+2))/2(abp-1) \rfloor$
$(K_a \circ bK_1) \square K_p$	$\lfloor (abp(ab+p-b-1))/2(abp-1) \rfloor$
$(K_a \circ bK_1) \square (K_p \circ qK_1)$	$\lfloor (abpq(ab+pq-b-q))/2(abpq-1) \rfloor$

$(q-1)/2$ and hence have diameter 2. Let k be a positive integer. Then a *k-tree* is a graph defined recursively as follows: the smallest k-tree is the complete graph with k vertices, and a k-tree with $n+1$ vertices where $n \geq k$ is obtained by adding a new vertex adjacent to each of the k arbitrarily selected but mutually adjacent vertices of a k-tree with n vertices.

The Harary graph $H_{n,d}$ is constructed by arranging the n vertices in a circular order and spreading the d edges around the boundary in a nice way, keeping the chords as short as possible. *Harary graph* $H_{n,d}$ is a d-connected graph on n vertices that has exactly $\lfloor \frac{dn}{2} \rfloor$ edges, and the structure of $H_{n,d}$ depends on the parities of d and n:

Case 1: d even. Let $d = 2r$. Then $H_{n,2r}$ is constructed as follows. It has vertices $0, 1, \cdots, n-1$, and two vertices i and j are jointed if $i - r \leq j \leq i + r$ (where addition is taken modulo n).
Case 2: d odd, n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$.
Case 3: d odd and n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and i to vertex $i + \frac{n+1}{2}$ for $1 \leq i \leq \frac{n-1}{2}$.

An *r-dimensional mesh* is the Cartesian product of r linear arrays. An r-dimensional hypercube is a special case of an r-dimensional mesh, in which the r linear arrays are all of size 2. An *r-dimensional mesh-connected k-ary hypercube* is a special case of an r-dimensional mesh, in which the r linear arrays are all of the same size k, where $k \geq 3$. An *r-dimensional generalized hypercube* is the Cartesian product of r cliques. An *r-dimensional mesh-connected tree* is the Cartesian product of r complete binary trees of the same height $h \geq 2$. An *r-dimensional hyper Petersen network* HP_r is the Cartesian product of the well-known Petersen graph and Q_{r-3}, where $r \geq 3$ and Q_{n-3} denotes an $(r-3)$-dimensional hypercube. Let $\mathscr{F}_n$ be a family of graphs of order n with vertex set $[n] = \{1, 2, \ldots, n\}$ and some specified probability distribution. The family is *quasi-random* if with probability approaching 1 as $n \to \infty$, we have $|E_G[S, \bar{S}]| = \frac{|S||\bar{S}|}{2} + o(n^2)$ for every subset $S \subseteq [n]$.

Peng, Chen, and Koh [116] obtained the following bounds of $\lambda_n(G)$.

Theorem 2.1.1 ([116]). *Let G be a graph of order n and edge-connectivity λ. Then* $\lfloor \frac{n\lambda}{2(n-1)} \rfloor \leq \lambda_n(G) \leq \lambda$.

The following result is an immediate consequence of Theorem 2.1.1.

Theorem 2.1.2 ([119]). *Let G be a graph of order n and edge-connectivity λ. Then* $\lambda_n(G) = \left\lfloor \frac{n\lambda}{2(n-1)} \right\rfloor$ *if and only if G is λ-regular.*

Corollary 2.1.3 ([119]).

(1) *If G is a r-regular graph of order n and* $r \geq \lfloor n/2 \rfloor$, *then* $\lambda_n(G) = \left\lfloor \frac{nr}{2(n-1)} \right\rfloor$. *If* $r = (n-1)/2$, *then* $\lambda_n(G) = \lambda_n(\overline{G}) = \left\lfloor \frac{nr(n-1)}{2} \right\rfloor$.
(2) *If G is a vertex-transitive graph or a regular edge-transitive graph of order n and size m, then* $\lambda_n(G) = \lfloor \frac{m}{n-1} \rfloor$.

Corollary 2.1.4 ([119]). *Let* G_i $(i = 1, 2)$ *be a connected graph of order* n_i*, size* m_i*, and edge-connectivity* λ_i*. If* G_i *is* λ_i*-regular, then*

$$\lambda_n(G_1 \Box G_2) = \left\lfloor \frac{m_1 n_2 + m_2 n_1}{n_1 n_2 - 1} \right\rfloor.$$

We first state the following result, which will play a key role for studying the spanning-tree packing number.

Theorem 2.1.5 ([119]). *Let G be a graph. Then $|E(G)|/(|V(G)|-1) \leq |X|/(\omega(G-X) - 1)$ for every edge cutset X of G if and only if*

$$\frac{|E(H)|}{|V(H)| - 1} \leq \frac{|E(G)|}{|V(G)| - 1}$$

for every subgraph H of G.

Proof. Let n and m, respectively, be the order and size of G. Suppose $|E(G)|/(|V(G)| - 1) \leq |X|/(\omega(G-X)) - 1)$ for every edge cutset X of G. Let H be any subgraph with n' vertices and m' edges. If $n' = n$, then $m'/(n'-1) \leq m/(n-1)$. Otherwise, let x be the cardinality of the edge cutset $X = E(G) - E(H)$, which separates G into H and some isolated vertices. Then $m = m' + x$ and $\omega(G-X) \geq n - n' + 1$. Therefore, we have

$$\frac{m}{n-1} \leq \frac{x}{n-n'},$$

i.e., $m(n-n') \leq (m-m')(n-1)$. Thus

$$\frac{m'}{n'-1} \leq \frac{m}{n-1},$$

as required.

Conversely, let X be any edge cutset separating G into t components $H_1, H_2, \ldots, H_t$. Then for each component H_i, we have

$$\frac{|E(H_i)|}{|V(H_i)| - 1} \leq \frac{m}{n-1}.$$

Thus

$$m = \sum_{i=1}^{t} |E(H_i)| + |X| \leq \sum_{i=1}^{t} \frac{m}{n-1}(|V(H_i)| - 1) + |X| = \frac{m}{n-1}(n-t) + |X|.$$

Therefore

$$\frac{m}{n-1} \leq \frac{|X|}{t-1},$$

as required. $\square$

Corollary 2.1.6 ([119]). *Let G be a connected graph of order n and size m. Then $\lambda_n(G) = \lfloor \frac{m}{n-1} \rfloor$ if and only if $|E(H)| \leq \lfloor \frac{m}{n-1} \rfloor (|V(H)| - 1)$ for every subgraph H of G.*

The edge toughness of a complete r-partite graph has been determined by Peng, Chen, and Koh [116]. Peng and Tay [119] later gave a simpler proof to the result.

Theorem 2.1.7 ([119]). *If G is a complete r-partite graph with n vertices and m edges, then $\lambda_n(G) = \lfloor \frac{m}{n-1} \rfloor$.*

We shall use Theorem 2.1.5 to prove Theorem 2.1.7. To do so, we first establish the following lemma, from which Theorem 2.1.7 follows as an easy consequence.

Lemma 2.1.8 ([119]). *Let $H = K_{m_1, m_2, \ldots, m_r}$ be a complete r-partite graph where $r \geq 2$ and $m_r \geq m_{r-1} \geq \ldots \geq m_1 \geq 1$. If G is a complete r-partite or $(r+1)$-partite graph obtained from H by adding one vertex, then*

$$\frac{|E(H)|}{|V(H)| - 1} \leq \frac{|E(G)|}{|V(G)| - 1}. \tag{2.1}$$

Proof. Note that $|E(H)| = \sum_{1 \leq i < j \leq r} m_i m_j$, $|V(H)| = \sum_{1 \leq i \leq r} m_i$, and $|V(G)| = |V(H)| + 1$. Then

$$\begin{aligned}
\frac{|E(H)|}{|V(H)| - 1} &\leq \frac{|E(G)|}{|V(G)| - 1} \\
&\iff |E(H)||V(G)| - |E(H)| \leq |E(G)||V(H)| - |E(G)| \\
&\iff |E(H)|(|V(H)| + 1) \leq |E(G)||V(H)| - |E(G)| + |E(H)| \\
&\iff |E(H)| \leq (|E(G)| - |E(H)|)(|V(H)| - 1).
\end{aligned}$$

Thus (2.1) is equivalent to the following:

$$\frac{|E(H)|}{|V(H)| - 1} \leq |E(G)| - |E(H)|. \tag{2.2}$$

Now

$$\begin{aligned}
\sum_{1 \leq i < j \leq r} m_i m_j &= m_1(m_2 + m_3 + \ldots + m_r) + m_2(m_3 + \ldots + m_r) + m_{r-1}(m_r) \\
&\leq \sum_{i=1}^{r-1} m_i \left(\sum_{j=1}^{r} m_j - 1 \right).
\end{aligned}$$

Hence

$$\frac{|E(H)|}{|V(H)| - 1} \leq \sum_{i=1}^{r-1} m_i. \tag{2.3}$$

But the addition of one vertex to H to create G increases the number of edges by at least $\sum_{i=1}^{r-1} m_i$. Therefore, (2.2) follows from (2.3). □

The following corollary is immediate.

Corollary 2.1.9 ([119]). *Let G be a complete r-partite graph. If H is an induced subgraph of G, then* $|E(H)|(|V(G)|-1) \leq |E(G)|(|V(H)|-1)$.

Using the above results, one can deduce the following proof:

Proof of Theorem 2.1.7. Let H be any subgraph of complete r-partite graph G. Then there exists a complete multipartite graph F such that H is a spanning subgraph of F. Clearly, we have

$$\frac{|E(H)|}{|V(H)|-1} \leq \frac{|E(F)|}{|V(F)|-1}. \tag{2.4}$$

From Lemma 2.1.8, we have

$$\frac{|E(F)|}{|V(F)|-1} \leq \frac{|E(G)|}{|V(G)|-1} = \frac{m}{n-1}, \tag{2.5}$$

since G can be obtained from F by adding $n - |V(F)|$ vertices successively. From (2.4) and (2.5), we conclude that $|E(H)| \leq \frac{m}{n-1}(|V(H)|-1)$ for any subgraph of G. From Corollary 2.1.6, we have $\lambda_n(G) = \lfloor \frac{m}{n-1} \rfloor$. □

2.2 Results for the Generalized k-Connectivity

In this section we will show how to compute the generalized k-connectivity of special graphs for general k, $2 \leq k \leq n$. Chartrand, Okamoto, and Zhang in [28] proved that if G is the complete 3-partite graph $K_{3,4,5}$, then $\kappa_3(G) = 6$. They also got the exact value of the generalized k-connectivity for complete graph K_n.

Theorem 2.2.1 ([28]). *For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil k/2 \rceil$.*

Proof. Let $S = \{u_1, u_2, \ldots, u_k\}$ and $V(K_n) - S = \{w_1, w_2, \ldots, w_{n-k}\}$ (if $k < n$). Also, let $\mathscr{T}$ be a maximum set of internally disjoint S-Steiner trees in K_n. Let $\mathscr{T}_1$ be the set of Steiner trees in $\mathscr{T}$ whose vertex set is S, and let $\mathscr{T}_2$ be the set of Steiner trees T in $\mathscr{T}$ for which S is a proper subset of $V(T)$. Thus $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. As every tree in $\mathscr{T}_2$ contains at least one vertex not belonging to S, it follows that $|\mathscr{T}_2| \leq |V(K_n) - S| = n - k$, and furthermore, $|\mathscr{T}_1| \leq \lfloor \binom{k}{2}/(k-1) \rfloor$. Hence $|\mathscr{T}| \leq n - \lceil k/2 \rceil$ and so $\kappa_k(K_n) \leq n - \lceil k/2 \rceil$.

For any $S \subseteq V(K_n)$ and $|S| = k$, if k is even, then $K_n[S]$ can be factored into $k/2$ Hamilton paths, whereas if k is odd, then $K_n[S]$ can be factored into $(k-1)/2$ Hamilton cycle, and so $K_n[S]$ contains $(k-1)/2$ edge-disjoint Hamilton paths.

Therefore, K_n contains $\lfloor k/2 \rfloor$ edge-disjoint Hamilton paths. If $k < n$, then for $1 \le i \le n-k$, let $T_i' = K_{1,k}$ with vertex set $S \cup \{w_i\}$ whose center is w_i. As these $n-k$ stars and the $\lfloor k/2 \rfloor$ Hamilton paths described earlier are internally disjoint S-Steiner trees, $\kappa_k(K_n) \ge (n-k) + \lfloor k/2 \rfloor = n - \lceil k/2 \rceil$. Therefore, $\kappa_k(K_n) = n - \lceil k/2 \rceil$. □

In [90], Li, Mao, and Sun obtained the explicit value of $\lambda_k(K_n)$. One may not expect that it is the same as $\kappa_k(K_n)$.

Theorem 2.2.2 ([90]). *For every two integers n and k with $2 \le k \le n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

From Theorems 2.2.1 and 2.2.2, we get that $\lambda_k(G) = \kappa_k(G)$ for a complete graph G. However, this is a very special case. Actually, $\lambda_k(G) - \kappa_k(G)$ could be very large. For example, let G be a graph obtained from two copies of the complete graph K_n by identifying one vertex in each of them. For $k \le n$, $\lambda_k(G) = n - \lceil k/2 \rceil$, but $\kappa_k(G) = 1$.

Okamoto and Zhang [112] investigated the generalized k-connectivity of a balanced complete bipartite graph $K_{a,a}$. Naturally, one may ask whether we can compute the value of generalized k-connectivity of a complete bipartite graph $K_{a,b}$ or even a complete multipartite graph. Actually, Li, Li, and Li obtained the value of generalized k-connectivity of all complete bipartite graphs for $2 \le k \le a+b$.

Theorem 2.2.3 ([79]). *Let a, b, k be three positive integers such that $a \le b$ and $2 \le k \le a+b$. If $k > b-a+2$ and $a-b+k$ is odd, then*

$$\kappa_k(K_{a,b}) = \frac{a+b-k+1}{2} + \left\lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \right\rfloor.$$

If $k > b-a+2$ and $a-b+k$ is even, then

$$\kappa_k(K_{a,b}) = \frac{a+b-k}{2} + \left\lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \right\rfloor.$$

If $k \le b-a+2$, then $\kappa_k(K_{a,b}) = a$.

The proof of Theorem 2.2.3 is long and complicated, which uses the following seven lemmas, Lemmas 2.2.4 through 2.2.10. The proofs of them are omitted. Note that $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$. Actually, all vertices in X are equivalent and all vertices in Y are equivalent. So instead of considering all k-element subsets S of $V(G)$, we can restrict our attention to the k-element subsets $S_i = \{x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_{k-i}\}$ for $0 \le i \le k$. Notice that if $i > a$ or $k-i > b$, then S_i does not exist. So we need only to consider S_i for $\max\{0,\ k-b\} \le i \le \min\{a,\ k\}$.

Now, let $\mathscr{T}$ be a maximum set of internally disjoint S_i-Steiner trees. Let $\mathbb{T}_0$ be the set of S_i-Steiner trees whose vertex set is S_i; let $\mathbb{T}_1$ be the set of S_i-Steiner trees whose vertex set is $S_i \cup \{u\}$, where $u \notin S_i$; and let $\mathbb{T}_2$ be the set of S_i-Steiner trees whose vertex set is $S_i \cup \{u, v\}$, where $u, v \notin S_i$ and they belong to different partitions.

Lemma 2.2.4 ([79]). *Let $\mathscr{T}$ be a maximum set of internally disjoint S_i-Steiner trees. Then we can always find a set $\mathscr{T}'$ of internally disjoint S_i-Steiner trees such that $|\mathscr{T}| = |\mathscr{T}'|$ and $\mathscr{T}' \subset \mathbb{T}_0 \cup \mathbb{T}_1 \cup \mathbb{T}_2$.*

We can assume that the maximum set $\mathscr{T}$ of internally disjoint S_i-Steiner trees is contained in $\mathbb{T}_0 \cup \mathbb{T}_1 \cup \mathbb{T}_2$. We will define the standard structure of Steiner trees in $\mathbb{T}_0$, $\mathbb{T}_1$, and $\mathbb{T}_2$, respectively. Every tree in $\mathbb{T}_0$ is of standard structure. A Steiner tree T in $\mathbb{T}_1$ with vertex set $V(T) = S_i \cup \{u\}$, where $u \in X - S_i$, is of standard structure, if u is adjacent to every vertex in $S_i \cap Y$. Since $|E(T)| = |V(T)| - 1 = k$ and $d_T(u) = |S_i \cap Y| = k - i$, there remain i edges incident with $S_i \cap X$. We know that $|S_i \cap X| = i$ and each vertex must have degree at least 1 in T. So every vertex in $S_i \cap X$ has degree 1. A Steiner tree T in $\mathbb{T}_1$ with vertex set $V(T) = S_i \cup \{v\}$, where $v \in Y - S_i$, is of standard structure, if v is adjacent to every vertex in $S_i \cap X$. Similarly, every vertex in $S_i \cap Y$ has degree 1. A Steiner tree T in $\mathbb{T}_2$ with vertex set $V(T) = S_i \cup \{u, v\}$, where $u \in X - S_i$ and $v \in Y - S_i$, is of standard structure, if u is adjacent to every vertex in $S_i \cap Y$, v is adjacent to every vertex in $S_i \cap X$, and u is adjacent to v. We then denote the resulting tree T by $T_{u,v}$. Denote the set of trees in $\mathbb{T}_0$, $\mathbb{T}_1$, and $\mathbb{T}_2$ with the standard structure by $\mathfrak{T}_0$, $\mathfrak{T}_1$, and $\mathfrak{T}_2$, respectively. Clearly, $\mathfrak{T}_0 = \mathbb{T}_0$.

Lemma 2.2.5 ([79]). *Let $\mathscr{T}$ be a maximum set of internally disjoint S_i-Steiner trees, $\mathscr{T} \subset \mathbb{T}_0 \cup \mathbb{T}_1 \cup \mathbb{T}_2$. Then we can always find a set $\mathscr{T}''$ of internally disjoint S_i-Steiner trees, such that $|\mathscr{T}| = |\mathscr{T}''|$ and $\mathscr{T}'' \subset \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \mathfrak{T}_2$.*

So, we can assume that the maximum set $\mathscr{T}$ of internally disjoint S_i-Steiner trees is contained in $\mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \mathfrak{T}_2$. Namely, all Steiner trees in $\mathscr{T}$ are of standard structure. For simplicity, we denote the union of the vertex sets of all Steiner trees in set $\mathscr{T}$ by $V(\mathscr{T})$ and the union of the edge sets of all Steiner trees in set $\mathscr{T}$ by $E(\mathscr{T})$. Let $\mathscr{T}_0 := \mathscr{T} \cap \mathfrak{T}_0$, $\mathscr{T}_1 := \mathscr{T} \cap \mathfrak{T}_1$, and $\mathscr{T}_2 := \mathscr{T} \cap \mathfrak{T}_2$. Then $\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_1 \cup \mathscr{T}_2$.

Lemma 2.2.6. *Let $\mathscr{T} \subset \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \mathfrak{T}_2$ be a maximum set of internally disjoint S_i-Steiner trees. Then either $X \subseteq V(\mathscr{T})$ or $Y \subseteq V(\mathscr{T})$.*

We conclude that if $\mathscr{T}$ is a maximum set of internally disjoint S_i-Steiner trees, then $X \subseteq V(\mathscr{T})$ or $Y \subseteq V(\mathscr{T})$.

Lemma 2.2.7 ([79]). *Let $\mathscr{T} \subset \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \mathfrak{T}_2$ be a maximum set of internally disjoint S_i-Steiner trees and $\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_1 \cup \mathscr{T}_2$. If there is a vertex $x \in X - V(\mathscr{T})$ and a tree $T \in \mathscr{T}_1$ with vertex set $S_i \cup \{y\}$, where $y \in Y - S_i$, then we can find a maximum set $\mathscr{T}' = \mathscr{T}_0' \cup \mathscr{T}_1' \cup \mathscr{T}_2'$ of internally disjoint S_i-Steiner trees, such that $\mathscr{T}_0' = \mathscr{T}_0$, $|\mathscr{T}_1'| = |\mathscr{T}_1| - 1$, and $|\mathscr{T}_2'| = |\mathscr{T}_2| + 1$.*

The case that there is a vertex $y \in Y - V(\mathscr{T})$ and a tree $T \in \mathscr{T}_1$ with vertex set $S_i \cup \{x\}$, where $x \in X - S_i$, is similar. We will show that we can always find a maximum set $\mathscr{T}$ of internally disjoint S_i-Steiner trees such that all vertices in $V(\mathscr{T}_1) - S_i$ belong to the same partition.

Lemma 2.2.8 ([79]). *Let p, q be two nonnegative integers. If $p(k-1)+qi \leq i(k-i)$ and there are q vertices $u_1, u_2, \ldots, u_q \in X - S_i$, then we can always find p Steiner* *trees $T_1, T_2, \ldots, T_p$ in $\mathfrak{T}_0$ and q Steiner trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathfrak{T}_1$, such that $V(T_j) = S_i$ for $1 \leq j \leq p$, $V(T_{p+m}) = S_i \cup \{u_m\}$ for $1 \leq m \leq q$, and T_r, T_s are edge disjoint for $1 \leq r < s \leq p+q$. Similarly, if $p(k-1) + q(k-i) \leq i(k-i)$, and there are q vertices $v_1, v_2, \ldots, v_q \in Y - S_i$, then we can always find p Steiner trees $T_1, T_2, \ldots, T_p$ in $\mathfrak{T}_0$ and q Steiner trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathfrak{T}_1$, such that $V(T_j) = S_i$ for $1 \leq j \leq p$, $V(T_{p+m}) = S_i \cup \{v_m\}$ for $1 \leq m \leq q$, and T_r, T_s are edge disjoint for $1 \leq r < s \leq p+q$.*

Lemma 2.2.9 ([79]). *Let $\mathscr{T} \subset \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \mathfrak{T}_2$ be a maximum set of internally disjoint S_i-Steiner trees and $\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_1 \cup \mathscr{T}_2$. If there are s Steiner trees $T_1, T_2, \ldots, T_s \in \mathscr{T}_1$ with vertex set $S_i \cup \{u_1\}$, $S_i \cup \{u_2\}$, …, $S_i \cup \{u_s\}$, respectively, where $u_j \in X - S_i$ for $1 \leq j \leq s$, and t Steiner trees $T_{s+1}, T_{s+2}, \ldots, T_{s+t} \in \mathscr{T}_1$ with vertex set $S_i \cup \{v_1\}$, $S_i \cup \{v_2\}$, …, $S_i \cup \{v_t\}$, respectively, where $v_j \in Y - S_i$ for $1 \leq j \leq t$, then we can find a set $\mathscr{T}' = \mathscr{T}_0' \cup \mathscr{T}_1' \cup \mathscr{T}_2'$ of internally disjoint S_i-Steiner trees such that $|\mathscr{T}| = |\mathscr{T}'|$ and all vertices in $V(\mathscr{T}_1') - S_i$ belong to the same partition.*

From Lemmas 2.2.7 and 2.2.9, if $\mathscr{T}'$ is a set of internally disjoint S_i-Steiner trees which we find currently, $X - V(\mathscr{T}) \neq \emptyset$ and $Y - V(\mathscr{T}) \neq \emptyset$, then no matter how many edges there are in $E(K_{a,b}[S_i]) \setminus E(\mathscr{T}')$, we always add to $\mathscr{T}'$ the trees in $\mathfrak{T}_2$ rather than the trees in $\mathfrak{T}_1$ to form a larger set of internally disjoint S_i-Steiner trees.

Lemma 2.2.10 ([79]). *Let $\mathscr{T} \subset \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \mathfrak{T}_2$ be a maximum set of internally disjoint S_i-Steiner trees and $\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_1 \cup \mathscr{T}_2$. If $V(\mathscr{T}) \subset V(G)$ and $\mathscr{T}_0 \neq \emptyset$, then we can find a maximum set $\mathscr{T}' = \mathscr{T}_0' \cup \mathscr{T}_1' \cup \mathscr{T}_2'$ of internally disjoint S_i-Steiner trees such that $|\mathscr{T}_0'| = |\mathscr{T}_0| - 1$, $|\mathscr{T}_1'| = |\mathscr{T}_1| + 1$, and $\mathscr{T}_2' = \mathscr{T}_2$.*

We can assume that for the maximum set $\mathscr{T}$ of internally disjoint S_i-Steiner trees, either $V(\mathscr{T}) = V(G)$ or $\mathscr{T}_0 = \emptyset$. Moreover, if $\mathscr{T}'$ is a set of internally disjoint S_i-Steiner trees which we find currently, $V(\mathscr{T}') \subset V(G)$, and the edges in $E(K_{a,b}[S_i]) \setminus E(\mathscr{T}')$ can form a tree T in $\mathfrak{T}_0$, then we will add the tree T'' into $\mathscr{T}'$ in Lemma 2.2.10 rather than the tree T to form a larger set of internally disjoint S_i-Steiner trees.

Using the above seven lemmas, the proof of Theorem 2.2.3 then goes as follows:

Proof of Theorem 2.2.3. Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$. As we have mentioned, we can restrict our attention to the k-element subsets $S_i = \{x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_{k-i}\}$ for $\max\{0, k-b\} \leq i \leq \min\{a, k\}$.

From the above lemmas, we can determine our rule to find the maximum set of internally disjoint Steiner trees connecting S_i. Namely, we first find as many Steiner trees in $\mathfrak{T}_2$ as possible, next we find as many Steiner trees in $\mathfrak{T}_1$ as possible, and finally we find as many Steiner trees in $\mathfrak{T}_0$ as possible. Let $\mathscr{T}$ be the maximum set of internally disjoint S_i-Steiner trees we finally find. We now compute $|\mathscr{T}|$:

Case 1. $k \leq b - a + 2$.

Note that $\kappa(S_0) = a$. For S_1, since $k \leq b - a + 2$, it follows that $b - (k-1) = b - k + 1 \geq a - 2 + 1 = a - 1$, and hence $|\mathscr{T}_2| = a - 1$. If $b - k + 1 = a - 1$, then

$|\mathscr{T}_1| = 0$ and $|\mathscr{T}_0| = 1$. If $b-k+1 > a-1$, then $|\mathscr{T}_1| = 1$ and $|\mathscr{T}_0| = 0$. No matter which case happens, we have $\kappa(S_1) = |\mathscr{T}_2| + |\mathscr{T}_1| + |\mathscr{T}_0| = a$. For S_i $(i \geq 2)$, since $k \leq b-a+2$, it follows that $b-(k-i) = b-k+i \geq a-2+i > a-i$, and hence $|\mathscr{T}_2| = a-i$. Since $b-k+i-(a-i) = b-a-k+2i \geq -2+2i \geq i$, it follows that $|\mathscr{T}_1| = i$ and $|\mathscr{T}_0| = 0$. Thus $\kappa(S_i) = |\mathscr{T}_2| + |\mathscr{T}_1| + |\mathscr{T}_0| = a$. In summary, if $k \leq b-a+2$, then $\kappa_k(G) = a$.

Case 2. $k > b-a+2$.

First, let us compare $\kappa(S_i)$ with $\kappa(S_{k-i})$ for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. If $a = b$, then $\kappa(S_i) = \kappa(S_{k-i})$. So we may assume that $a < b$. For $i = 0$, $\kappa(S_0) = a < b = \kappa(S_k)$. For $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, we will give the expressions of $\kappa(S_i)$ and $\kappa(S_{k-i})$.

For S_i, since every pair of vertices $u \in X-S_i$ and $v \in Y-S_i$ can form a tree $T_{u,v}$, then $|\mathscr{T}_2| = \min\{a-i, b-(k-i)\}$, and hence

$$|\mathscr{T}_2| = \begin{cases} a-i & \text{if } i \geq \frac{a-b+k}{2} \\ b-k+i & \text{if } i < \frac{a-b+k}{2}. \end{cases}$$

Since every tree T in $\mathscr{T}_1$ has a vertex in $V(K_{a,b}) - (S_i \cup V(\mathscr{T}_2))$, we have

$$|\mathscr{T}_1| \leq \begin{cases} b-k+i-(a-i) & \text{if } i \geq \frac{a-b+k}{2} \\ a-i-(b-k+i) & \text{if } i < \frac{a-b+k}{2}. \end{cases}$$

On the other hand, if the tree T has vertex set $S_i \cup \{u\}$, where $u \in X-S_i$, then every vertex in $S_i \cap X$ is incident with one edge in $E(S_i)$, where $E(S_i)$ denotes the set of edges whose ends are both in S_i. And if the tree T has vertex set $S_i \cup \{v\}$, where $v \in Y-S_i$, then every vertex in $S_i \cap Y$ is incident with one edge in $E(S_i)$. Since every vertex in $S_i \cap X$ is incident with $k-i$ edges in $E(S_i)$ and every vertex in $S_i \cap Y$ is incident with i edges in $E(S_i)$, we have

$$|\mathscr{T}_1| \leq \begin{cases} i & \text{if } i \geq \frac{a-b+k}{2} \\ k-i & \text{if } i < \frac{a-b+k}{2}. \end{cases}$$

Combining the two inequalities, we get

$$|\mathscr{T}_1| = \begin{cases} \min\{b-a-k+2i, i\} & \text{if } i \geq \frac{a-b+k}{2} \\ \min\{a-b+k-2i, k-i\} & \text{if } i < \frac{a-b+k}{2}, \end{cases}$$

and hence

$$|\mathscr{T}_1| = \begin{cases} i & \text{if } i \geq a-b+k \\ b-a-k+2i & \text{if } \frac{a-b+k}{2} \leq i < a-b+k \\ a-b+k-2i & \text{if } i < \frac{a-b+k}{2}. \end{cases}$$

From Lemma 2.2.8, we have

$$|\mathcal{T}_0| = \begin{cases} \left\lfloor \frac{i(k-i)-|\mathcal{T}_1|(k-i)}{k-1} \right\rfloor & \text{if } \ i \geq \frac{a-b+k}{2} \\ \left\lfloor \frac{i(k-i)-|\mathcal{T}_1|i}{k-1} \right\rfloor & \text{if } \ i < \frac{a-b+k}{2}, \end{cases}$$

and hence

$$|\mathcal{T}_0| = \begin{cases} 0 & \text{if } \ i \geq a-b+k \\ \left\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \right\rfloor & \text{if } \ \frac{a-b+k}{2} \leq i < a-b+k \\ \left\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \right\rfloor & \text{if } \ i < \frac{a-b+k}{2}. \end{cases}$$

So we have

$$\kappa(S_i) = \begin{cases} a & \text{if } \ i \geq a-b+k \\ b-k+i+\left\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \right\rfloor & \text{if } \ \frac{a-b+k}{2} \leq i < a-b+k \\ a-i+\left\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \right\rfloor & \text{if } \ i < \frac{a-b+k}{2}. \end{cases}$$

Since $i \geq 1$, it follows that $k-i \leq k-1$. If $\frac{a-b+k}{2} \leq i < a-b+k$, then

$$\left\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \right\rfloor \leq i-(b-a-k+2i) = a-b+k-i,$$

and hence $\kappa(S_i) \leq b-k+i+a-b+k-i = a$. If $i < \frac{a-b+k}{2}$, then $a-b+k-2i > 0$ and $k-i-(a-b+k-2i) < k-i \leq k-1$, and hence

$$\left\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \right\rfloor \leq i.$$

So $\kappa(S_i) \leq a-i+i = a$. We conclude that $\kappa(S_i) \leq a$ for $i \geq 1$.

We now consider S_{k-i}. Similarly, we have $|\mathcal{T}_2| = \min\{a-(k-i), b-i\}$. Since $a < b$ and $i \leq \lfloor \frac{k}{2} \rfloor \leq \lceil \frac{k}{2} \rceil \leq k-i$, it follows that $b-i > a-(k-i)$, and hence $|\mathcal{T}_2| = a-k+i$ and $|\mathcal{T}_1| = \min\{b-i-(a-k+i), k-i\}$. Therefore, we have

$$|\mathcal{T}_1| = \begin{cases} k-i & \text{if } \ i \leq b-a \\ b-a+k-2i & \text{if } \ i > b-a. \end{cases}$$

Moreover,

$$|\mathcal{T}_0| = \begin{cases} 0 & \text{if } \ i \leq b-a \\ \left\lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \right\rfloor & \text{if } \ i > b-a. \end{cases}$$

So, we have

$$\kappa(S_{k-i}) = \begin{cases} a & \text{if } i \le b-a \\ b-i+\left\lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \right\rfloor & \text{if } i > b-a. \end{cases}$$

Now, we can compare $\kappa(S_i)$ with $\kappa(S_{k-i})$. For $i \le b-a$, $\kappa(S_{k-i}) = a \ge \kappa(S_i)$. For $i > b-a$, there must be $b-a < k-i$, that is, $i < a-b+k$. If $\frac{a-b+k}{2} \le i < a-b+k$, then

$$\begin{aligned}
&\kappa(S_{k-i}) - \kappa(S_i) \\
&= b-i+\left\lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \right\rfloor - \left\{ b-k+i+\left\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \right\rfloor \right\} \\
&\ge (k-2i)+\left\lfloor \frac{(k-2i)(b-a-k)}{k-1} \right\rfloor \ge (k-2i)+\left\lfloor \frac{(k-2i)(1-k)}{k-1} \right\rfloor = 0.
\end{aligned}$$

So, we have $\kappa(S_{k-i}) \ge \kappa(S_i)$. If $i < \frac{a-b+k}{2}$, then $\kappa(S_{k-i}) - \kappa(S_i) \ge (b-a) + \left\lfloor \frac{2i(a-b)}{k-1} \right\rfloor$. Since $i < \frac{a-b+k}{2}$, it follows that $2i \le k-1$, and hence $\frac{(2i)(a-b)}{k-1} \ge a-b$. So, we have $\kappa(S_{k-i}) - \kappa(S_i) \ge b-a+a-b = 0$, that is, $\kappa(S_{k-i}) \ge \kappa(S_i)$. We conclude that $\kappa(S_{k-i}) \ge \kappa(S_i)$ for $0 \le i \le \lfloor \frac{k}{2} \rfloor$. So, in order to get $\kappa_k(G)$, it is enough to consider $\kappa(S_i)$ for $0 \le i \le \lfloor k/2 \rfloor$.

Next, let us compare $\kappa(S_i)$ with $\kappa(S_{i+1})$ for $0 \le i \le \lfloor k/2 \rfloor - 1$. For $i = 0$, $\kappa(S_i) = a \ge \kappa(S_{i+1})$. For $1 \le i \le \lfloor k/2 \rfloor - 1$, we have

$$\kappa(S_i) = \begin{cases} a & \text{if } i \ge a-b+k \\ b-k+i+\left\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \right\rfloor & \text{if } \frac{a-b+k}{2} \le i < a-b+k \\ a-i+\left\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \right\rfloor & \text{if } i < \frac{a-b+k}{2}. \end{cases}$$

and

$$\kappa(S_{i+1}) = \begin{cases} a & \text{if } i \ge a-b+k-1 \\ b-k+i+1+\left\lfloor \frac{[i+1-(b-a-k+2i+2)](k-i-1)}{k-1} \right\rfloor & \text{if } \frac{a-b+k}{2}-1 \le i \\ & \text{and } i < a-b+k-1 \\ a-i-1+\left\lfloor \frac{[k-i-1-(a-b+k-2i-2)](i+1)}{k-1} \right\rfloor & \text{if } i < \frac{a-b+k}{2}-1. \end{cases}$$

So, $\kappa(S_{a-b+k}) = \kappa(S_{a-b+k+1}) = \cdots = \kappa(S_{\min\{a,k\}}) = a$. If $i < \frac{a-b+k}{2} - 1$, then

$$\kappa(S_i) - \kappa(S_{i+1}) \ge 1 + \left\lfloor \frac{a-b-2i-1}{k-1} \right\rfloor \ge 1 + \left\lfloor \frac{1-k}{k-1} \right\rfloor = 0.$$

So, $\kappa(S_i) \geq \kappa(S_{i+1})$. If $a-b+k$ is odd, then $\kappa(S_0) \geq \kappa(S_1) \geq \cdots \geq \kappa(S_{\frac{a-b+k-3}{2}}) \geq \kappa(S_{\frac{a-b+k-1}{2}})$. If $a-b+k$ is even, then $\kappa(S_0) \geq \kappa(S_1) \geq \cdots \geq \kappa(S_{\frac{a-b+k-4}{2}}) \geq \kappa(S_{\frac{a-b+k-2}{2}})$.

If $a-b+k$ is even, then $\kappa(S_{\frac{a-b+k}{2}-1}) = \frac{a+b-k}{2} + 1 + \lfloor\frac{(b-a+k-2)(a-b+k-2)}{4(k-1)}\rfloor$, and $\kappa(S_{\frac{a-b+k}{2}}) = \frac{a+b-k}{2} + \lfloor\frac{(b-a+k)(a-b+k)}{4(k-1)}\rfloor$. Since $(a-b+k)(b-a+k)-(b-a+k-2)(a-b+k-2) = (a-b+k)(b-a+k) - [(a-b+k)(b-a+k) - 2(b-a+k) - 2(a-b+k-2)] = 4(k-1)$, we have $\kappa(S_{\frac{a-b+k}{2}-1}) = \kappa(S_{\frac{a-b+k}{2}})$. If $a-b+k$ is odd, then $\kappa(S_{\frac{a-b+k-1}{2}}) = \frac{a+b-k+1}{2} + \lfloor\frac{(b-a+k-1)(a-b+k-1)}{4(k-1)}\rfloor = \kappa(S_{\frac{a-b+k+1}{2}})$.

If $\frac{a-b+k}{2} \leq i < a-b+k-1$, then

$$\kappa(S_{i+1}) - \kappa(S_i) \geq 1 + \left\lfloor \frac{b-a-2k+2i+1}{k-1} \right\rfloor \geq 1 + \left\lfloor \frac{1-k}{k-1} \right\rfloor = 0.$$

So, $\kappa(S_{i+1}) \geq \kappa(S_i)$. Namely, if $a-b+k$ is odd, we have $\kappa(S_{\frac{a-b+k+1}{2}}) \leq \kappa(S_{\frac{a-b+k+3}{2}}) \leq \cdots \leq \kappa(S_{a-b+k-1}) \leq a = \kappa(S_{a-b+k})$, and if $a-b+k$ is even, we have $\kappa(S_{\frac{a-b+k}{2}}) \leq \kappa(S_{\frac{a-b+k+2}{2}}) \leq \cdots \leq \kappa(S_{a-b+k-1}) \leq a = \kappa(S_{a-b+k})$.

Thus, if $k > b-a+2$ and $a-b+k$ is odd, then $\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k-1}{2}}) = \frac{a+b-k+1}{2} + \left\lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \right\rfloor$. If $k > b-a+2$ and $a-b+k$ is even, then $\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k}{2}}) = \frac{a+b-k}{2} + \left\lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \right\rfloor$. The proof is complete. □

For complete multipartite graphs, Li, Li, Li obtained a result only for complete equipartition 3-partite graphs.

Theorem 2.2.11 ([80]). *Let $b \geq 2$ be a positive integer. Then, for the complete equipartition 3-partite graph $K_3 \circ bK_1$, we have*

$$\kappa_k(K_3 \circ bK_1) = \begin{cases} \left\lfloor \frac{\lceil k^2/3 \rceil + k^2 - 2kb}{2(k-1)} \right\rfloor + 3b - k & \text{if } k \geq \frac{3b}{2} \\ \left\lfloor \frac{3bk+3b-k+1}{2k+1} \right\rfloor & \text{if } \frac{3b}{4} < k < \frac{3b}{2} \text{ and } k = 1 \pmod 3 \\ \left\lfloor \frac{3bk+6b-2k+1}{2k+2} \right\rfloor & \text{if } b \leq k < \frac{3b}{2} \text{ and } k = 2 \pmod 3 \\ \left\lfloor \frac{3b}{2} \right\rfloor & \text{if } k < \frac{3b}{2} \text{ and } k = 0 \pmod 3 \\ \left\lfloor \frac{3b+1}{2} \right\rfloor & \text{otherwise} \end{cases}$$

Let $U = \{u_1, u_2, \cdots, u_a\}$ and $V = \{v_1, v_2, \cdots, v_b\}$ be the two parts of a complete bipartite graph $K_{a,b}$. Set $S_i = \{u_1, u_2, \cdots, u_i, v_1, v_2, \cdots, v_{k-i}\}$ for $0 \leq i \leq k$. If $k > b-a+2$ and $a-b+k$ is odd, then $\kappa_k(K_{a,b}) = \kappa(S_{(a-b+k-1)/2})$, in the part X there are $\frac{a-b+k-1}{2}$ vertices not in S, and in the part Y there are $\frac{a-b+k-1}{2}$ vertices not in S. The number of vertices not in S of each part is almost the same. And if

$k > b - a + 2$ and $a - b + k$ is even, then $\kappa_k(K_{a,b}) = \kappa(S_{(a-b+k)/2})$, in the part X there are $\frac{a-b+k}{2}$ vertices not in S, and in the part Y there are $\frac{a+b-k}{2}$ vertices not in S. The number of vertices not in S of each part is the same.

Similarly, let $U = \{u_1, u_2, \cdots, u_b\}$, $V = \{v_1, v_2, \cdots, v_b\}$, and $W = \{w_1, w_2, \cdots, w_b\}$ be the three parts of a complete equipartition 3-partite graph $K_3 \circ bK_1$. Set $S_{r,s,t} = \{u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s, v_1, v_2, \cdots, v_t\}$ for $0 \le r, s, t \le k$ with $r + s + t = k$. If $k = 0 \pmod 3$, then $\kappa_k(K_3 \circ bK_1) = \kappa(S_{k/3,k/3,k/3})$, in the part U there are $b - k/3$ vertices not in S, in the part V there are $b - k/3$ vertices not in S, and in the part W there are $b - k/3$ vertices not in S. The number of vertices in each part but not in S is the same. If $k = 1 \pmod 3$, then $\kappa_k(K_3 \circ bK_1) = \kappa(S_{(k+2)/3,(k-1)/3,(k-1)/3})$. And if $k = 2 \pmod 3$, then $\kappa_k(K_3 \circ bK_1) = \kappa(S_{(k+1)/3,(k+1)/3,(k-2)/3})$. In both cases, the number of vertices in each part but not in S is almost the same.

So, for the complete equipartition a-partite graph $K_a \circ bK_1$, W. Li proposed the following two conjectures in her Ph.D. thesis [85].

Conjecture 2.2.12 ([85]). *For a complete equipartition a-partite graph G with partition $(X_1, X_2, \cdots, X_a)$ and integer $k = ab + c$, where b, c are integers and $0 \le c \le a - 1$, we have $\kappa_k(G) = \kappa(S)$, where S is a k-subset of $V(G)$ such that $|S \cap X_1| = \cdots = |S \cap X_c| = b + 1$ and $|S \cap X_{c+1}| = \cdots = |S \cap X_a| = b$.*

Conjecture 2.2.13 ([85]). *For a complete multipartite graph G, we have $\kappa_k(G) = \kappa(S)$, where S is a k-subset of $V(G)$ such that the number of vertices not in S of each part is almost the same.*

Let X be a finite Abelian group and its operation be called addition, denoted by $+$. Let A be a subset of $X \setminus \{0\}$ such that $a \in A$ implies $-a \in A$, where 0 is the identity element of X. The *Cayley graph* $Cay(X, A)$ is defined to have vertex set X such that there is an edge between x and y if and only if $x - y \in A$. A *circulant graph* is a Cayley graph on a cyclic group. Cayley graphs are important objects of study in algebraic graph theory; see [4, 11].

Sun and Zhou [131] studied the generalized connectivity of Cayley graphs.

Theorem 2.2.14 ([131]). *Let G be a cubic connected Cayley graph on an Abelian group with order $n \ge 8$. Then*

$$\kappa_k(G) = \lambda_k(G) = \begin{cases} 2 & \text{if } 3 \le k \le 6 \\ 1 & \text{if } 7 \le k \le n. \end{cases}$$

Theorem 2.2.15 ([131]). *Let G be a connected Cayley graph of degree 4 on an Abelian group with order $n \ge 3$. Then*

$$\kappa_k(G) = \begin{cases} 3 & \text{if } k = 3 \\ 1 \text{ or } 2 & \text{if } 8 \le k \le n-2 \\ 2 & \text{if } k = n-1, n \end{cases}$$

and

$$\lambda_k(G) = \begin{cases} 3 & \text{if } k = 3 \\ 2 \text{ or } 3 & \text{if } 4 \le k \le 7 \\ 2 & \text{otherwise.} \end{cases}$$

In [84], Li, Tu, and Yu considered Cayley graphs $Cay(X, S)$ when the group X is a permutation group. Denote by $Sym(n)$ the group of all permutations on $\{1, \dots, n\}$. Let $(p_1 p_2 \dots p_n)$ denote a permutation on $\{1, \cdots, n\}$ and (ij), which is called a *transposition*, denote the permutation that swaps the objects at positions i and j (not swapping element i and j), that is, $(p_1 \dots p_i \dots p_j \dots p_n)(ij) = (p_1 \dots p_j \dots p_i \dots p_n)$. Let $\mathscr{T}$ be a set of transpositions and $G(\mathscr{T})$ be the graph on n vertices $\{1, 2, \dots, n\}$ such that there is an edge ij in $G(\mathscr{T})$ if and only if the transposition $(ij) \in \mathscr{T}$. The graph $G(\mathscr{T})$ is called the *transposition generating graph* of $Cay(Sym(n), \mathscr{T})$.

Moreover, if $G(\mathscr{T})$ is a tree, we call $G(\mathscr{T})$ a *transposition tree* and denote $Cay(Sym(n), \mathscr{T})$ by Γ_n. Specially, if $G(\mathscr{T}) \cong K_{1,n-1}$, then $Cay(Sym(n), \mathscr{T})$ is called a *star graph* S_n, and $Cay(Sym(n), \mathscr{T})$ is called a *bubble-sort graph* B_n if $G(\mathscr{T}) \cong P_n$.

Li, Tu, and Yu [84] studied generalized 3-connectivity of the star graph S_n and the bubble-sort graph B_n and got the following result.

Theorem 2.2.16 ([84]).

(1) *For any* $n \ge 3$, $\kappa_3(S_n) = n - 2$.
(2) *For any* $n \ge 3$, $\kappa_3(B_n) = n - 2$.

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