

# Chapter 2

## Theory

In this chapter we briefly glance at basic concepts of porous medium theory (Sect. 2.1.1) and thermal processes of multiphase media (Sect. 2.1.2). We will study the mathematical description of thermal processes in the context of continuum mechanics and numerical methods for solving the underlying governing equations (Sect. 2.2).

### 2.1 Continuum Mechanics of Porous Media

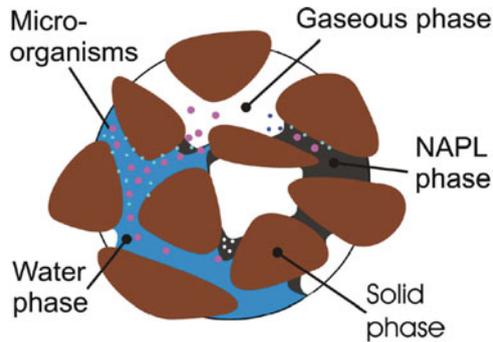
There is a great body of existing literature in the field of continuum mechanics and thermodynamics of porous media existing, you should look at, e.g. Carslaw and Jaeger (1959), Häfner et al. (1992), Bear (1972), Prevost (1980), de Boer (2000), Ehlers and Bluhm (2002), Kolditz (2002), Lewis et al. (2004), Goerke et al. (2012), Diersch (2014).

#### 2.1.1 Porous Medium Model

“The Theory of Mixtures as one of the basic approaches to model the complex behavior of porous media has been developed over decades (concerning basic assumptions see e.g. Bowen 1976; Truesdell and Toupin 1960). As the Theory of Mixtures does not incorporate any information about the microscopic structure of the material,<sup>1</sup> it has been combined with the Concept of Volume Fractions by e.g.

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<sup>1</sup>Within the context of the Theory of Mixtures the ideal mixture of all constituents of a multiphase medium is postulated. Consequently, the realistic modeling of the mutual interactions of the constituents is difficult.



Bowen (1980), de Boer and Ehlers (1986), Lewis and Schrefler (1998), Prevost (1980). Within the context of this enhanced Theory of Mixtures (also known as Theory of Porous Media), all kinematical and physical quantities can be considered at the macroscale as local statistical averages of their values at the underlying microscale. Concerning a detailed overview of the history of the modeling of the behavior of multiphase multicomponent porous media, the reader is referred to e.g. de Boer (2000). Comprehensive studies about the theoretical foundation and numerical algorithms for the simulation of coupled problems of multiphase continua are given in e.g. de Boer (2000), Ehlers and Bluhm (2002), Lewis and Schrefler (1998) and the quotations therein.” (Kolditz et al. 2012)

- A porous medium consists of different phases, i.e. a solid and at least one fluid phase,
- heat transfer processes are diffusion (solid phase), advection, dispersion,
- heat transfer between solid grains can occur by radiation when the fluid is a gas.
- A valid averaging volume for a porous medium is denoted as a representative elementary volume (REV).

Since the geometry of porous media in reality is not known exactly, a continuum approach comes into play. Figure 2.1 depicts the general idea of the porous medium approach. We do not need to know all the details about the microscopic porous medium structure but the portions of each phase which can be described macroscopically by **porosity** and **saturation**.

### 2.1.2 Thermal Processes

The assumption of local thermodynamic equilibrium is important and valid for many geothermal applications. At low Reynolds number flows and with small grain diameters, we may neglect the difference in temperature between the individual

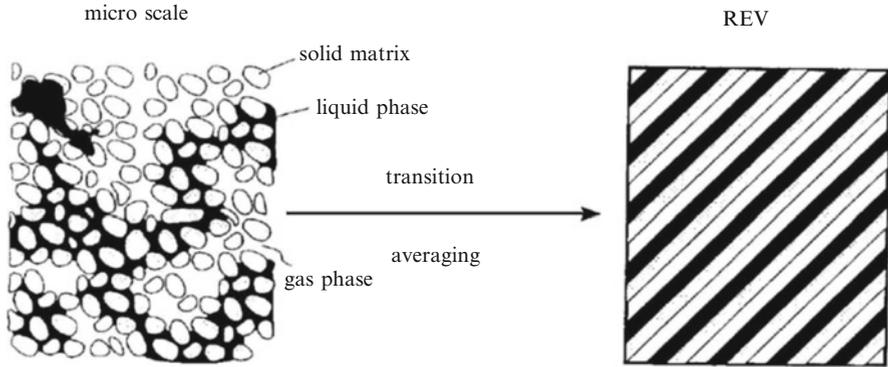


Fig. 2.1 REV concept (Helmig 1997)

phases, i.e. all phase temperatures are assumed to be equal. Physically this means, that the energy exchange between the phases is significantly faster than the energy transport within a phase.

The most important thermal processes within the context of geothermal energy systems are heat diffusion, heat advection, and heat storage. Additionally we might have to consider:

- heat radiation
- latent heat
- heat sources and sinks

### 2.1.2.1 Heat Diffusion

Diffusion processes basically are resulting from the Brownian molecular motion.

Heat diffusion basically is the transportation of heat by molecular activity. The diffusive heat flux is described by the famous Fourier's<sup>2</sup> law (2.1) (Fig. 2.2).

$$\mathbf{j}_{\text{diff}} = -\lambda^{\text{eff}} \nabla T \tag{2.1}$$

where  $\mathbf{j}_{\text{diff}}$  is the conductive heat flux, and  $\lambda^{\text{eff}}$  is the effective thermal conductivity of a porous media. The simplest definition of a porous medium property is given by volume fraction weighting. In case of a fully water saturated porous medium with porosity  $n$ ,  $\lambda^{\text{eff}}$  reads

$$\lambda^{\text{eff}} = n\lambda^w + (1 - n)\lambda^s \tag{2.2}$$

<sup>2</sup>A bit history of Fourier comes in the lecture.

**Fig. 2.2** Transport of heat by molecular diffusion. (Picture from [http://www.ehow.de/experimente-konduktion-strategie\\_9254/](http://www.ehow.de/experimente-konduktion-strategie_9254/))



**Table 2.1** Typical values of thermal conductivity in [ $\text{W m}^{-1} \text{K}^{-1}$ ] at 20 °C for some natural media

Material	Value
Iron	73
Limestone	1.1
Water	0.58
Dry air	0.025

where superscripts <sup>s</sup> and <sup>w</sup> stand for solid phase and water phase, respectively. For unsaturated media, a general formulation for effective heat conductivity can be written as

$$\lambda^{\text{eff}} = n \sum_f S^f \lambda^f + (1 - n) \lambda^s \quad (2.3)$$

where the superscript f stands for any fluid phase residing in the medium. A few typical examples of thermal conductivity for some natural media is listed in Table 2.1.

### 2.1.2.2 Heat Advection

Heat advection is the transportation of heat by fluid motion.<sup>3</sup>

The advective heat flux is described as

$$\mathbf{j}_{\text{adv}} = c^w \rho^w T \mathbf{v} \quad (2.4)$$

where  $c^w$  is specific heat and  $\rho^w$  is density of the water phase, and  $\mathbf{v}$  is the Darcy velocity (discharge per unit area) (Fig. 2.3).

<sup>3</sup>Note: “Heat” can be even cold in case the temperature is lower than the ambient one ;-).

**Fig. 2.3** Transport of heat by fluid motion. (Picture from <http://notutopiacom.ipage.com/wordpress/about-2/about/>)



### 2.1.2.3 Heat Storage

Heat storage in a porous medium can be expressed by the amount of heat  $Q$  in [J] within a balance volume  $V$  in [m<sup>3</sup>]

$$Q = (c\rho)^{\text{eff}} VT \quad (2.5)$$

where  $c$  is the specific heat capacity of the medium with

$$c_p = \left. \frac{\partial h}{\partial T} \right|_{p=\text{const}} \quad c_v = \left. \frac{\partial u}{\partial T} \right|_{v=\text{const}} \quad (2.6)$$

where  $c_p$  and  $c_v$  are isobaric and isochoric heat capacities, respectively. The specific heat capacity  $c$  is the heat necessary to increase the temperature of a unit mass of a medium by 1 K. The heat capacity of a porous medium is composed by its phase properties and can be expressed as an effective parameter, obtained analogously to (2.2):

$$(c\rho)^{\text{eff}} = nc^w\rho^w + (1-n)c^s\rho^s \quad (2.7)$$

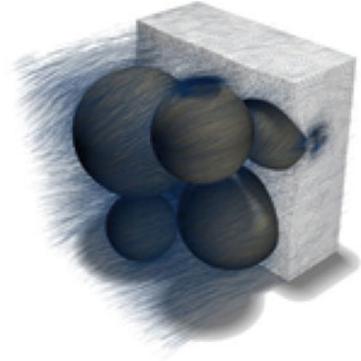
### 2.1.2.4 Heat Dispersion

Similar to mass dispersion, the structure of a porous medium results in a dispersive transport of heat. The basic idea behind the classic hydrodynamic dispersion theory by Scheidegger (Scheidegger 1961) is a normal distribution pattern through a regular porous medium (Fig. 2.4).

$$\mathbf{j}_{\text{disp}} = -\rho^w c^w \left( \alpha_T n |\mathbf{v}| \delta_{ij} + (\alpha_L - \alpha_T) n \frac{v_i v_j}{|\mathbf{v}|} \right) \nabla T \quad (2.8)$$

where  $\alpha_T$  and  $\alpha_L$  are transversal and longitudinal dispersivity.

**Fig. 2.4** Source: Naumov, personal communication



### 2.1.2.5 Heat Balance

Flow rates of heat (energy) in porous media can be described by balance equations. The heat balance equation is expressing an equilibrium of thermal processes, i.e. heat storage, diffusive and advective fluxes as well as heat sources and sinks.

$$c\rho \frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{j}_{\text{diff}} + \mathbf{j}_{\text{adv}} + \mathbf{j}_{\text{disp}}) = Q_T \quad (2.9)$$

where  $Q_T$  is the heat production term in  $[\text{J m}^{-3} \text{s}^{-1}]$ .

## 2.2 Governing Equations of Heat Transport in Porous Media

### 2.2.1 Energy Balance

The equation of energy conservation is derived from the first law of thermodynamics which states that the variation of the total energy of a system is due to the work of acting forces and heat transmitted to the system.

The total energy per unit mass  $e$  (specific energy) can be defined as the sum of internal (thermal) energy  $u$  and specific kinetic energy  $\mathbf{v}^2/2$ . Internal energy is due to molecular movement. Gravitation is considered as an energy source term, i.e. a body force which does work on the fluid element as it moves through the gravity field. The conservation quantity for energy balance is total energy density

$$\psi^e = \rho e = \rho(u + \mathbf{v}^2/2) \quad (2.10)$$

Using mass and momentum conservation we can derive the following balance equation for the internal energy

$$\rho \frac{du}{dt} = \rho q^u - \nabla \cdot (\mathbf{j}_{\text{diff}} + \mathbf{j}_{\text{disp}}) + \boldsymbol{\sigma} : \nabla \mathbf{v} \quad (2.11)$$

where  $q^u$  is the internal energy (heat) source,  $\mathbf{j}_{\text{diff}}$  and  $\mathbf{j}_{\text{disp}}$  are the diffusive and dispersive heat fluxes, respectively. Utilizing the definition of the material derivative

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \quad (2.12)$$

and neglecting stress power, we obtain the heat energy balance equation for an arbitrary phase

$$\rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T - \nabla \cdot \boldsymbol{\lambda} \nabla T = \rho q_T \quad (2.13)$$

where  $\boldsymbol{\lambda}$  contains both the diffusive and dispersive heat conduction parts.

### 2.2.2 Porous Medium

The heat balance equation for the porous medium consisting of several solid and fluid phases is given by

$$\sum_{\alpha} \epsilon^{\alpha} c^{\alpha} \rho^{\alpha} \frac{\partial T}{\partial t} + \nabla \cdot \left( \sum_{\gamma} n S^{\gamma} \rho^{\gamma} c^{\gamma} \mathbf{v}^{\gamma} T - \sum_{\alpha} \epsilon^{\alpha} \boldsymbol{\lambda}^{\alpha} \nabla T \right) = \sum_{\alpha} \epsilon^{\alpha} \rho^{\alpha} q_{\text{th}} \quad (2.14)$$

where  $\alpha$  is all phases and  $\gamma$  is fluid phases, and  $\epsilon^{\alpha}$  is the volume fraction of the phase  $\alpha$ .

Most important is the assumption of local thermodynamic equilibrium, meaning that all phase temperatures are equal and, therefore, phase contributions can be superposed. The phase change terms are canceled out with the addition of the individual phases.

With the following assumptions:

- local thermal equilibrium,
- fully saturated porous medium,
- neglecting viscous dissipation effects,

the governing equations for heat transport in a porous medium can be further simplified.

$$(c\rho)^{\text{eff}} \frac{\partial T}{\partial t} + (c\rho)^{\text{fluid}} \mathbf{v} \cdot \nabla T - \nabla \cdot (\boldsymbol{\lambda}^{\text{eff}} \nabla T) = q_T \quad (2.15)$$

with

$$(c\rho)^{\text{eff}} = \sum_{\alpha} \epsilon^{\alpha} c^{\alpha} \rho^{\alpha} \quad (2.16)$$

$$(c\rho)^{\text{fluid}} = n \sum_{\gamma} S^{\gamma} c^{\gamma} \rho^{\gamma} \quad (2.17)$$

$$\boldsymbol{\lambda}^{\text{eff}} = \sum_{\alpha} \epsilon^{\alpha} \boldsymbol{\lambda}^{\alpha} \quad (2.18)$$

For isotropic heat conduction without heat sources and we have the following classic diffusion equation

$$\frac{\partial T}{\partial t} = \nabla \cdot (\boldsymbol{\alpha}^{\text{eff}} \nabla T) \quad (2.19)$$

with heat diffusivity  $\boldsymbol{\alpha}^{\text{eff}} = \boldsymbol{\lambda}^{\text{eff}} / (c\rho)^{\text{eff}}$

### 2.2.2.1 Boundary Conditions

In order to specify the solution for the heat balance equation (2.9) we need to prescribe boundary conditions along all boundaries. Normally we have to consider three types of boundary conditions:

1. Prescribed temperatures (Dirichlet condition)

$$T = \bar{T} \quad \text{on} \quad \Gamma_T \quad (2.20)$$

2. Prescribed heat fluxes (Neumann condition)

$$q_n = \mathbf{j}_{\text{diff}} \cdot \mathbf{n} \quad \text{on} \quad \Gamma_q \quad (2.21)$$

3. Convective heat transfer (Robin condition)

$$q_n = a(T - T_{\infty}) \quad \text{on} \quad \Gamma_a \quad (2.22)$$

where  $a$  is the heat transfer coefficient in  $[\text{W K}^{-1} \text{m}^{-2}]$ .

### 2.2.3 Darcy's Law

For linear momentum conservation in porous media with a rigid solid phase we assume, in general, that inertial forces can be neglected (i.e.  $d\mathbf{v}/dt \approx 0$ ) and body forces are gravity at all. Assuming furthermore that internal fluid friction is small in comparison to friction on the fluid-solid interface and that turbulence effects can be neglected we obtain the Darcy law for each fluid phase  $\gamma$  in multiphase flow.

$$\mathbf{q}^\gamma = nS^\gamma \mathbf{v}^\gamma = -nS^\gamma \left( \frac{k_{\text{rel}}^\gamma \mathbf{k}}{\mu^\gamma} (\nabla p^\gamma - \rho^\gamma \mathbf{g}) \right) \quad (2.23)$$



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