Preface

1 Introduction

This volume grew out of two Simons Symposia on “Nonarchimedean and tropical geometry” which took place on the island of St. John in April 2013 and in Puerto Rico in February 2015. Each meeting gathered a small group of experts working near the interface between tropical geometry and nonarchimedean analytic spaces for a series of inspiring and provocative lectures on cutting edge research, interspersed with lively discussions and collaborative work in small groups. Although the participants were few in number, they brought widely ranging expertise, a high level of energy, and focused engagement. The articles collected here, which include high-level surveys as well as original research, give a fairly accurate portrait of the main themes running through the lectures and the mathematical discussions of these two symposia.

Both tropical geometry and nonarchimedean analytic geometry in the sense of Berkovich produce “nice” (e.g., Hausdorff, path connected, locally contractible) topological spaces associated to varieties over valued fields. These topological spaces are the main feature which distinguishes tropical geometry and Berkovich theory from other approaches to studying varieties over valued fields, such as rigid analytic geometry, the geometry of formal schemes, or Huber’s theory of adic spaces. All of these approaches are interrelated, however, and the papers in the present volume touch on all of them. The topological spaces produced by tropical geometry and Berkovich’s theory are also linked to one another; in many contexts, nonarchimedean analytic spaces are limits of tropical varieties, and tropical varieties are often best understood as finite polyhedral approximations to Berkovich spaces.

Topics of active research near the interface between tropical and nonarchimedean geometry include:

- Differential forms, currents, and solutions of differential equations on Berkovich spaces and their skeletons
- The homotopy types of nonarchimedean analytifications

- The existence of “faithful tropicalizations” which encode the topology and geometry of analytifications
- Relations between nonarchimedean analytic spaces and algebraic geometry, including logarithmic schemes, birational geometry, and linear series on algebraic curves
- Adic tropical varieties relate to Huber’s theory of adic spaces analogously to the way that usual tropical varieties relate to Berkovich spaces
- Relations between non-archimedean geometry and combinatorics, including deep and fascinating connections between matroid theory, tropical geometry, and Hodge theory

2 Contents

The survey paper of *Gubler* presents a streamlined version of the theory of *differential forms and currents on nonarchimedean analytic spaces* due to Antoine Chambert-Loir and Antoine Ducros, in the important special case of analytifications of algebraic varieties. Starting with the formalism of superforms due to Lagerberg, Gubler establishes or outlines the key results in the theory, including nonarchimedean analogs of Stokes’ formula, the projection formula, and the Poincaré–Lelong formula. Gubler also proves that these formulas are compatible with well-known results from tropical algebraic geometry, such as the Sturmfels–Tevelev multiplicity formula, and indicates how the results generalize from analytifications of algebraic varieties to more general analytic spaces.

The theory of differential forms and currents on analytifications of algebraic varieties was developed in parallel, using rather different methods, by *Boucksom, Favre, and Jonsson*, who provide a survey of their work in this volume. Using their foundational work, they are able to investigate a nonarchimedean analog of the *Monge–Ampère equation* on complex varieties. The uniqueness and existence of solutions in the complex setting are famous theorems of Calabi and Yau, respectively. The nonarchimedean analog of the Monge–Ampère equation was first considered by Kontsevich and Tschinkel, and the uniqueness of solutions (analogous to Calabi’s theorem) was established by Yuan and Zhang. The article of Boucksom, Favre, and Jonsson outlines the authors’ proof of existence in a wide range of cases and concludes with a treatment of the special case of toric varieties.

The survey paper by *Kedlaya* is devoted to another topic of much recent research activity, the radii of convergence of solutions for *p-adic differential equations on curves*. A number of classical results, starting with the work of Dwork and Robba in the 1970s, have recently been improved using a fruitful new point of view, introduced by Baldassarri, based on Berkovich spaces. One studies the radius of convergence as a function on the Berkovich analytification and proves that the behavior of this function is governed by its retraction to a suitable skeleton. Kedlaya discusses the state of the art in this active field, including the recent joint papers of

Poineau and Pulita and the forthcoming work of Baldassari and Kedlaya. He also discusses applications to ramification theory, Artin–Schreier theory, and the Oort conjecture.

The survey paper by *Ducros* gives an introduction to the fundamental recent work of Hrushovski and Loeser on *tameness properties* of the topological spaces underlying Berkovich analytifications. Using model theory, and in particular the theory of *stably dominated types*, Hrushovski and Loeser prove that Berkovich analytifications of algebraic varieties and semi-algebraic sets are locally contractible and have the homotopy type of finite simplicial complexes. (Related results, but with different hypotheses, were proven earlier by Berkovich using completely different methods.) Ducros provides the reader with a gentle introduction to the model theory needed to understand the work of Hrushovski and Loeser.

The research article by *Cartwright* pertains to the general question "What are the possible homotopy types of a Berkovich analytic space?" One way of determining the homotopy type of a Berkovich space is to find a deformation retract onto a *skeleton*, such as the dual complex of the special fiber in a regular semi-stable model. Cartwright has developed a theory of *tropical complexes*, decorating these dual complexes with additional numerical data that makes them behave locally like tropicalizations (so that one can make sense, e.g., of chip-firing moves on divisors in higher dimensions). It is well known that any finite graph can be realized as the dual complex of the special fiber in a regular semi-stable degeneration of curves. Cartwright's article uses his theory of tropical complexes to prove that a wide range of two-dimensional simplicial complexes, including triangulations of orientable surfaces of genus at least 2, cannot be realized as dual complexes of special fibers of regular semi-stable degenerations.

Tropicalizations of embeddings of algebraic varieties in toric varieties depend on the choice of an embedding. Unless an embedding is chosen carefully, the homotopy type of the analytification might be quite different from that of a given tropicalization. For this reason, one often hunts for *faithful* tropicalizations, in which a fixed skeleton maps homeomorphically onto its image in a manner which preserves the integer affine structure. The article by *Werner* in this volume surveys the state of the art in the hunt for faithful tropicalizations, including Werner's work with Gubler and Rabinoff generalizing the earlier work of Baker, Payne, and Rabinoff, as well as her work with Häbich and Cueto showing that the tropicalization of the Plücker embedding of the Grassmannian $G(2,n)$ is faithful.

Curves of genus at least 1 over $\mathbf{C}((t))$ have *canonical* minimal skeletons, obtained by taking a minimal regular model over the valuation ring and taking the dual complex of the special fiber. For higher-dimensional varieties, there is no longer a unique minimal regular model. Nevertheless, canonical skeletons do exist in many cases, including for varieties of log-general type (varieties having "sufficiently many pluricanonical forms"). The survey paper by *Nicaise* presents two elegant constructions of this *essential skeleton*, based respectively on Nicaise's joint work with Mustaţă and Xu. This work relies crucially on deep facts from the minimal model program and suggests the existence of further relations between birational geometry and the topology of Berkovich spaces yet to be discovered.

The essential skeleton of the analytification of a variety X/K, where K is a discretely valued field, is defined using a certain *weight function* attached to pluricanonical forms. The definition of the weight function uses arithmetic intersection theory and only makes sense over a discretely valued field. The research article by *Temkin* gives a new construction of the essential skeleton which makes sense when K is an *arbitrary* nonarchimedean field and which agrees with the Mustaţă–Nicaise construction when K is discretely valued of residue characteristic zero. The new construction of Temkin is based on the so-called *Kähler seminorm* on sheaves of relative differential pluriforms. Temkin carefully lays the foundations for the theory of seminorms on sheaves of rings or modules and, as an application, proves generalizations of the main theorems of Mustaţă and Nicaise.

Both Berkovich's theory and tropical geometry work equally well over trivially valued fields, but in these cases, one does not have an interesting theory of degenerations to produce skeletons from dual complexes of special fibers. The article by *Abramovich, Chen, Marcus, Ulirsch, and Wise* explains how logarithmic structures on varieties over valued fields produce skeletons of Berkovich analytifications and, moreover, how these skeletons can be endowed with the structure of an *Artin fan*. The authors explain how, following Ulirsch, an Artin fan can be thought of as the nonarchimedean analytification of an Artin stack that locally looks like the quotient of a toric variety by its dense torus. The final section presents a series of intriguing questions for future research.

As mentioned above, in many cases, Berkovich spaces can be understood as limits of tropicalizations. The article by *Foster* gives an expository treatment of recent progress in this direction, presenting joint work with Payne in which the adic analytifications of Huber are realized as limits of *adic tropicalizations*. The underlying topological space of an adic tropicalization is the disjoint union of all initial degenerations. Just as Berkovich spaces are maximal Hausdorff quotients of Huber adic spaces, ordinary tropicalizations are maximal Hausdorff quotients of adic tropicalizations. One technical advantage of adic tropicalizations is that they are *locally ringed spaces* (ordinary tropicalizations do carry a natural structure sheaf, the push-forward of the structure sheaf on the Berkovich analytic space, but the stalks of this sheaf are not local rings).

The wide-ranging survey article of *Baker and Jensen* covers the tropical approach to degenerations of linear series, along with applications to Brill and Noether theory and other problems in algebraic and arithmetic geometry. Starting from Jacobians of graphs, component groups of Néron models, the combinatorics of chip-firing, and tropical geometry of Riemann–Roch, the paper makes connections to Berkovich spaces and their skeletons and also with the classical theory of limit linear series due to Eisenbud and Harris. The concluding sections give overviews of several applications, including the tropical proofs of the Brill–Noether theorem, Gieseker–Petri theorem, and maximal rank conjecture for quadrics, as well as the recent work of Katz, Rabinoff, Zureick, and Brown on uniform bounds for the number of rational points on curves of small Mordell–Weil rank.

The volume ends with the encyclopedic survey article by *Katz*, which provides an introduction to matroid theory aimed at an audience of algebraic geometers.

Highlights of the survey include equivalent descriptions of matroids in terms of matroid polytopes and cohomology classes on the permutahedral toric variety, as well as a discussion of realization spaces and connections to tropical geometry. The article concludes with an exposition of the Huh–Katz proof of Rota's *log-concavity conjecture* for characteristic polynomials of matroids in the representable case.[1]

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[1]While this book was in press, Adiprasito, Huh, and Katz announced a proof of the full Rota conjecture.

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