Chapter 2
Local Limit Cycles of Degenerate Foci in Cubic Systems

Terence R. Blows

Abstract The problem of determining the stability of a weak focus in a quadratic or cubic system has been the focus of much research. Here we outline a simple but imperfect approach to the study of degenerate foci and use the method to give an example of a cubic system with four local limit cycles about a degenerate focus.

Keywords Cubic system • Limit cycles degenerate focus

2.1 Introduction

From his famous list of problems the second part of Hilbert's Sixteenth Problem was the topic of much interest in the 1980s and 1990s. The papers of Shi [1] and of Chen and Wang [2] which gave examples of quadratic systems with four limit cycles were a catalyst for this, but a major contributor to the increased work in this area was the rise of computer algebra systems that allowed lengthy algebraic manipulations to be carried out by a machine. With advances in bifurcation theory happening at the same time, see Rousseau [3], this was a rich period for research in planar polynomial systems.

A fixed point of a planar system of differential equations is called a *center* if a neighborhood of the fixed point is filled with closed orbits. Centers can occur in two ways; as well as the much studied case when the critical point is a *weak focus* (purely imaginary eigenvalues) there is the case where the critical point is a *degenerate focus*. The simplest type of the latter occurs under certain conditions when the linearization about the critical point is nilpotent but non-zero. These conditions are described with proof in Andronov et al. [4] and summarized in Perko [5].

Andronov's condition for monodromicity does not specify whether the fixed point is a center or focus, and in this sense the situation is similar to that of a weak focus. The problem of determining the stability of a weak focus has been well-studied and dates back to Poincaré. One approach is to construct a Liapunov

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function, and this can be done using an algorithm which is easily implemented using symbolic computing. See, for example, Blows and Lloyd [6]. Here we use a similar approach and use a Liapunov function to determine the stability of a degenerate focus. The method described is imperfect—it does not determine the stability for every degenerate focus—but we are able to use the method with some success. In particular we describe and apply the method to degenerate foci of cubic systems and extend a result of Andreev et al. [7].

It should be noted that another possibility is that the localization about the critical point has no linear terms. An example of this was studied in Blows and Rousseau [8] where the localization was about the point at infinity of a cubic system of a certain type.

2.2 Method

We consider cubic systems that have a degenerate focus at the origin. These may be written such that the linear part has a canonical form corresponding to a Jordan block with double zero eigenvalue:

$$\begin{cases} x' = y + P_2(x,y) + P_3(x,y) \\ y' = Q_2(x,y) + Q_3(x,y) \end{cases} \tag{2.1}$$

Also for monodromicity it is necessary that ([4, 5]) $Q_2(x, 0) = 0$ and $Q_3(x, 0) < 0$.

To study the stability of the origin we seek to construct a Liapunov function of the form

$$V(x,y) = V_2(x,y) + V_3(x,y) + V_4(x,y) + \cdots + V_n(x,y) + \cdots$$

where $V_k(x, y)$ is homogeneous of degree k. This gives

$$V' = \frac{\partial V_2}{\partial x} y + \cdots$$

and to be one-signed we therefore need $V_2(x, 0) = 0$. For V itself to be positive in a neighborhood of the origin it is therefore necessary that $V_2(x, y) = cy^2$ for some $c > 0$ and we make the arbitrary and convenient choice $c = ½$ to get

$$V_2 = \frac{1}{2} y^2 \tag{2.2}$$

We have

$$\begin{aligned} V' = &\left(\frac{\partial V_2}{\partial x} + \frac{\partial V_3}{\partial x} + \frac{\partial V_4}{\partial x} + \cdots\right)(y + P_2(x,y) + P_3(x,y)) \\ &+ \left(\frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial y} + \frac{\partial V_4}{\partial y} + \cdots\right)(Q_2(x,y) + Q_3(x,y)), \end{aligned}$$

and gathering like terms gives

$$
\begin{aligned}
V' = &\left(\frac{\partial V_2}{\partial x}\right) y \\
&+ \left(\frac{\partial V_3}{\partial x}\right) y + \left(\frac{\partial V_2}{\partial x}\right) P_2(x,y) + \left(\frac{\partial V_2}{\partial y}\right) Q_2(x,y) \\
&+ \left(\frac{\partial V_4}{\partial x}\right) y + \left(\frac{\partial V_3}{\partial x}\right) P_2(x,y) + \left(\frac{\partial V_3}{\partial y}\right) Q_2(x,y) + \left(\frac{\partial V_2}{\partial x}\right) P_3(x,y) \\
&+ \left(\frac{\partial V_2}{\partial y}\right) Q_3(x,y) \\
&+ \left(\frac{\partial V_5}{\partial x}\right) y + \left(\frac{\partial V_4}{\partial x}\right) P_2(x,y) + \left(\frac{\partial V_4}{\partial y}\right) Q_2(x,y) + \left(\frac{\partial V_3}{\partial x}\right) P_3(x,y) \\
&+ \left(\frac{\partial V_3}{\partial y}\right) Q_3(x,y) + \cdots
\end{aligned}
$$

In order to guarantee that the quadratic and cubic terms of V' are both zero, the choice (2.2) then implies that

$$V_3(x,y) = -\int Q_2(x,y)\, dx$$

Indeed we have $V' \equiv 0$ if we can recursively choose V_k such that

$$
\begin{aligned}
\left(\frac{\partial V_k}{\partial x}\right) y = &-\left(\frac{\partial V_{k-1}}{\partial x}\right) P_2(x,y) - \left(\frac{\partial V_{k-1}}{\partial y}\right) Q_2(x,y) - \left(\frac{\partial V_{k-2}}{\partial x}\right) P_3(x,y) \\
&- \left(\frac{\partial V_{k-2}}{\partial y}\right) Q_3(x,y)
\end{aligned}
$$

for all integers $k \geq 4$. However the term on the right-hand side may contain terms of the form x^{k+1} and so the best we can do when choosing the V_k is to have

$$V' = \eta_5 x^5 + \eta_6 x^6 + \eta_7 x^7 + \cdots + \eta_k x^k + \cdots$$

If the leading non-zero η_k is such that k is even, then V' is one-signed in a neighborhood of the origin, and the stability of the origin is determined by the sign of η_k. If all η_k terms are zero, then the origin is a center. However if the leading non-zero η_k is such that k is odd, then the construction fails to give a Liapunov function. Such cases will require a different method. See, for example, Sadovskii [9].

The center problem parallels the case of a weak focus. Although there are an infinite number of η_k, this set has a finite basis which we denote $<L(1), L(2), L(3) \ldots L(N)>$ where the Liapunov numbers $L(k)$ are numbered in order as they arise from the η_k. Calculating the η_k and reducing them to a finite set of Liapunov numbers is a difficult problem, and it is likely, as with the case of a weak focus, that the full solution to the problem may lie out of reach even with fast computers and Gröbner basis methods.

Another connection with weak foci lies in the generation of small amplitude limit cycles by perturbation methods. This is described below in the proof of Theorem 2.

2.3 Results

Using a judicious linear coordinate change, we may assume without loss of generality that $\boldsymbol{P_3(x,\ 0) = 0}$ and $\boldsymbol{Q_3(x,\ 0) = -1}$ in (2.1). We therefore consider systems of the form

$$\begin{cases} \boldsymbol{x' = y + Cx^2 + Dxy + Fy^2 + Nx^2y + Qxy^2 + Ry^3} \\ \boldsymbol{y' = Axy + By^2 - x^3 + Kx^2y + Lxy^2 + My^3} \end{cases}$$

Applying the algorithm we find that

$$\eta_5 = 1/2(A + 2C)(AC + 1)$$
$$\eta_6 = -\left(5AB + 14BC + 5A^2BC + 17ABC^2 + 6BC^3 + 2AD - 6CD + 5A^2CD + 12AC^2D + 2K - ACK + 6C^2K\right)/6$$

However the solution to $\eta_5 = \eta_6 = 0$ is far from simple, and we are already faced with computational difficulties that we do not wish to get into here. Instead we make the convenient choice $A = C = 0$ to easily get $\eta_5 = 0$. It is easy to see that then $\eta_6 = -K/3$. Under the conditions $A = C = K = 0$ we find using *Mathematica* 8 that

$$\eta_7 = F/4$$
$$\eta_8 = (-19BF + 8DF - 12M - 4Q)/20$$
$$\eta_9 = \left(209B^2F - 60D^2F + 55FL + 90DM + 40FN + 207BDF + 42BM - 66BQ + 40DQ\right)/120$$
$$\eta_{10} = \left(-1509B^3F + 480D^3F - 818DFL - 720DFN - 660D^2M - 360D^2Q - 3261B^2DF + 18B^2M + 726B^2Q - 1982BD^2F + 439BFL - 22BDM + 1090BFN + 956BDQ + 552LM + 480MN + 264LQ + 240NQ\right)/840$$

In terms of Liapunov quantities, where $L(1) = \eta_5$ and $L(2) = -K/3$ have been set to zero, we have

$$L(3) = F/4$$
$$L(4) = (3M + Q)/5$$
$$L(5) = (D + 2B)M/4$$

We have a choice from L(5): Either $D+2B = 0$ or $M = 0$. However, as we show in the proof of Theorem 1, the latter gives a center. So we assume M is non-zero. We make the choices $F = 0$, $Q = -3M$, and $D = -2B$ to get

$$\eta_{10} = 2M\,(L + N)\,/7$$
$$\eta_{11} = 3BM\left(2300B^2 + 216L + 181N\right)/112$$

Substituting $N = -L$ from η_{10} gives

$$L(7) = 0$$

If B or M is equal to zero, then, as we show in the proof of Theorem 1, we have a center. Otherwise

$$\eta_{12} = -M\big(4436580B^4 + 387976B^2L - 1504L^2 + 263331B^2N - 3968LN - 2464N^2\big)/5040$$

And subbing $L = -14B^2$ and $N = 4B^2$ gives

$$L(8) = (108161/560)\ B^4M$$

So $\eta_5 = \eta_6 = \eta_7 = \eta_8 = \eta_9 = \eta_{10} = \eta_{11} = \eta_{12} = 0$ implies a center.

Theorem 1 *The origin of the system*

$$\begin{cases} x' = y + Dxy + Fy^2 + Nx^2y + Qxy^2 + Ry^3 \\ y' = By^2 - x^3 + Kx^2y + Lxy^2 + My^3 \end{cases}$$

is a center if and only if one of the following two conditions holds:

1) $K = F = M = Q = 0$
2) $K = F = B = D = 0; \quad Q = -3M, N = -L$

Proof Necessity has already been shown. For the sufficiency of 1) note that in this case the origin is a center due to the symmetry $(x, y, t) \to (x, -y, -t)$. Condition 2) gives a Hamiltonian system.

In the following theorem we start with the weakest possible degenerate focus, namely when $\eta_5 = \eta_6 = \eta_7 = \eta_8 = \eta_9 = \eta_{10} = \eta_{11} = 0$ but $\eta_{12} \neq 0$, we may perturb η_{10}, η_8, and η_6 away from zero in turn to get three local limit cycles in the same manner as using multiple Hopf bifurcation from a weak focus. We then perturb λ non-zero to get a fourth in a manner that is new. It is possible that other perturbations will produce more than one local limit cycle; this would require a complete analysis of the unfolding of the degenerate critical point in a manner similar to that of Rousseau and Zhu [10] for an elliptic nilpotent singularity in quadratic systems.

Theorem 2 *The system*

$$\begin{cases} x' = y - 2Bxy + Nx^2y + (\delta - 3M)\,xy^2 + Ry^3 \\ y' = \lambda\,(-x + sgn\,(M)\,y) + By^2 - x^3 + \mu x^2y + Lxy^2 + My^3 \end{cases}$$

where $M \neq 0, B \neq\ 0,\ \delta M < 0, \mu M > 0,\ O << |\lambda| << |\mu| << |\delta| << |\varepsilon| << 1$ *has at least three local limit cycles in a neighborhood of the origin.*

Proof With $\lambda = \mu = \delta = 0$, the origin is a degenerate focus whose stability is given by the sign of M. The perturbations of δ, and μ away from zero in turn each cause a change in stability and produce local limit cycles. At this point, the origin has stability given by the sign opposite to M. Finally perturbing $\lambda \neq 0$ produces a strong focus at the origin whose stability is given by the sign of M to produce one final local limit cycle.

Appendix: Mathematica 8

Mathematica was used interactively to produce the results in Sect. 2.3. Firstly the base functions are put in place:

$$\text{P2} = cx\hat{}2 + Dxy + F\hat{}2$$

$$\text{Q2} = Axy + By\hat{}2$$

$$\text{P3} = Nx\hat{}2y + Qxy\hat{}2 + Ry\hat{}3$$

$$\text{Q3} = -x\hat{}3 + Kx\hat{}2y + Lxy\hat{}2 + My\hat{}3$$

$$\text{V2} = 1/2y\hat{}2$$

$$\text{V3} = -\text{Integrate}\,[\text{Q2}, x]$$

After this each iteration of the algorithm has a sequence of similar steps. The first set is as follows:

$$T4 = -D\,[\text{V3}, x]\,\text{P2} - D\,[\text{V3}, y]\,\text{Q2} - D\,[\text{V2}, x]\,\text{P3} - D\,[\text{V2}, y]\,\text{Q3}$$

$$\text{Collect}\,[\%, \{x, y\}]$$

$$\text{X4} = \text{Coefficient}\,[\%, x\hat{}4]$$

$$\text{V5} = \text{Simplify}\,[(\text{T4} - \text{X4}\,x\hat{}4)\,/y]$$

Each X terms give us a focal value η, and the V terms give us the homogeneous pieces of the Liapunov function that we are constructing. We continue through as many of these steps as is necessary.

References

1. Shi, S.-l.: A concrete example of the existence of four limit cycles for plane quadratic systems. Sci. Sin. 153–158 (1980)
2. Chen, L.S., Wang, M.S.: The relative position and number of limit cycles of a quadratic differential system. Acta Math. Sin. **22**, 751–758 (1979)
3. Rousseau, C.: Bifurcation methods in polynomial systems. In: Proceedings of the Nato Advanced Study Institute (Séminaire de Mathématiques Supérieures), “Bifurcations and periodic orbits of vector fields”, pp. 383–428. Kluwer Academic Publishers (1992)
4. Andronov, A.A., Leontovich, E.A., Gordon, I.I., Meier, A.G.: Qualitative Theory of Second-order Dynamical Systems. Wiley, New York (1973)
5. Perko, L.M.: Differential Equations and Dynamical Systems. Springer, New York (1991)
6. Blows, T.R., Lloyd, N.G.: The number of limit cycles of certain polynomial differential equations. Proc. R. Soc. Edinburgh Sect. A **98**, 215–239 (1984)
7. Andreev, A.F., Sadovskii, A.P., Tsikalyuk, V.A.: The center-focus problem for a system with homogeneous nonlinearities in the case of zero eigenvalues of the linear part. Differ. Equ. **39**, 155–164 (2003)
8. Blows, T.R., Rousseau, C.: Bifurcations at infinity in polynomial vector fields. J. Differ. Equ. **104**, 215–242 (1993)
9. Sadovskii, A.P.: Solution of the center-focus problem for some systems of nonlinear oscillations. Diff. Uravn. **14**, 268–269 (1978)
10. Rousseau, C., Zhu, H.: PP-graphics with a nilpotent elliptic singularity in quadratic systems and Hilbert’s 16th problem. J. Differ. Equ. **196**, 169–208 (2004)

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