Chapter 2
Output–Input Ratio Efficiency Measures

Ever since the industrial revolution, people have been working to use the smallest effort to produce the largest output, so that resources, including human, are utilized more efficiently. Manufacturing companies develop standards to help achieve this, such as the number of items that should be produced with one unit of a certain type of input, in order to better control the production process and increase productivity. Similarly, service companies aim to increase the number of customers served by one employee in a unit of time.

Productivity is generally defined as the amount of output produced by one unit of input. Theoretically, there is a maximum productivity which can be achieved only under perfect conditions. The productivity of a production unit divided by the maximum productivity is the efficiency of this particular unit. In this context, efficiency is always less than or equal to unity.

Consider a production activity that applies multiple inputs to produce multiple outputs. Let $X_{ij},\ i = 1,\ \ldots,\ m$, be the quantity of input i employed by unit j in a period of time, and $Y_{rj},\ \ r = 1,\ \ldots,\ s$, be the quantity of output r produced in the same period. Productivity is thus ratio of the aggregate output to the aggregate input, expressed as:

$$P_j = \frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{m} v_i X_{ij}} \tag{2.1}$$

where P_j is the productivity of unit j, and u_r and v_i are the conversion factors (or weights) of output r and input i, respectively (Bitran and Chang 1984). The key point in this measure is the determination of the weights, so that all the inputs and outputs are restated in their common denominations. Prices are commonly used in such calculations, such that the productivity indicates the amount of money that can be generated from each dollar consumed. This formula is quite simple and easy to calculate. However, the problem is that in many cases some inputs and outputs do not have market values, which makes the aggregation of the inputs and outputs

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difficult. For example, in calculating the productivity of a forest, it is difficult to determine the monetary values of soil conservation, carbon dioxide absorption, and wildlife habitation. For this reason, the measurement of productivity was for a long time limited to the ratio of one output to one input when there is the problem of commensurability.

2.1 CCR Model

Charnes et al. (1978) proposed a fractional programming model, commonly referred to as the CCR model, in which the problem of non-commensurability was solved. The idea is to allow the focal production unit (generalized as the decision making unit, DMU, in their study) to select the most favorable weights (or multipliers) u_r and v_i to calculate the productivity ratio. The only restriction is that the productivity ratios of all DMUs calculated from the multipliers selected by this DMU must be less than or equal to one. Since this ratio is between zero and one, and it can be shown that the ratio is equal to the actual output of a DMU to the maximum output that can be produced with the same amount of input of this DMU, it is a measure of efficiency. It is also a relative measure when the maximum output is obtained from a sample.

2.1.1 Input Model

The CCR model for measuring the relative efficiency of a DMU indexed by 0 is:

$$
\begin{aligned}
E_0 = \max.\ & \frac{\sum_{r=1}^{s} u_r Y_{r0}}{\sum_{i=1}^{m} v_i X_{i0}} \\
\text{s.t.}\ & \frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{m} v_i X_{ij}} \le 1, \qquad j = 1, \ldots, n \\
& u_r, v_i \ge \varepsilon, \qquad r = 1, \ldots, s, \quad i = 1, \ldots, m
\end{aligned} \tag{2.2}
$$

Note that the multipliers u_r and v_i are required to be greater than a small positive number ε, to avoid some unfavorable factors being ignored by assigning zero to the corresponding multipliers (Charnes et al. 1979). This small number ε is called a non-Archimedean number (Charnes and Cooper 1984). If $E_0 = 1$, then this DMU is in a state of Pareto optimality, also called Pareto efficiency (Koopmans 1951). Originally, Pareto efficiency referred to a state that augments the value of one variable necessarily reduces the value of another. Koopmans (1951) extended it to productive efficiency to refer to a state that an improvement in any factor, i.e., an increase in an output or a decrease in an input, requires a deterioration of at least one other factor, i.e., a decrease in at least one output or an increase in at least one input.

It thus is also called Pareto-Koopmans efficiency (Charnes and Cooper 1961). If the lower bound ε is removed, and one still has $E_0 = 1$, then this DMU is only weakly efficient (Charnes et al. 1986, 1991), because in this case one can reduce the amount of some input or increase the amount of some output, and still have $E_0 = 1$. In contrast to the weakly efficient condition, the normal case of $E_0 = 1$, with the lower bound ε imposed upon the multipliers, is called strongly efficient.

Model (2.2) is commonly called a ratio model. It is a linear fractional program, which, based on the ideas set out in Charnes and Cooper (1962), can be transformed into the following linear model:

$$\begin{aligned} E_0 = \max.\quad & \sum_{r=1}^{s} u_r Y_{r0} \\ \text{s.t.}\quad & \sum_{i=1}^{m} v_i X_{i0} = 1 \\ & \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \le 0, \qquad j = 1, \ldots, n \\ & u_r, v_i \ge \varepsilon,\ r = 1, \ldots, s, \quad i = 1, \ldots, m \end{aligned} \tag{2.3}$$

This model is called a multiplier model. When there is only one input and one output this model uses a straight line, passing through the origin and superimposing upon all DMUs, as the production frontier. Since the production frontier passes through the origin, this implies that a proportional change in the input leads to the same proportional change in the output. One thus has a situation of constant returns to scale.

Model (2.3) has a dual, which can be formulated as:

$$\begin{aligned} E_0 = \min.\quad & \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\ \text{s.t.}\quad & \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = \theta X_{i0}, \qquad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = Y_{r0}, \qquad r = 1, \ldots, s \\ & \lambda_j,\ s_i^-,\ s_r^+ \ge 0,\ j = 1, \ldots, n,\ i = 1, \ldots, m,\ r = 1, \ldots, s \\ & \theta \text{ unrestricted in sign.} \end{aligned} \tag{2.4}$$

Since $s_i^-,\ s_r^+ \ge 0$, the first two sets of constraints imply $\sum_{j=1}^{n} \lambda_j X_{ij} \le \theta X_{i0}$ and $\sum_{j=1}^{n} \lambda_j Y_{rj} \ge Y_{r0}$, which indicate that all observations have a larger amount of inputs and smaller amount of outputs than the point $\left(\sum_{j=1}^{n} \lambda_j X_{ij},\ \sum_{j=1}^{n} \lambda_j Y_{rj}\right)$ on the production frontier. In other words, the observations are enveloped by the production frontier. Model (2.4) is thus called an envelopment model. Moreover, the second constraint set indicates that the DMU fixes its outputs at the current level of Y_{r0} (or an adjusted amount of $Y_{r0} + s_r^+$, to be exact) to look for the reduction ratio θ that the amount of inputs can be reduced. Model (2.4), or Model (2.2), is thus an input model. Another point to be noted is that although mathematically θ is not restricted to be positive, it will always be a positive number, less than or equal to one, due to the basic properties of the problem (as the smallest reduction ratio is zero).

Consider five DMUs, *A*, *B*, *C*, *D*, and *E*, each applying input *X* to produce output *Y*, with the data shown in Table 2.1, and as depicted in Fig. 2.1. Ray *OR*, lying above all observations with the smallest slope, is the production frontier constructed from these DMUs. The efficiency of each DMU is the ratio of the minimum input needed to produce the same amount of output of this DMU (on the production frontier) to the actual amount of input used by this DMU. For example, the efficiency of DMU *C* is $X_U/X_C = 0.6$. By applying Model (2.2),

Table 2.1 CCR efficiency of an example with one input and one output

DMU	Input *X*	Output *Y*	Efficiency
A	3	2	2/3
B	4	4	1
C	5	3	0.6
D	6	6	1
E	10	8	0.8

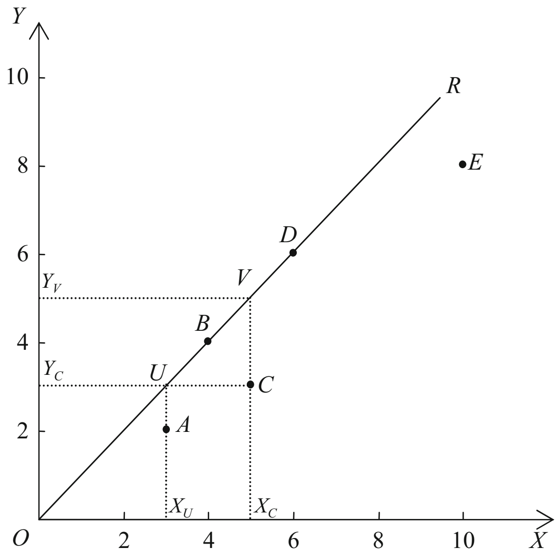


Fig. 2.1 CCR efficiency with one input and one output

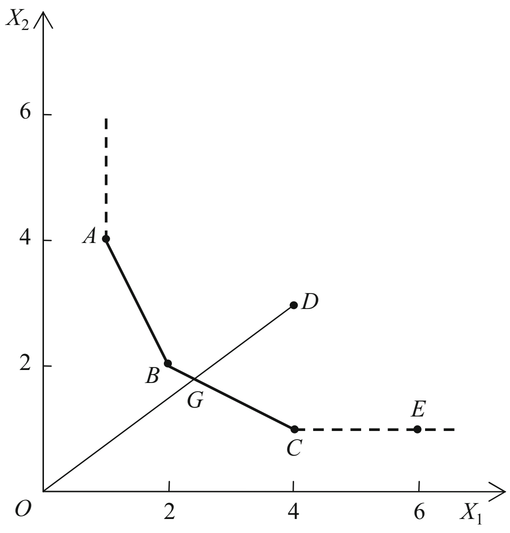
the efficiency of all DMUs can be calculated, with the results shown in the last column of Table 2.1.

Consider another case, where five DMUs, A, B, C, D, and E, applying different combinations of two inputs X_1 and X_2 to produce one unit of output Y, with the data shown in Table 2.2. Figure 2.2 shows the isoquant of $Y = 1$, constructed from these DMUs, where DMUs A, B, and C are efficient, as they lie on the isoquant. DMU D has an efficiency score of $OG/OD = 0.6$. DMU E lies on the part of the isoquant extended horizontally from DMU C, which indicates that it is weakly efficient. This is a situation in which by assigning zero to multiplier v_1 in Model (2.2), one obtains $E_E = 1$. As a matter of fact, DMU E is dominated by DMU C, as it consumes two more units of X_1 than DMU C to produce the same amount of Y. The associated optimal solution is $u^* = 1 - 2\varepsilon$, $v_1^* = \varepsilon$ and $v_2^* = 1 - 6\varepsilon$.

Table 2.2 CCR efficiency of an example with two inputs and one output

DMU	Input X_1	Input X_2	Output Y	Efficiency
A	1	4	1	1
B	2	2	1	1
C	4	1	1	1
D	4	3	1	0.6
E	6	1	1	$1 - 2\varepsilon$

Fig. 2.2 CCR efficiency measured from the isoquant



2.1.2 Output Model

Efficiency can also be measured from the output side. The CCR efficiency in this case is represented in the reciprocal form of $1/E_0$. The full output model is:

$$\begin{aligned} \frac{1}{E_0} = \min. \quad & \frac{\sum_{i=1}^{m} v_i X_{i0}}{\sum_{r=1}^{s} u_r Y_{r0}} \\ \text{s.t.} \quad & \frac{\sum_{i=1}^{m} v_i X_{ij}}{\sum_{r=1}^{s} u_r Y_{rj}} \geq 1, \quad j = 1, \ \ldots, \ n \\ & u_r, v_i \geq \varepsilon, \qquad r = 1, \ \ldots, \ s, \ i = 1, \ \ldots, \ m. \end{aligned} \tag{2.5}$$

This model is exactly the same as Model (2.2), except that the objective function is represented in reciprocal form. Its linear transformation is:

$$\begin{aligned} \frac{1}{E_0} = \min. \quad & \sum_{i=1}^{m} v_i X_{i0} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r Y_{r0} = 1 \\ & \sum_{i=1}^{m} v_i X_{ij} - \sum_{r=1}^{s} u_r Y_{rj} \geq 0, \qquad j = 1, \ \ldots, \ n \\ & u_r, v_i \geq \varepsilon, \ \ r = 1, \ \ldots, \ s, \quad i = 1, \ \ldots, \ m, \end{aligned} \tag{2.6}$$

and the corresponding envelopment model, which is the dual of Model (2.6), is:

$$\begin{aligned} \frac{1}{E_0} = \max. \quad & \varphi + \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = X_{i0}, \qquad i = 1, \ \ldots, \ m \\ & \sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = \varphi Y_{r0}, \quad r = 1, \ldots, s \\ & \lambda_j, \ s_i^-, \ s_r^+ \geq 0, \ j = 1, \ \ldots, \ n, \ i = 1, \ \ldots, \ m, \ r = 1, \ \ldots, \ s \\ & \varphi \text{ unrestricted in sign.} \end{aligned} \tag{2.7}$$

The constraints of this model indicate that this model fixes the inputs at the current level of X_{i0} (or the adjusted amount of $X_{i0} - s_i^-$, to be exact), and looks for the largest extent φ that the outputs can be expanded. For this reason, this model is an output model.

Referring to the DMUs in Fig. 2.1, the output model measures the efficiency of DMU C by using point V on the production frontier as the benchmark, which

Table 2.3 CCR efficiency of an example with one input and two outputs

DMU	Input X	Output Y_1	Output Y_2	Efficiency
A	1	2	3	2/3
B	1	2	5	1
C	1	4	1	$1/(1.25+0.75\varepsilon)$
D	1	4	4	1
E	1	5	2	1

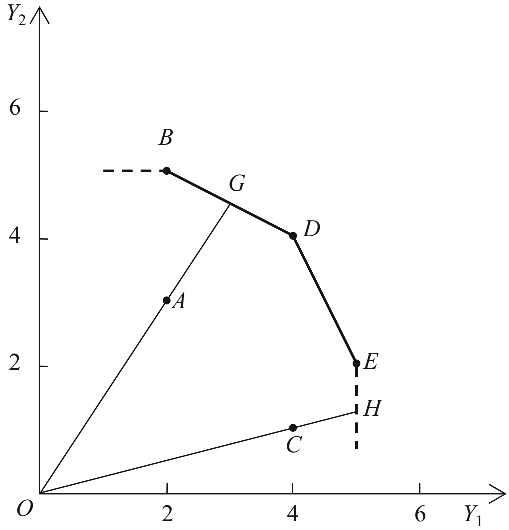


Fig. 2.3 CCR efficiency measured from the product transformation curve

applies the same amount of input as DMU C does to produce the largest possible amount of output. Geometrically, the inverse of the efficiency, as calculated from Model (2.6), is the ratio of Y_V to Y_C. Under constant returns to scale, which is reflected by the linear production frontier passing through the origin, the efficiency measured from the input model, X_U/X_C, is the same as that measured from the output model, Y_C/Y_V, although the benchmarks selected by the input and output models are different.

To discuss efficiency measurement from the output side in higher dimensions, consider five DMUs, A, B, C, D, and E, each applying one unit of input X to produce different amounts of two outputs, Y_1 and Y_2, with the data shown in Table 2.3. The product transformation curve constructed from these DMUs is depicted in Fig. 2.3, where DMUs B, D, and E, lying on the product transformation curve, are efficient. This figure shows that DMU A uses point G, on the product transformation curve, as the benchmark to measure its efficiency, with $E_A = OA/OG = 2/3$. DMU C uses point H, on the segment of the product transformation curve extended almost vertically from DMU E, to measure efficiency, to get an efficiency score of $OC/OH = 1/(1.25 + 0.75\varepsilon)$. The optimal solution is $v^* = 1.25 + 0.75\varepsilon$, $u_1^* = 0.25(1 - \varepsilon)$, $u_2^* = \varepsilon$. Note that 0.75ε in the denominator is caused by the very small scale of the vertical line tilting to the right from point E, as required by $u_2 \geq \varepsilon$.

When a set of DMUs uses the same amount of input to produce different amounts of output, it is obvious that the one with the largest output has a relative efficiency equal to one, and all others have a relative efficiency equal to the ratio of their amount of output to the largest amount. The efficiency measured from the CCR model satisfies this definition. Consider a situation of n DMUs, all applying the same amount of m inputs, $X_{ik} = X_{ij}, j = 1, \ldots, n, i = 1, \ldots, m$, to produce different amounts of one output Y. From Fig. 2.1 it is clear that the DMU with the largest amount of output, $Y_{\max}$, must lie on the production frontier. Let $\left(u^*, v_i^*, i = 1, \ldots, m\right)$ be the optimal solution obtained from Model (2.2) in calculating the efficiency of a DMU. For the DMU with the largest amount of output $Y_{\max}$, one has $u^* Y_{\max} = \sum_{i=1}^m v_i^* X_{i\max} = \sum_{i=1}^m v_i^* X_{i0}$. The efficiency of the DMU being evaluated can be expressed as: $E_0 = u^* Y_0 / \sum_{i=1}^m v_i^* X_{i0} = u^* Y_0 / u^* Y_{\max} = Y_0 / Y_{\max}$, as expected.

Consider another situation, where all DMUs apply different amounts of one input X to produce the same amount of s outputs, i.e., $Y_{r0} = Y_{rj}$, $j = 1, \ldots, n, r = 1, \ldots, s$. The efficiency of the DMU being evaluated, from the input point of view, is the ratio of the minimum input level, $X_{\min}$, divided by the actual amount of input consumed: $X_{\min}/X_0$. Let $\left(u_r^*, r = 1, \ldots, s, v^*\right)$ be the optimal solution obtained from Model (2.2) in calculating the efficiency of a DMU. By the same token, one has $\sum_{r=1}^s u_r^* Y_{r0} = \sum_{r=1}^s u_r^* Y_{r\min} = v^* X_{\min}$. The efficiency of the DMU being evaluated can then be expressed as: $E_0 = \sum_{r=1}^s u_r^* Y_{r0} / v^* X_0 = v^* X_{\min} / v^* X_0 = X_{\min} / X_0$, as expected. This verifies that the efficiency measured from the CCR model follows the conventional definition of relative efficiency.

2.2 BCC Model

The CCR model assumes constant returns to scale, in that the output increases in the same proportion as the input. For the case of a single input and single output, the production frontier is a straight line passing through the origin. In production economics, due to the effect of fixed inputs, returns to scale usually increase in the early stage of production, where the amount of variable input is relatively small. As the amount of variable input increases, returns to scale diminish to constant, and finally become decreasing. Taking this phenomenon into consideration, Banker et al. (1984) extended the CCR model to allow for variable returns to scale, referred to as the BCC model. Conceptually, they allow the production frontier to move away from the origin by introducing a constant in aggregating either the inputs or outputs. The constant plays the role of the intercept in the linear production frontier. This model also has two forms, input and output.

2.2.1 Input Model

The idea of the output-input ratio efficiency measure is to aggregate the outputs into a virtual output and the inputs into a virtual input, and to take their ratio to be the measure of efficiency. In the input model the outputs are treated as the explanatory variables to calculate the expected (minimum) virtual input. The ratio of the minimum virtual input to the actual virtual input, aggregated from the multiple inputs, is the input efficiency. The model developed by Banker et al. (1984) to measure the efficiency from the input side is:

$$\begin{aligned} E_0 = \max.\quad & \frac{\sum_{r=1}^{s} u_r Y_{r0} - u_0}{\sum_{i=1}^{m} v_i X_{i0}} \\ \text{s.t.}\quad & \frac{\sum_{r=1}^{s} u_r Y_{rj} - u_0}{\sum_{i=1}^{m} v_i X_{ij}} \leq 1, \qquad j = 1, \ldots, n \\ & u_r,\ v_i \geq \varepsilon, \quad r = 1, \ldots, s, \quad i = 1, \ldots, m \\ & u_0 \text{ unrestricted in sign.} \end{aligned} \tag{2.8}$$

The difference between this model and Model (2.2), the one under constant returns to scale, is the inclusion of the intercept u_0.

The linear fractional objective function in Model (2.8) can be linearized by assigning the denominator to one, and leaving the numerator as the objective function. This is because Model (2.8) has multiple solutions, in that if $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is an optimal solution, then so is $(c\boldsymbol{u}^*, c\boldsymbol{v}^*)$, for $c > 0$. Assigning the denominator to one to reduce one degree of freedom will thus not alter the optimal objective value, E_0, although the optimal solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ may be different. The linear fractional constraints are easily linearized by multiplying both sides by the denominator to obtain the following linear programming model:

$$\begin{aligned} E_0 = \max.\quad & \sum_{r=1}^{s} u_r Y_{r0} - u_0 \\ \text{s.t.}\quad & \sum_{i=1}^{m} v_i X_{i0} = 1 \\ & \sum_{r=1}^{s} u_r Y_{rj} - u_0 - \sum_{i=1}^{m} v_i X_{ij} \leq 0, \qquad j = 1, \ldots, n \\ & u_r,\ v_i \geq \varepsilon, \quad r = 1, \ldots, s, \qquad i = 1, \ldots, m \\ & u_0 \text{ unrestricted in sign.} \end{aligned} \tag{2.9}$$

This linear model can also be obtained by applying the idea in Charnes and Cooper (1962) for transforming a linear fractional program into a linear program.

Table 2.4 Technical and scale efficiencies of an example with one input and one output

DMU	Input X	Output Y	CCR Efficiency	BCC Input Efficiency		BCC Output Efficiency	
				Technical	Scale	Technical	Scale
A	3	2	2/3	1	2/3	1	2/3
B	4	4	1	1	1	1	1
C	5	3	0.6	0.7	6/7	0.6	1
D	6	6	1	1	1	1	1
E	10	8	0.8	1	0.8	1	0.8

Model (2.9) has a dual, which can be formulated as:

$$
\begin{aligned}
E_0 = \min. \quad & \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = \theta X_{i0}, \qquad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = Y_{r0}, \qquad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j,\ s_i^-,\ s_r^+ \geq 0,\ j = 1, \ldots, n, \quad i = 1, \ldots, m,\ r = 1, \ldots, s. \\
& \theta \text{ unrestricted in sign.}
\end{aligned}
\tag{2.10}
$$

This model is almost the same as Model (2.4), the one under the assumption of constant returns to scale, except that a convexity constraint of $\sum_{j=1}^{n} \lambda_j = 1$ is added.

To see the difference between the BCC model and the CCR model in measuring efficiencies from the input side, consider the example in Table 2.1, where five DMUs apply one input X to produce one output Y. The first four columns of Table 2.4 are copied from Table 2.1 for easy comparison, and the positions of the five DMUs in the X-Y plane are re-drawn in Fig. 2.4 for detailed explanation. By applying Model (2.8), one obtains the efficiencies of the five DMUs, as shown in the fifth column of Table 2.4, where DMUs A, B, D, and E are efficient, and only DMU C is inefficient. This implies that the production frontier constructed from these DMUs by this model is composed of the piecewise line segments $ABDE$, as shown in Fig. 2.4. Since this problem has only one input, we can divide the multipliers in the numerator by that in the denominator to obtain the following linear program and have a clearer geometric interpretation:

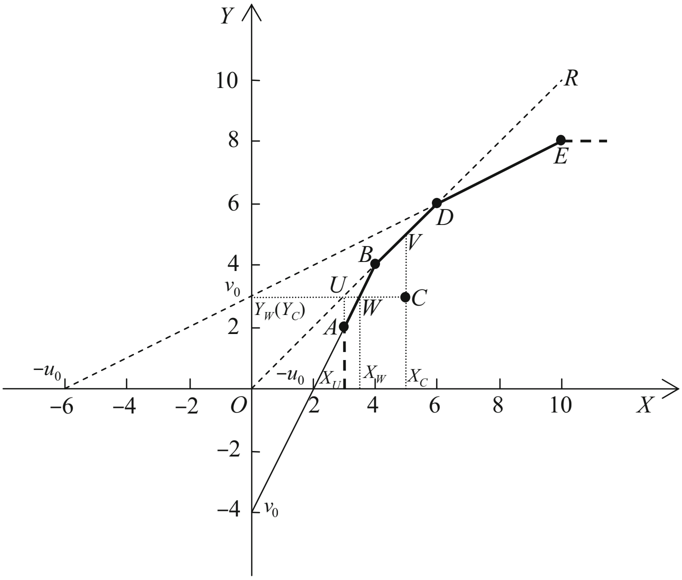


Fig. 2.4 Efficiency measurement under variable returns to scale

$$
\begin{aligned}
E_0 = \max. \quad & (u_1 Y_0 - u_0)/X_0 \\
\text{s.t.} \quad & \left(u_1 Y_j - u_0\right)/X_j \le 1, j = A,\ B,\ C,\ D,\ E \\
& u_1 \ge \varepsilon,\ u_0 \text{ unrestricted in sign.}
\end{aligned}
\tag{2.11}
$$

The multiplier u_1 in this case is the slope of the line segment corresponding to the DMU being evaluated by considering Y as the horizontal axis, and u_0 is the intercept of the line segment extending to the X-axis. For example, in calculating the efficiency of DMU A, one obtains $u_1^* = 0.5$ and $-u_0^* = 2$, corresponding to the line segment AB. From Fig. 2.4 it is noted that this is not the unique solution, and $\left(u_1^*,\ -u_0^*\right) = (\varepsilon,\ 3 - 2\varepsilon)$, which represents a vertical line extended from DMU A, is obviously another one. As a matter of fact, any line passing through DMU A with a slope u_1 between ε and 0.5, and the corresponding intercept of $-u_0 = 3 - 2u_1$, can be the frontier. By the same token, when Model (2.11) is applied to measure the efficiency of DMUs B, D, and E, one also obtains alternative solutions. Based on Fig. 2.4, the alternative solutions for DMU B are $0.5 \le u_1^* \le 1$, with $-u_0^* = 4 - 4u_1^*$, for DMU D they are $1 \le u_1^* \le 2$, with $-u_0^* = 6 - 6u_1^*$, and for DMU E they are $2 \le u_1^* < \infty$, with $-u_0^* = 10 - 8u_1^*$.

In calculating the efficiency of DMU C by applying Model (2.11), the optimal solution is unique, with $\left(u_1^*,\ -u_0^*\right) = (0.5,\ 2)$, in that the line segment AB is the frontier, and point W is the benchmark, that DMU C uses when measuring efficiency. The efficiency is $X_W/X_C = 0.7$. Recall that the frontier used to calculate efficiency under constant returns to scale is the ray OR, with the CCR efficiency of 0.6. From Fig. 2.4 it is clear that for the whole production frontier the region under the line segment *BD* (in terms of the output) has the most productive scale, in that the average input consumed per unit of output is the smallest. For scales (in terms of output) smaller than that of DMU *B* or greater than that of DMU *D*, the average amount of input used to produce one unit of output is larger, and the larger amount of input needed in these two regions is due to inadequate scales. The extra amount of input required relative to the minimum amount is the inefficiency due to improper scale. For example, point *W*, with a BCC efficiency one, is technically efficient; however, it is not efficient from the viewpoint of scale, because under the most productive scale an input level of X_U is enough to produce the output level Y_W. X_U/X_W is thus the scale efficiency of *W*. Since DMU *C* has the same output level of *W*, it has the same scale efficiency as *W* has, X_U/X_W. Different from *W*, DMU *C* is technically inefficient, because it requires X_C units of input, rather than X_W, to produce the same amount of output Y_C, and X_C/X_W is its technical efficiency. By generalizing this idea to multiple-input and multiple-output cases, one has the BCC efficiency as the technical efficiency, and the ratio of the CCR efficiency to the BCC efficiency as the scale efficiency for the DMU being evaluated. Column six of Table 2.4 shows the scale efficiencies of the five DMUs.

In higher dimensions the optimal solution ($\boldsymbol{u}^*$, $\boldsymbol{v}^*$) for a technically efficient DMU has $\sum_{r=1}^{s} u_r^* y_r - u_0^* - \sum_{i=1}^{m} v_i^* x_i = 0$, where x_i, y_r are now variables, which is a supporting hyperplane. Banker et al. (1984) showed that when the solution is unique, negative, zero, and positive values of u_0^* indicate that the associated DMU is in the regions of increasing, constant, and decreasing returns to scale, respectively. This is also seen in the graphical interpretation shown in Fig. 2.4. Banker and Thrall (1992) presented a proof for this when the focal DMU is technically efficient, and Banker et al. (1996) later removed this condition.

2.2.2 Output Model

In contrast to the input model, where the minimum amount of inputs needed to produce the specified output levels is obtained to measure efficiency, the output model looks for the maximum amount of outputs that can be produced from the given amount of inputs to measure efficiency. Based on this concept, the output BCC model for measuring the efficiency of a DMU has the following form:

$$\begin{aligned} \frac{1}{E_0} = \min. \quad & \frac{\sum_{i=1}^{m} v_i X_{i0} + v_0}{\sum_{r=1}^{s} u_r Y_{r0}} \\ \text{s.t.} \quad & \frac{\sum_{i=1}^{m} v_i X_{ij} + v_0}{\sum_{r=1}^{s} u_r Y_{rj}} \geq 1, \qquad j = 1, \ldots, n \\ & u_r, \ v_i \geq \varepsilon, \quad r = 1, \ldots, s, \quad i = 1, \ldots, m \\ & v_0 \text{ unrestricted in sign.} \end{aligned} \tag{2.12}$$

Similar to the input model, the numerator of the fractional in Model (2.12) calculates the maximal virtual output that can be produced from the actual amount of inputs for the DMU being evaluated, and the denominator calculates the virtual output aggregated from the actual amount of outputs. The ratio is then the reciprocal of the efficiency, from the output point of view.

The equivalent linear program to the ratio model (2.12) is:

$$\begin{aligned} \frac{1}{E_0} = \text{min.} \quad & \sum_{i=1}^{m} v_i X_{i0} + v_0 \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r Y_{r0} = 1 \\ & \sum_{i=1}^{m} v_i X_{ij} + v_0 - \sum_{r=1}^{s} u_r Y_{rj} \geq 0, \quad j = 1, \ldots, n \\ & u_r,\ v_i \geq \varepsilon, r = 1, \ldots, s, \qquad i = 1, \ldots, m \\ & v_0 \text{ unrestricted in sign,} \end{aligned} \tag{2.13}$$

and the corresponding envelopment model, which is the dual of Model (2.13), is:

$$\begin{aligned} \frac{1}{E_0} = \text{max.} \quad & \varphi + \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = X_{i0}, \qquad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = \varphi Y_{r0}, \quad r = 1, \ldots, s. \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j,\ s_i^-,\ s_r^+ \geq 0, j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \\ & \varphi \text{ unrestricted in sign} \end{aligned} \tag{2.14}$$

This envelopment model also has an assumption of variable returns to scale, and the difference of this from that under constant returns to scale, i.e., Model (2.7), is the convexity constraint of $\sum_{j=1}^{n} \lambda_j = 1$.

Model (2.12) can also be used to discuss the region of returns to scale, where the DMU being evaluated is located. Consider the example in Fig. 2.4. The production frontier constructed from the five DMUs by Model (2.12) is the same set of line segments *ABDE* as that constructed by Model (2.8). Similar to the discussion in the input model of Sect. 2.2.1, where the only multiplier v in the denominator can be absorbed by the multipliers u_1 and u_0 in the numerator, Model (2.12) can be simplified as follows by merging the only multiplier u in the denominator into v_1 and v_0 in the numerator:

$$
\begin{aligned}
\frac{1}{E_0} = \text{min.} \quad & (v_1 X_k + v_0)/Y_0 \\
\text{s.t.} \quad & \left(v_1 X_j + v_0\right)/Y_j \geq 1,\ j = A,\ B,\ C,\ D,\ E \\
& v_1 \geq \varepsilon,\ v_0 \,\text{unrestricted in sign},
\end{aligned}
\tag{2.15}
$$

where v_1 is the slope, and v_0 is the intercept, of the linear frontier that the DMU being evaluated uses to measure efficiency. For example, to measure the efficiency of DMU A, one solution is $\left(v_1^*,\ v_0^*\right) = (2,\ -4)$, which corresponds to the line segment AB. In measuring the efficiencies of DMUs B and D, one solution is $\left(v_1^*,\ v_0^*\right) = (1,\ 0)\big)$, which corresponds to the line segment BD. Finally, in calculating the efficiency of DMU E, one solution is $\left(v_1^*,\ v_0^*\right) = (0.5,\ 3)$. As discussed in the input model, for the whole production frontier, the region of constant returns to scale, i.e., line segment BD, with $v_0 = 0$, has the largest average return (amount of output per unit input), and the marginal return (the additional amount of output produced by an additional unit of input) is equal to the average return. In the region of smaller scales (in terms of input), with $v_0 < 0$, the average return is smaller than that of the region of constant returns to scale, although the marginal return of the former is larger than that of the latter. In contrast, in the region of larger scales, with $v_0 > 0$, both the average and marginal returns are smaller than those of the region of constant returns to scale. The intercept v_0 serves as an indicator of the type of returns to scale of the DMU.

DMU A lies on the production frontier, and it is thus technically efficient. However, if it is measured by the CCR model, then it is not efficient, with a CCR efficiency of 2/3, based on the benchmark U. This inefficiency is obviously due to an inadequate scale. The BCC model (2.12) thus measures the technical efficiency, while the CCR model (2.5) measures the overall efficiency, and the ratio of the CCR efficiency to the BCC efficiency is the scale efficiency. It is worth noting that DMU C has a scale efficiency of one, although it is technically inefficient. The second-to-last column of Table 2.4 shows the technical efficiencies of the five DMUs from the output side, and the last column shows their scale efficiencies. The BCC efficiencies from the input and output sides, including technical and scale, are not necessarily the same.

The graphical interpretation in this section shows that when the number of DMUs is relatively small, most of them appear on the production frontier, with a perfect efficiency score of one, which apparently overstates their efficiency. This raises the question of how many DMUs are needed to construct an empirical frontier which does not deviate from the true one by too much, in order to obtain meaningful efficiency measures. A rule of thumb is to have at least three times the total number of inputs and outputs, that is $n \geq 3(m + s)$ (Banker et al. 1989). Time series data can be used for cases in which the number of all possible DMUs does not satisfy this rule, by considering the same DMU at different time periods as different DMUs, thus increasing the number of DMUs in the calculation, referred to as window analysis in Charnes et al. (1985).

2.3 Restrictions on Multipliers

One issue that was widely discussed in the early development of the DEA approach was the value used for the small non-Archimedean number ε. If this is too small, then it will be ignored in computer calculations due to rounding. If, on the other hand, it is not small enough, then a Pareto efficiency DMU may become inefficient. As to what value should be assigned to ε, Lewin and Morey (1981) recommended 10^{-6}. However, since different units of measurement for the input and output factors will affect the function of ε, e.g., centimeters versus kilometers, it is inappropriate to assign the same value to ε for factors of different scales. Charnes and Cooper (1984) thus suggested using $\varepsilon = 10^{-5}$ when efficiency is expressed as a percentage (e.g., using $E_0 = 100$ rather than 1.0), and input and output entries are kept in the range of 1 to 100. There are also other suggestions for setting the value of ε (Färe and Hunsaker 1986).

In many cases there is prior information regarding the importance of the factors that requires the corresponding multipliers to lie in specific ranges in the form of (Dyson and Thanassoulis 1988):

$$\begin{aligned} &L_i^I \le v_i \le U_i^I, \quad && i = 1, \ldots, m \\ &L_r^O \le u_r \le U_r^O, \quad && r = 1, \ldots, s \end{aligned} \tag{2.16}$$

Consider a case of n DMUs, where each DMU applies different amounts of two inputs X_1 and X_2 to produce one unit of one output Y. The CCR input model (2.2), with restrictions on the range of the multipliers included, can be expressed as:

$$\begin{aligned} E_0 = \text{max.} \quad & 1/(v_1X_{10} + v_2X_{20}) \\ \text{s.t.} \quad & v_1X_{1j} + v_2X_{2j} \ge 1, \quad j = 1, \ldots, n \\ & L_i^I \le v_i \le U_i^I, \qquad i = 1, 2 \end{aligned} \tag{2.17}$$

where (X_{1j}, X_{2j}) is the input observation of a DMU. The lower bound constraint $L_i^I \le v_i$, $i = 1, 2$, can be expressed as $v_1(1/L_1^I) + v_2(0) \ge 1$ and $v_1(0) + v_2(1/L_2^I) \ge 1$. These two constraints imply that two more DMUs, with observations $(1/L_1^I, 0)$ and $(0, 1/L_2^I)$, are included to construct the frontier.

To handle the upper bound of $v_i \le U_i^I$, we substitute it into the constraint of $v_1X_{1j} + v_2X_{2j} \ge 1$ to obtain $U_1^IX_{1j} + v_2X_{2j} \ge 1$ and $v_1X_{1j} + U_2^IX_{2j} \ge 1$, or $v_2 \ge (1 - U_1^IX_{1j})/X_2$ and $v_1 \ge (1 - U_2^IX_{2j})/X_{1j}$. These lower bounds on v_i imply the addition of two sets of n DMUs, with observations $(0, X_{2j}/(1 - U_1^IX_{1j}))$ and $(X_{1j}/(1 - U_2^IX_{2j}), 0)$, $j = 1, \ldots, n$, to the original DMUs to construct the frontier together.

As more DMUs are included, the constructed frontier will be raised higher in the X-Y plane (or expanded towards the origin in the input space, or expanded outwards in the output space). The efficiency of every DMU will thus either remain the same or decrease. For the data contained in Table 2.2, suppose the restrictions of

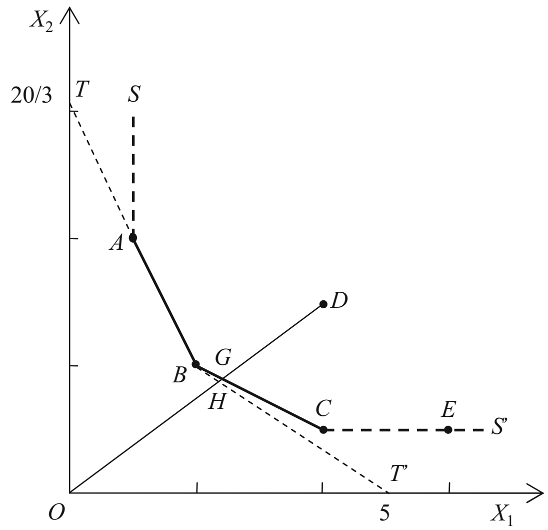


Fig. 2.5 Frontiers adjusted by absolute bounds on multipliers

$0.2 \leq v_1 \leq 0.4$ and $0.1 \leq v_2 \leq 0.5$ are imposed. The new DMUs generated by the lower bounds are (5, 0) and (0, 10). The two upper bounds are able to generate ten new DMUs; however, only three are feasible, (8, 0), (0, 20/3), and (0, 10). Referring to Fig. 2.2, which is redrawn as Fig. 2.5, the line segments $SABCS'$ are the frontier constructed from the original five DMUs, and TBT' are those constructed from the new set of DMUs. Based on this new frontier, the original efficient DMU C and the weak efficient DMU E become inefficient, the efficiency of DMU D decreases from OG/OD to OH/OD, whereas the original efficient DMUs A and B are still efficient.

The range in Expression (2.16) for each multiplier is in absolute scale, which has different effects for measures of different scales, and may obtain misleading results (Podinovski 1999). To eliminate this undesirable effect, Thompson et al. (1986) proposed the concept of an assurance region, with the following form, to restrict the range of the multipliers:

$$
\begin{aligned}
&L_i^I \leq \frac{v_i}{v_1} \leq U_i^I, \qquad i = 2, \ \ldots, \ m \\
&L_r^O \leq \frac{u_r}{u_1} \leq U_r^O, \quad r = 2, \ \ldots, \ s
\end{aligned} \tag{2.18}
$$

In this form the importance of each input/output factor is expressed in relation to that of the first one. The absolute bounds in (2.16) are special cases of the relative bounds, with $v_1 = u_1 = 1$.

To see how these constraints affect the frontier, consider the example in Table 2.3, where all five DMUs apply one unit of input X to produce different amounts of two outputs Y_1 and Y_2. The constraints of the CCR output model (2.5) for this problem are $u_1 Y_{1j} + u_2 Y_{2j} \leq 1, \ j = A, \ \ldots, \ E$. Figure 2.6 is redrawn from

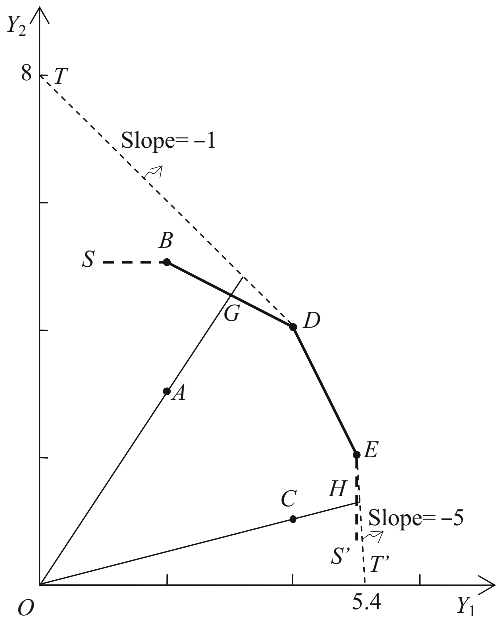


Fig. 2.6 Frontiers adjusted by relative bounds on multipliers

Fig. 2.3, in which the line segments $SBDES'$ are the frontier constructed from the original five DMUs. The frontier line segments BD and DE can be expressed by the equation $u_1y_1 + u_2y_2 = 1$, with a slope of $-u_1/u_2$ equal to -0.5 and -2, respectively. Note that here y_1 and y_2 are variables. Suppose an assurance region of $1 \leq u_1/u_2 \leq 5$ is imposed. This implies that two frontiers with slopes of -1 and -5 are added. These two frontiers correspond to line segments TD and ET' in Fig. 2.6, and the new frontier becomes the line segments $TDET'$. Under this new frontier, the originally efficient DMU B becomes inefficient, the inefficient DMUs A and C have lower efficiency scores, and the efficient DMUs D and E are still efficient.

The assurance region for restricting the relative range of either input or output multipliers can be linked to be more general (Thompson et al. 1990), based on the concept of a cone ratio (Charnes et al. 1989). The most general linear form of restrictions on multipliers is: $\alpha_1 u_1 + \ldots + \alpha_s u_s + \beta_1 v_1 + \ldots + \beta_m v_m \geq 0$, and Wong and Beasley (1990) had an application for this. Tracy and Chen (2005) introduced a formulation which provides generalized treatment for weight restrictions.

2.4 Ranking

An issue closely related to restrictions on multipliers is ranking. The DEA technique identifies efficient DMUs, and there is usually more than one DMU that is efficient. All efficient DMUs have a perfect efficiency score of one, which makes ranking of them difficult. Imposing tighter ranges for the multipliers, as discussed in the preceding section, may help discriminate the efficient DMUs. In addition to weight restrictions, there are many other approaches to ranking.

DMUs with higher efficiency scores are usually considered more efficient, and thus have higher ranks. However, some scholars believe that DMUs using different frontier facets to measure efficiency are not comparable, and only those using the same frontier facet can be compared. The same frontier facet means the same value of multipliers for calculating efficiency scores. Based on this idea, some studies use the same weight for all DMUs to calculate efficiency. This is the most stringent case of the assurance region, in which there is only one set of multipliers that can be selected in the region.

The first article to propose this idea was Roll et al. (1991), using the multipliers that yield the largest average efficiency score from all DMUs to calculate the efficiency of every DMU. The model under constant returns to scale is:

$$\begin{aligned} \text{max.} \quad & \sum_{j=1}^{n} E_j/n \\ \text{s.t.} \quad & E_j = \frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{m} v_i X_{ij}} \le 1, \qquad j = 1, \ldots, n \\ & u_r,\ v_i \ge \varepsilon, \quad r = 1, \ldots, s, \quad i = 1, \ldots, m \end{aligned} \tag{2.19}$$

At optimality, the efficiency of DMU j is calculated as $E_j = \sum_{r=1}^{s} u_r^* Y_{rj} / \sum_{i=1}^{m} v_i^* X_{ij}$. The efficiency scores thus measured then have a common basis for ranking. The common-weight frontier is a hyperplane that superimposes upon all DMUs, and all DMUs use this frontier to calculate efficiency.

Kao and Hung (2005) proposed the idea of using the compromise solution to determine the set of common weights that minimizes the total difference between the ideal efficiency (calculated from the conventional CCR or BCC models) and the actual efficiency (calculated from the common weight) of all DMUs to determine the multipliers. The model is:

$$\begin{aligned} \text{min.} \quad & \sum_{j=1}^{n} \left(E_j^* - \frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{m} v_i X_{ij}} \right)^p \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{m} v_i X_{ij}} \le 1, \qquad j = 1, \ldots, n \\ & u_r,\ v_i \ge \varepsilon, r = 1, \ldots, s, \quad i = 1, \ldots, m, \end{aligned} \tag{2.20}$$

where E_j^* in this case is the efficiency of DMU j calculated from the CCR model, and $p > 1$ is the distance parameter. When $p = 1$ this model is equivalent to Model (2.19), and it can thus be considered as an extension of the idea of maximizing the average efficiency of all DMUs. Kao and Hung (2005) recommended using $p = 2$, as this value produces a result of minimum variance.

Another idea related to a common weight is cross efficiency (Doyle and Green 1994), and this approach uses the multipliers selected by DMU j to calculate the efficiency of all other DMUs, in addition to itself. Therefore, every DMU has n cross efficiencies calculated from n sets of multipliers selected by n DMUs. The averages of the n cross-efficiencies from every DMU are then used for ranking. Since there are multiple solutions for using either the CCR or BCC models to measure the efficiency of every DMU, and improperly selected multipliers can lead to misleading results, one approach is to select the multipliers that produce the maximum weighted average efficiency of all DMUs. This procedure uses a conventional DEA model to calculate the efficiency of a DMU to obtain the efficiency E_0. Then the multipliers that maximize the weighted average efficiency of the n DMUs, while maintaining the efficiency of this DMU at E_0, are sought via the following model:

$$\begin{aligned}
\max.\quad & \frac{\sum_{j=1}^{n}\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{j=1}^{n}\sum_{i=1}^{m} v_i X_{ij}} \\
\text{s.t.}\quad & \sum_{r=1}^{s} u_r Y_{r0} = E_0 \sum_{i=1}^{m} v_i X_{i0} \\
& \frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{m} v_i X_{ij}} \le 1, \qquad j = 1, \ldots, n \\
& u_r,\ v_i \ge \varepsilon,\ r = 1, \ldots, s,\ i = 1, \ldots, m.
\end{aligned} \tag{2.21}$$

Note that the objective function is the average of the efficiencies $E_j = \sum_{r=1}^{s} u_r Y_{rj} / \sum_{i=1}^{m} v_i X_{ij},\ j = 1, \ldots, n$, weighted by the proportion of their aggregate input in the total aggregate input, $\sum_{i=1}^{m} v_i X_{ij} / \sum_{j=1}^{n} \sum_{i=1}^{m} v_i X_{ij}$. The optimal solution (u^*, v^*) is then used to calculate the efficiency of every DMU d, E_{d0}. This procedure is repeated for every DMU f to obtain the cross efficiency for every DMU d, E_{df}. The average efficiency for DMU d, $\overline{E}_d = \sum_{f=1}^{n} E_{df}/n$, is then used for ranking.

Most of the ranking methods are based on the idea of restricting the range of the multipliers that are used to calculate the efficiencies. In contrast, Andersen and Petersen (1993) proposed eliminating the focal DMU to construct the frontier from the remaining $n - 1$ DMUs in order to calculate an efficiency index for ranking. However, this method is only for ranking efficient DMUs. Since the DMUs being eliminated are efficient ones, they will fall outside of the region encompassed by the new frontier, and their efficiency scores calculated based on this frontier will be greater than one. This is why this efficiency index is said to measure super efficiency. The following is an output model for calculating the super efficiency of the focal DMU under variable returns to scale:

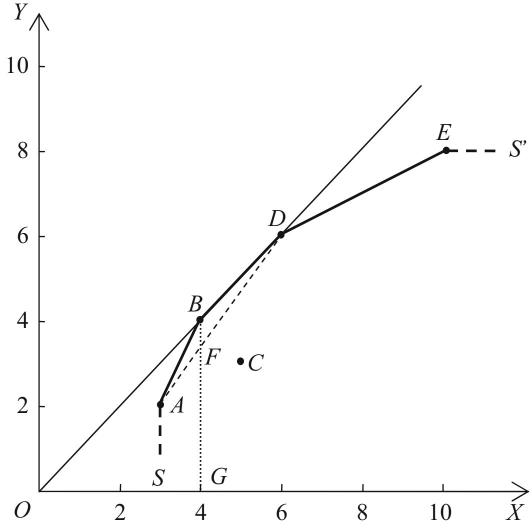


Fig. 2.7 Calculation of super efficiency

$$
\begin{aligned}
\frac{1}{E_0} = \min. \quad & \frac{\sum_{i=1}^{m} v_i X_{i0} + v_0}{\sum_{r=1}^{s} u_r Y_{r0}} \\
\text{s.t.} \quad & \frac{\sum_{i=1}^{m} v_i X_{ij} + v_0}{\sum_{r=1}^{s} u_r Y_{rj}} \geq 1, \qquad j = 1, \ldots, n, \quad j \neq 0 \\
& u_r, v_i \geq \varepsilon, r = 1, \ldots, s, \quad i = 1, \ldots, m \\
& v_0 \text{ unrestricted in sign.}
\end{aligned} \tag{2.22}
$$

Figure 2.7 is a graphical interpretation of super efficiency using the example in Fig. 2.4, where five DMUs, A, B, C, D, and E, use different amounts of input X to produce different amounts of output Y. The frontier constructed from these DMUs is the line segments $SABDES'$, and DMUs A, B, D, and E are efficient. In order to rank these four efficient DMUs, they are each eliminated in turn to construct new frontiers to calculate their super efficiencies. For example, the super efficiency of DMU B is measured from the frontier $SADES'$ constructed from the other four DMUs, A, C, D, and E, as BG/FG. The super efficiencies of DMUs D and E can be calculated similarly. To calculate the super efficiency of DMU A, however, one will obtain an unbounded value, because DMU A does not have a line segment with which to calculate efficiency. This situation will not occur under constant returns to scale. While several methods have been proposed to solve this problem (Li et al. 2007, Cook et al. 2009), super efficiency does not seem to be a suitable method for ranking (Banker and Chang 2006).

2.5 Supplementary Literature

The major difference between the CCR and BCC models is the effect of scale, and many articles address this issue. Seiford and Zhu (1999) reviewed three basic methods for determining returns to scale and the effects of multiple solutions. Banker et al. (2004) discussed returns to scale for several DEA models. Some other works related to this topic are Jahanshahloo and Soleimani-Damaneh (2004), Zarepisheh and Soleimani-Damaneh (2009), Fukuyama (2000), and Førsund and Hjalmarsson (2004).

The issue of weight restrictions is also widely discussed in the DEA literature, and there are different ways of classifying the related methods. The following studies have reviewed the literature on this topic: Roll and Golany (1993), Allen et al. (1997), Angulo-Meza and Estellita Lins (2002), Joro and Viitala (2004), and Sarrico and Dyson (2004). The works of Podinovski and Thanassoulis (2007), Khalili et al. (2010), Podinovski and Bouzdine-Chameeva (2013), and Førsund (2013), also discuss this issue.

Soltanifar and Lotfi (2011) discussed the strengths and weaknesses of several ranking methods, and proposed the voting analytic hierarchy process method. Wu et al. (2012) compared different cross efficiency methods for ranking. Other comprehensive reviews of the methods used for ranking include Adler et al. (2002), Singh and Chand (2007), Jablonsky (2012), and Hosseinzadeh et al. (2013).

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