The Non-Archimedean Monge–Ampère Equation

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Abstract We give an introduction to our work on the solution to the non-Archimedean Monge–Ampère equation and make comparisons to the complex counterpart. These notes are partially based on talks at the 2015 Simons Symposium on Tropical and Nonarchimedean Geometry.

Keywords Monge–Ampère equation • Non-Archimedean geometry • Berkovich spaces • Calabi-Yau theorem • Complex geometry • Metrics on line bundles

1 Introduction

The purpose of these notes is to discuss the Monge–Ampère equation

$$\mathrm{MA}(\phi) = \mu$$

in both the complex and non-Archimedean setting. Here μ is a positive measure[1] on the analytification of a smooth projective variety, ϕ is a semipositive metric on an ample line bundle on X, and MA is the Monge–Ampère operator. All these terms will be explained below.

In the non-Archimedean case, our presentation is based on the papers [13, 14] to which we refer for details. In the complex case, we follow [6] rather closely. Generally speaking, we avoid technicalities or detailed proofs.

[1] All measures in this paper will be assumed to be Radon measures.

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2 Metrics on Lines Bundles

Let K be a field equipped with a complete multiplicative norm and let X be a smooth projective variety over K. To this data we can associate an analytification X^{an}. When K is the field of complex numbers with its usual norm, X^{an} is a compact complex manifold. When the norm is non-Archimedean, X^{an} is a K-analytic space in the sense of Berkovich [3]. In either case, it is a compact Hausdorff space.

Let L be a line bundle on X. It also admits an analytification L^{an}. A *metric* on L^{an} is a rule that to a local section $s : U \to L^{\mathrm{an}}$, where $U \subset X^{\mathrm{an}}$, associates a function $\|s\|$ on U, subject to the condition $\|fs\| = |f| \cdot \|s\|$, for any analytic function f on U. The metric is *continuous* if $\|s\|$ is continuous on U for every s.

For our purposes it is convenient to use additive notation for metrics and line bundles. Given an open cover U_α of X^{an} and local trivializations of L^{an} on each U_α, we can identify a section s of L with a collection $(s_\alpha)_\alpha$ of analytic functions. A metric ϕ is then a collection of functions $(\phi_\alpha)_\alpha$ in such a way that $\|s\|_\phi = |s_\alpha| e^{-\phi_\alpha}$ on U_α. With this convention, if ϕ is a metric on L^{an}, any other metric is of the form $\phi + f$, where f is a function on X^{an}. If ϕ_i is a metric on L_i, $i = 1, 2$, then $\phi_1 + \phi_2$ is a metric on $L_1 + L_2$.

Over the complex numbers, smooth metrics ϕ (i.e., each ϕ_α is smooth) play an important role. Of similar status, for K non-Archimedean, are *model metrics* defined as follows.[2] Let R be the valuation ring of K and k the residue field. A *model* of X is a normal scheme $\mathcal{X}$, flat and projective over $\operatorname{Spec} R$ and with generic fiber isomorphic to X. A model of L is a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ whose restriction to X is isomorphic to L. It defines a continuous metric $\phi_{\mathcal{L}}$ on L in such a way that any local nonvanishing section of a multiple of $\mathcal{L}$ has norm constantly equal to one. Model functions, that is, model metrics on $\mathcal{O}_X$, are dense in $C^0(X^{\mathrm{an}})$. We refer to [21] or [14] for a more thorough discussion.

Over $\mathbf{C}$, a smooth metric ϕ on L^{an} is semipositive (positive) if its curvature form $dd^c\phi$ is a semipositive (positive) $(1,1)$-form. Here $dd^c\phi = dd^c\phi_\alpha = \frac{i}{\pi}\partial\overline{\partial}\phi_\alpha$ for any α. Such metrics only exist when L is nef.

In the non-Archimedean setting we say that a model metric $\phi_{\mathcal{L}}$ on L^{an} is semipositive if the line bundle $\mathcal{L}$ is relatively nef, that is, its degree is nonnegative on any proper curve contained in the special fiber $\mathcal{X}_0$. This implies that L is nef.

In both the complex and non-Archimedean case we say that a continuous metric ϕ is semipositive if there exists a sequence $(\phi_m)_1^\infty$ of semipositive smooth/model metrics such that $\lim_{m\to\infty} \sup_{X^{\mathrm{an}}} |\phi_m - \phi| = 0$. In the non-Archimedean case, this notion was first introduced by Zhang [51] and Gubler [28]. In the complex case, it is more natural to say that a continuous metric ϕ is semipositive if its curvature current $dd^c\phi$ is a positive closed current. At least when L is ample, one can then

[2]Model metrics are not smooth in the sense of Chambert-Loir and Ducros [22] but nevertheless, for our purposes, play the same role as smooth metrics in the complex case.

prove (see Sect. 7 below) that ϕ can be approximated by smooth metrics; such an approximation is furthermore crucial for many arguments in pluripotential theory.

In the non-Archimedean case, Chambert-Loir and Ducros have introduced a notion of forms and currents on Berkovich spaces. However, it is not known whether a continuous metric whose curvature current (in their sense) is semipositive can be approximated by semipositive model metrics.

In both the complex and non-Archimedean case we denote by $\mathrm{PSH}^0(L^{\mathrm{an}})$ the space of continuous semipositive metrics on L^{an}. Here the superscript refers to continuity (C^0) whereas "PSH" reflects the fact that in the complex case, semipositive metrics are global versions of plurisubharmonic functions.

3 The Monge–Ampère Operator

In the complex case, the Monge–Ampère operator is a second order differential operator: we set $\mathrm{MA}(\phi) = (dd^c\phi)^n$ for a smooth metric ϕ. It is a nonlinear operator if $n > 1$. When ϕ is semipositive, $\mathrm{MA}(\phi)$ is a smooth positive measure on X^{an} of mass (L^n). It is a *volume form*, that is, equivalent to Lebesgue measure, if ϕ is positive.

Next we turn to the non-Archimedean setting. *From now on we assume that K is discretely valued*. Pick a uniformizer t of the maximal ideal in the valuation ring R of K.

Consider a model metric $\phi_{\mathcal{L}}$, associated to a model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $\operatorname{Spec} R$. Write the special fiber as $\mathcal{X}_0 = \mathrm{div}(t) = \sum_{i\in I} b_i E_i$, where E_i are the irreducible components of $\mathcal{X}_0$ and $b_i \in \mathbf{Z}_{>0}$. To each E_i is associated a unique (divisorial) point $x_i \in X^{\mathrm{an}}$. We then define

$$\mathrm{MA}(\phi) := \sum_{i\in I} b_i (\mathcal{L}|_{E_i})^n \delta_{x_i}.$$

If $\phi_{\mathcal{L}}$ is semipositive, $\mathcal{L}|_{E_i}$ is nef; hence, $(\mathcal{L}|_{E_i})^n \geq 0$ and $\mathrm{MA}(\phi_{\mathcal{L}})$ is a positive measure. Its total mass is

$$\int_{X^{\mathrm{an}}} 1 \cdot \mathrm{MA}(\phi_{\mathcal{L}}) = \sum_{i\in I} b_i (\mathcal{L}|_{E_i})^n = (\mathcal{L}^n \cdot \mathcal{X}_0) = (\mathcal{L}^n \cdot \mathcal{X}_\eta) = (L^n).$$

Here the second to last equality follows from the flatness of $\mathcal{X}$ over $\operatorname{Spec} R$, and the last equality from $\mathcal{X}_\eta \simeq X$ and $\mathcal{L}_\eta \simeq L$.

From now on assume that L is *ample*, that is, we have a *polarized* pair (X, L). In both the complex and non-Archimedean case we define $\mathrm{MA}(\phi)$ for a continuous semipositive metric by $\mathrm{MA}(\phi) := \lim_{m\to\infty} \mathrm{MA}(\phi_m)$ for any sequence $(\phi_m)_1^\infty$ converging uniformly to ϕ. Of course, it is not obvious that the limit exists or independent of the sequence $(\phi_m)_1^\infty$. In the complex case this is a very special case of the Bedford–Taylor theory developed in [8, 9]. The analogous analysis in the non-Archimedean case is due to Chambert-Loir [20].

4 The Complex Monge–Ampère Equation

Theorem 4.1. *Let (X, L) be a polarized complex projective variety of dimension n and let μ be a positive measure on X^{an} of total mass (L^n).*

(i) *If μ is a volume form, then there exists a smooth positive metric ϕ on L^{an} such that* $\mathrm{MA}(\phi) = \mu$.
(ii) *If μ is absolutely continuous with respect to Lebesgue measure, with density in L^p for some $p > 1$, then there exists a (Hölder) continuous metric ϕ on L^{an} such that* $\mathrm{MA}(\phi) = \mu$.
(iii) *The metrics in (i) and (ii) are unique up to additive constants.*

The uniqueness statement in the setting of (i) is due to Calabi. The much harder existence part was proved by Yau [48], using PDE techniques. The combined result is often called the Calabi–Yau theorem.

The general setting of (ii) and (iii) was treated by Kołodziej [37, 38] who used methods of pluripotential theory together with a nontrivial reduction to Yau's result. Guedj and Zeriahi [34] more generally established the existence of solutions of $\mathrm{MA}(\phi) = \mu$ for positive measures μ (of mass (L^n)) that do not put mass on pluripolar sets. In this generality, the metrics ϕ are no longer continuous but rather lie in a suitable energy class, modeled upon work by Cegrell [19]. Dinew [24], improving upon an earlier result by Błocki [10], proved the corresponding uniqueness theorem. All these existence and uniqueness results are furthermore valid (in a suitable formulation) in the transcendental case, when (X, ω) is a Kähler manifold.

The complex Monge–Ampère equation is of fundamental importance to complex geometry. For example, it implies that every compact complex manifold with vanishing first Chern class (such manifolds are now called Calabi–Yau manifolds) admits a Ricci flat metric in any given Kähler class. The complex Monge–Ampère equation also plays a key role in recent work on the space of Kähler metrics.

5 The Non-Archimedean Monge–Ampère Equation

As before, suppose $K \simeq k((t))$ is a discretely valued field with valuation ring $R \simeq k[[t]]$ and residue field k. We further assume that K has *residue characteristic zero*, $\mathrm{char}\, k = 0$. This implies that $R \simeq k[[t]]$ and $K \simeq k((t))$, where k is the residue field of K. More importantly, X then admits SNC models, that is, regular models $\mathcal{X}$ such that the special fiber $\mathcal{X}_0$ has simple normal crossings. The dual complex $\Delta_{\mathcal{X}}$, encoding intersections between irreducible components of $\mathcal{X}_0$, then embeds as a compact subset of X^{an}.

Theorem 5.1. *Let (X, L) be a polarized complex projective variety of dimension n over K. Assume that X is defined over a smooth k-curve. Let μ be a positive measure on X^{an} of total mass (L^n), supported on the dual complex of some SNC model.*

(i) *There exists a continuous metric* ϕ *on* L^{an} *such that* $\mathrm{MA}(\phi) = \mu$.
(ii) *The metric in (i) is unique up to an additive constant.*

Here the condition on X means that there exists a smooth projective curve C over k, a smooth projective variety Y over C, and a point $p \in C$ such that X is isomorphic to the base change $Y \times_k \operatorname{Spec} K$, where K is the fraction field of $\widehat{\mathcal{O}}_{C,p}$. This condition is presumably redundant, but is used in the proof: see Sect. 9.

To our knowledge, the first to consider the Monge–Ampère equation (or Calabi–Yau problem) in a non-Archimedean setting were Kontsevich and Tschinkel [40]. They outlined a strategy in the case when μ is a point mass.

The case of curves ($n = 1$) was treated in detail by Thuillier in his thesis [47]; see also [2, 26]. In this case, the Monge–Ampère equation is linear and one can construct fundamental solutions by exploring the topological structure of X^{an}.

In higher dimensions, Yuan and Zhang [50] proved the uniqueness statement (ii). Their proof, based on the method by Błocki, is valid in a more general context than stated above. The first existence result was obtained by Liu [43], who treated the case when X is a maximally degenerate abelian variety and μ is equivalent to Lebesgue measure on the skeleton of X. His approach amounts to solving a *real* Monge–Ampère equation on the skeleton. The existence result (i) above was proved by the authors in [13] and the companion paper [14]. We will discuss our approach below.

The geometric ramifications of the non-Archimedean Monge–Ampère equations remain to be developed.

6 A Variational Approach

We shall present a unified approach to solving the complex and non-Archimedean Monge–Ampère equations in any dimension. The method goes back to Alexandrov's work in convex geometry [1]. It was adapted to the complex case in [6] and to the non-Archimedean analogue in [13].

The general strategy is to construct an *energy functional*

$$E : \mathrm{PSH}^0(L^{\mathrm{an}}) \to \mathbf{R}$$

whose derivative is the Monge–Ampère operator, $E' = \mathrm{MA}$, in the sense that

$$\frac{d}{dt} E(\phi + tf)|_{t=0} = \int_{X^{\mathrm{an}}} f \, \mathrm{MA}(\phi),$$

for every continuous semipositive metric $\phi \in \mathrm{PSH}^0(L^{\mathrm{an}})$ and every smooth/model function f on X^{an}.

Grant the existence of this functional for the moment. Given a measure μ on X^{an}, consider the functional $F_\mu : \mathrm{PSH}^0(L^{\mathrm{an}}) \to \mathbf{R}$ defined by

$$F_\mu(\phi) = E(\phi) - \int \phi\, \mu.$$

Suppose we can find $\phi \in \mathrm{PSH}^0(L^{\mathrm{an}})$ that maximizes F_μ. Since the derivative of F_μ is equal to $F'_\mu = \mathrm{MA} - \mu$, we then have $0 = F'_\mu(\phi) = \mathrm{MA}(\phi) - \mu$ as required.

Now, there are at least three problems with this approach:

(1) There is a priori no reason why a maximizer should exist in $\mathrm{PSH}^0(L^{\mathrm{an}})$. We resolve this by introducing a larger space $\mathrm{PSH}(L^{\mathrm{an}})$ with suitable compactness properties and find a maximizer there.
(2) Granted the existence of a maximizer $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$, we are maximizing over a convex set rather than a vector space, so there is no reason why $F'_\mu(\phi) = 0$. Compare maximizing the function $f(x) = x^2$ on the real interval $[-1, 1]$: the maximum is not at a critical point.
(3) In the end we want to show that—after all—the maximizer is continuous, that is, $\phi \in C^0(L^{\mathrm{an}})$.

We shall discuss how to address (1) and (2) in the next two sections. The continuity result in (3) requires a priori capacity estimates due to Kołodziej, and will not be discussed in these notes.

7 Singular Semipositive Metrics

Plurisubharmonic (psh) functions are among the *objets souples* (soft objects) in complex analysis according to P. Lelong [42]. This is reflected in certain useful compactness properties. The global analogues of psh functions are semipositive singular metrics on holomorphic line bundles. Here "singular" means that vectors may have infinite length.

Theorem 7.1. *Let K be either* $\mathbf{C}$ *or a discretely valued field of residue characteristic zero, and let (X, L) be a smooth projective polarized variety over K. Then there exists a unique class* $\mathrm{PSH}(L^{\mathrm{an}})$*, the set of singular semipositive metrics, with the following properties:*

- $\mathrm{PSH}(L^{\mathrm{an}})$ *is a convex set which is closed under maxima and addition of constants;*
- $\mathrm{PSH}(L^{\mathrm{an}}) \cap C^0(L^{\mathrm{an}}) = \mathrm{PSH}^0(L^{\mathrm{an}})$*;*
- *if s_i, $1 \le i \le p$, are nonzero global sections of mL for some $m \ge 1$, then $\phi := \frac{1}{m} \max_i \log |s_i| \in \mathrm{PSH}(L^{\mathrm{an}})$; further, ϕ is continuous iff the sections s_i have no common zero;*

- *if* (ϕ_j) *is an arbitrary family in* $\mathrm{PSH}(L^{\mathrm{an}})$ *that is uniformly bounded from above, then the usc regularization of* $\sup_j \phi_j$ *belongs to* $\mathrm{PSH}(L^{\mathrm{an}})$*;*
- *if* (ϕ_j) *is a decreasing net in* $\mathrm{PSH}(L^{\mathrm{an}})$*, then either* $\phi_j \to -\infty$ *uniformly on* X^{an} *or* $\phi_j \to \phi$ *pointwise on* X^{an} *for some* $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$*;*
- ***Regularization****: for every* $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ *there exists a decreasing sequence* $(\phi_m)_{m=1}^{\infty}$ *of smooth/model metrics such that* ϕ_m *converges pointwise to* ϕ *on* X^{an} *as* $m \to \infty$*; and*
- ***Compactness****: the space* $\mathrm{PSH}(L^{\mathrm{an}})/\mathbf{R}$ *is compact.*

To make sense of the compactness statement we need to specify the topology on $\mathrm{PSH}(L^{\mathrm{an}})$. In the complex case, one usually fixes a volume form μ on X^{an} and takes the topology induced by the L^1-norm: $\|\phi - \psi\| = \int_{X^{\mathrm{an}}} |\phi - \psi|\mu$. In the non-Archimedean case, there is typically no volume form on X^{an}. Instead, we say that a net $(\phi_j)_j$ in $\mathrm{PSH}(L^{\mathrm{an}})$ converges to ϕ if $\lim_j \sup_{\Delta_{\mathcal{X}}} |\phi_j - \phi| = 0$ for every SNC model $\mathcal{X}$. Implicit in this definition is that the restriction to $\Delta_{\mathcal{X}}$ of every singular metric in $\mathrm{PSH}(L^{\mathrm{an}})$ is continuous: see Theorem 7.2 below.

In the complex case, one typically defines $\mathrm{PSH}(L^{\mathrm{an}})$ as the set of usc singular metrics ϕ that are locally represented by L^1 functions and whose curvature current $dd^c\phi$ (computed in the sense of distributions) is a positive closed current. Thus ϕ is locally given as the sum of a smooth function and a psh function. Most of the statements above then follow from basic facts about plurisubharmonic functions in $\mathbf{C}^n$. The regularization result is the most difficult. On $\mathbf{C}^n$ it is easy to regularize using convolutions. With some care, one can in the global (projective) case glue together local regularizations to obtain a global one. See [23] for a general result and [11] for a relatively simple argument applicable in our setting.

In the non-Archimedean case, we are not aware of any workable a priori definition of $\mathrm{PSH}(L^{\mathrm{an}})$. Chambert-Loir and Ducros [22] have a notion of forms and currents on Berkovich spaces, but it is unclear if it gives the right objects for the purposes of the theorem above. Instead, we prove the following result:

Theorem 7.2. *For any SNC model* $\mathcal{X}$*, the restriction of the dual complex* $\Delta_{\mathcal{X}} \subset X^{\mathrm{an}}$ *of the set of model metrics on* L^{an} *forms an equicontinuous family.*

This is proved using a rather subtle argument, involving intersection numbers on toroidal models dominating $\mathcal{X}$. It would be interesting to have a different proof. At any rate, Theorem 7.2 allows us to define $\mathrm{PSH}(L^{\mathrm{an}})$ as the set of usc singular metrics ϕ satisfying, for every sufficiently large SNC model $\mathcal{X}$,

(i) $(\phi - \phi_0) \circ r_{\mathcal{X}} \geq \phi - \phi_0$ and
(ii) the restriction of ϕ to $\Delta_{\mathcal{X}}$ is a uniform limit of a sequence $\phi_m|_{\Delta_{\mathcal{X}}}$, where each ϕ_m is a semipositive model metric.

Here ϕ_0 is a fixed model metric, determined by some model dominated by $\mathcal{X}$. The map $r_{\mathcal{X}} : X^{\mathrm{an}} \to \Delta_{\mathcal{X}} \subset X^{\mathrm{an}}$ is a natural retraction. Since ϕ is usc, condition (i) implies that $\phi = \phi_0 + \lim_{\mathcal{X}} (\phi - \phi_0) \circ r_{\mathcal{X}}$, so that ϕ is determined by its restrictions to all dual complexes.

With this definition, the compactness of $\mathrm{PSH}(L^{\mathrm{an}})/\mathbf{R}$ follows from Theorem 7.2 and Ascoli's theorem. Regularization, however, is quite difficult to show. We are not aware of any procedure that would replace convolution in the complex case. Instead we use algebraic geometry. Here is an outline of the proof.

Fix $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$. For any SNC model $\mathcal{X}$, ϕ naturally induces a model metric $\phi_{\mathcal{X}}$. The semipositivity of ϕ implies that the net $(\phi_{\mathcal{X}})_{\mathcal{X}}$ indexed by the collection of (isomorphism classes of) SNC models decreases to ϕ. Unfortunately, except in the curve case $n = 1$, $\phi_{\mathcal{X}}$ has no reason to be semipositive; this reflects the fact that the pushforward of a nef line bundle may fail to be nef. We address this by defining $\psi_{\mathcal{X}}$ as the supremum of all semipositive (singular) metrics dominated by $\phi_{\mathcal{X}}$. We then show that $\psi_{\mathcal{X}}$ is continuous and can be *uniformly* approximated by a sequence $(\psi_{\mathcal{X},m})_m^\infty$ of semipositive model metrics. From this data it is not hard to produce a decreasing net of semipositive model metrics converging to ϕ.

Let us say a few words on the construction of the semipositive model metrics $\phi_{\mathcal{X},m}$ since this is a key step in the paper [14]. For simplicity assume that L is base point free and that $\phi_{\mathcal{X}}$ is associated with a line bundle $\mathcal{L}$ (rather than an $\mathbf{R}$-line bundle) on $\mathcal{X}$. Let $\mathfrak{a}_m$ be the base ideal of $m\mathcal{L}$, cut out by the global sections; it is cosupported on the special fiber $\mathcal{X}_0$. The sequence $(\mathfrak{a}_m)_m$ is a *graded sequence* in the sense that $\mathfrak{a}_l \cdot \mathfrak{a}_m \subset \mathfrak{a}_{l+m}$. Each $\mathfrak{a}_m$ naturally defines a semipositive model metric $\psi_{\mathcal{X},m}$ on L^{an}. The fact that $\psi_{\mathcal{X},m}$ converges *uniformly* to $\psi_{\mathcal{X}}$ translates into a statement that the graded sequence $(\mathfrak{a}_m)_m$ is "almost" finitely generated. This in turn is proved using *multiplier ideals* and ultimately reduces to the Kodaira vanishing theorem; to apply the latter, it is crucial to work in residue characteristic zero.

The argument above proves that any $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ is the limit of a decreasing *net* of semipositive model metrics. When ϕ is continuous, the convergence is uniform by Dini's theorem, and we can use the sup-norm to extract a decreasing *sequence* of model metrics converging to ϕ. In the general case, the *Monge–Ampère capacity* developed in [13, § 4] (and modeled on [8, 33]) can similarly be used to extract a convergent sequence from a net.

8 Energy

In the complex case, the (Aubin–Mabuchi) energy functional is defined as follows. Fix a smooth semipositive reference metric ϕ_0 and set

$$E(\phi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\mathrm{an}}} (\phi - \phi_0)(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}. \tag{1}$$

for any smooth metric ϕ. Here $(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}$ is a *mixed Monge–Ampère measure*. It is a positive measure if ϕ is semipositive.

In the non-Archimedean case, mixed Monge–Ampère measures can be defined using intersection theory when ϕ and ϕ_0 are model metrics, and the energy of ϕ is then defined exactly as above.

For two smooth/model metrics ϕ and ψ we have

$$E(\phi) - E(\psi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\mathrm{an}}} (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}. \tag{2}$$

This is proved using integration by parts in the complex case and follows from basic intersection theory in the non-Archimedean case.

We can draw two main conclusions from (2). First, the derivative of the energy functional is the Monge–Ampère operator, in the sense that

$$\frac{d}{dt} E(\phi + tf)\Big|_{t=0} = \int_{X^{\mathrm{an}}} f \,\mathrm{MA}(\phi) \tag{3}$$

for a smooth/model metric ϕ on L^{an} and a smooth/model function f on X^{an}.

Second, $E(\psi) \geq E(\phi)$ when $\psi \geq \phi$ are semipositive. It then makes sense to set

$$E(\phi) := \inf\{E(\psi) \mid \psi \geq \phi,\ \psi \text{ a semipositive smooth/model metric on } L^{\mathrm{an}}\}.$$

for any singular semipositive metric $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$. The resulting functional

$$E : \mathrm{PSH}(L^{\mathrm{an}}) \to [-\infty, \infty)$$

has many good properties: E is concave, monotonous, and satisfies $E(\phi + c) = E(\phi) + c$ for $c \in \mathbf{R}$. Further, E is usc and continuous along decreasing nets.

The energy functional singles out a class $\mathcal{E}^1(L^{\mathrm{an}})$ of metrics with *finite energy*, $E(\phi) > -\infty$. This class has good properties. In particular, one can (with some effort) define mixed Monge–Ampère measures $(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}$ for $\phi, \psi \in \mathcal{E}^1(L^{\mathrm{an}})$, and (1) continues to hold.

Let us now go back to the variational approach to solving the Monge–Ampère equation. Fix a positive measure μ on X^{an} of mass (L^n). In the complex case we assume that μ is absolutely continuous with respect to Lebesgue measure, with density in L^p for some $p > 1$. In the non-Archimedean case we assume that μ is supported on some dual complex. In both cases, one can show that the functional $\phi \to \int(\phi - \phi_0)\,\mu$ is (finite and) continuous on $\mathrm{PSH}(L^{\mathrm{an}})$, where ϕ_0 is the same reference metric as in (1). Thus the functional $F_\mu\colon \mathrm{PSH}(L^{\mathrm{an}}) \to [-\infty, \infty)$ defined by

$$F_\mu(\phi) := E(\phi) - \int (\phi - \phi_0)\mu$$

is upper semicontinuous. It follows from (2) that F_μ does not depend on the choice of reference metric ϕ_0. We also have $F_\mu(\phi + c) = F_\mu(\phi)$ for $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$, $c \in \mathbf{R}$. Thus F_μ descends to a usc functional on the quotient space $\mathrm{PSH}(L^{\mathrm{an}})/\mathbf{R}$. By Theorem 7.1, the latter space is compact, so we can find $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ maximizing F_μ. It is clear that $\phi \in \mathcal{E}^1(L^{\mathrm{an}})$, so the mixed Monge–Ampère measures of ϕ and ϕ_0 are well defined. However, equation (3) no longer makes sense, since there is no reason for the metric $\phi + tf$ to be semipositive for $t \neq 0$. Therefore, it is not clear that $\mathrm{MA}(\phi) = \mu$, as desired. In the next section, we explain how to get around this problem.

9 Envelopes, Differentiability, and Orthogonality

We define the *psh envelope* of a (possibly singular) metric ψ on L^{an} by

$$P(\psi) := \sup\{\phi \in \mathrm{PSH}(L^{\mathrm{an}}) \mid \phi \leq \psi\}^*.$$

As before, ϕ^* denotes the usc regularization of a singular metric ϕ. In all cases, we need to consider, ψ will be the sum of a metric in $\mathcal{E}^1(L^{\mathrm{an}})$ and a continuous function on X^{an}. In particular, ψ is usc, $P(\psi) \in \mathcal{E}^1(L^{\mathrm{an}})$ and $P(\psi) \leq \psi$.

This envelope construction was in fact already mentioned at the end of Sect. 7 as it plays a key role in the regularization theorem. The psh envelope is an analogue of the convex hull; see Fig. 1.

The key fact about the psh envelope is that the composition $E \circ P$ is differentiable and that $(E \circ P)' = E' \circ P$. More precisely, we have

Theorem 9.1. *For any $\phi \in \mathcal{E}^1(L^{\mathrm{an}})$ and $f \in C^0(X^{\mathrm{an}})$, the function $t \mapsto E(P(\phi+tf))$ is differentiable at $t = 0$, with derivative $\frac{d}{dt}E(\phi + tf)|_{t=0} = \int f\,\mathrm{MA}(\phi)$.*

Granted this result, let us show how to solve the Monge–Ampère equation. Pick $\phi \in \mathcal{E}^1(L^{\mathrm{an}})$ that maximizes $F_\mu(\phi) = E(\phi) - \int(\phi - \phi_0)\mu$ and consider any $f \in C^0(X^{\mathrm{an}})$. For any $t \in \mathbf{R}$ we have

$$\begin{aligned} E(P(\phi + tf)) - \int(\phi + tf - \phi_0)\mu &\leq E(P(\phi + tf)) - \int(P(\phi + tf) - \phi_0)\mu \\ &\leq E(\phi) - \int(\phi - \phi_0)\mu. \end{aligned}$$

Since the left-hand side is differentiable at $t = 0$, the derivative must be zero, which amounts to $\int f\,\mathrm{MA}(\phi) - \int f\mu = 0$. Since $f \in C^0(X^{\mathrm{an}})$ was arbitrary, this means that $\mathrm{MA}(\phi) = \mu$, as desired.

The proof of this differentiability results proceeds by first reducing to the case when ϕ and f are continuous. A key ingredient is then

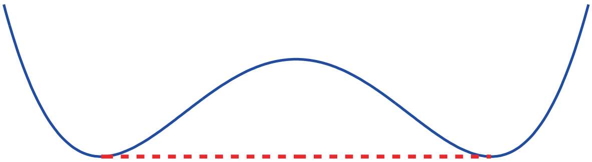

Fig. 1 The convex hull $P(f)$ of a continuous function f of one variable. Note that $P(f)$ is affine, i.e., $P(f)'' = 0$ where $P(f) \neq f$

Theorem 9.2. *For any continuous metric ϕ on L^{an} we have*

$$\int_{X^{\mathrm{an}}} (\phi - P(\phi))\, \mathrm{MA}(P(\phi)) = 0. \tag{4}$$

In other words, the Monge–Ampère measure $\mathrm{MA}(P(\phi))$ is supported on the locus $P(\phi) = \phi$. A version of this for functions of one variable is illustrated in Fig. 1.

To prove this result, we can reduce to the case when ϕ is a smooth/model metric. In the complex case, Theorem 9.2 was proved by Berman and the first author in [5] using the pluripotential theoretic technique known as "balayage." In the non-Archimedean setting, Theorem 9.2 is deduced in [13] from the asymptotic orthogonality of Zariski decompositions in [12] and is for this reason called the *orthogonality property*. The assumption in Theorem 5.1 that the variety X be defined over a smooth k-curve is used exactly in order to apply the result from [12].

The solution to the non-Archimedean Monge–Ampère equation $\mathrm{MA}(\phi) = \mu$ can be made slightly more explicit in the case when the support of μ is a singleton, $\mu = d_L\, \delta_x$, where $d_L := (L^n)$ and $x \in X^{\mathrm{an}}$ belongs to some dual complex; such points x are known as *quasimonomial* or *Abhyankar* points.

The fiber L_x^{an} of L^{an} above $x \in X^{\mathrm{an}}$ is isomorphic to the Berkovich affine line over the complete residue field $\mathcal{H}(x)$. Fix any nonzero $y \in L_x^{\mathrm{an}}$ and set

$$\phi_x := \sup\{\phi \in \mathrm{PSH}(L^{\mathrm{an}}) \mid \|y\|_\phi \geq 1\}.$$

By Boucksom et al. [13, Proposition 8.6], $\mathrm{MA}(\phi_x)$ is supported on x, so $\mathrm{MA}(\phi_x) = d_L\, \delta_x$. It would be interesting to find an example of a divisorial point $x \in X^{\mathrm{an}}$ such that ϕ_x is not a model function.

10 Curves

The disadvantage of the variational approach to the Monge–Ampère equation is that it gives very little control on the solution beyond continuity. Here we shall make the solution more concrete in the case of curves; the next section deals with toric varieties.

Thus assume that X is a smooth projective curve over K. In this case, the Monge–Ampère operator (which one would normally refer to as the Laplacian) is *linear*: if ϕ_i is a metric on L_i, $i = 1, 2$, then $\mathrm{MA}(\phi_1 + \phi_2) = \mathrm{MA}(\phi_1) + \mathrm{MA}(\phi_2)$. Furthermore, as we shall see, the Monge–Ampère operator is naturally defined on *any* singular semipositive metric on L^{an}, for an ample line bundle L, and we can solve $\mathrm{MA}(\phi) = \mu$ for *any* positive measure μ of mass $\deg L$.

Let us first explain this in the complex case; X^{an} is then a compact Riemann surface. Fix a smooth metric ϕ_0 on L^{an}. The curvature form $\omega_0 := dd^c\phi_0$ is a volume form of mass $\deg L$. A singular metric ϕ on L^{an} is then semipositive iff $\varphi := \phi - \phi_0$ is an ω_0*-psh function*, that is, a locally integrable function φ that is locally the sum of a smooth function and a psh function, and such that $\omega_0 + dd^c\varphi$ is a positive measure. We then set $\mathrm{MA}(\phi) := \omega_0 + dd^c\phi$; this definition does not depend on the choice of ϕ_0.

Now we explain how to solve the equation $\mathrm{MA}(\phi) = \mu$ for any positive measure μ of mass $d_L := \deg L$. Writing $\phi = \phi_0 + \varphi$ as above, we must solve $dd^c\varphi = \mu - \omega_0$, where $\omega_0 := dd^c\phi_0$. It suffices to do this when $\mu = d_L\delta_x$ for some $x \in X^{\mathrm{an}}$: indeed, if we normalize the solution φ_x to $dd^c\varphi_x = \mu - d_L\delta_x$ by $\int_{X^{\mathrm{an}}} \varphi_x\,\omega_0 = 0$, then the function φ_μ defined by $\varphi_\mu(y) := d_L^{-1} \int_{X^{\mathrm{an}}} \varphi_x(y)\,d\mu(x)$ satisfies $dd^c\varphi_\mu = \omega_0 - \mu$ and is normalized by $\int_{X^{\mathrm{an}}} \varphi_\mu\,\omega_0 = 0$.

The function φ_x can be "physically" interpreted as the voltage (suitably normalized) when putting a charge of $+d_L$ at the point x and a total charge of $-d_L$ spread out according to the measure ω_0. Mathematically, Perron's method describes it as the supremum of all ω_0-subharmonic functions φ on X^{an} satisfying $\int_{X^{\mathrm{an}}} \varphi\,\omega_0 = 0$ and $\varphi \le d_L \log|z| + O(1)$, where z is a local coordinate at x.

Now we consider the non-Archimedean case. As before, let us assume that K is a discretely valued field of residue characteristic zero, even though this is not really necessary in the one-dimensional case.[3]

The main point is that any Berkovich curve has the structure of a generalized[4] metric graph. We will not describe this in detail, but here is the idea. The dual graph $\Delta_{\mathcal{X}}$ of any SNC model $\mathcal{X}$ is a connected, one-dimensional simplicial complex. As before, we view it as a subset of X^{an}. It carries a natural integral affine structure, inducing a metric. If $\mathcal{X}'$ is an SNC model dominating $\mathcal{X}$ (in the sense that the canonical birational map $\mathcal{X} \dashrightarrow \mathcal{X}'$ is a morphism), then $\Delta_{\mathcal{X}}$ is a subset of $\Delta_{\mathcal{X}'}$ and the inclusion $\Delta_{\mathcal{X}} \to \Delta_{\mathcal{X}'}$ is an isometry. There is a also a (deformation) retraction $r_{\mathcal{X}} : X^{\mathrm{an}} \to \Delta_{\mathcal{X}}$, and $X^{\mathrm{an}} \simeq \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$. In this way, the metrics on the dual complexes induce a generalized metric on X^{an}.

The structure of each $\Delta_{\mathcal{X}}$ and of X^{an} as metric graphs allows us to define a Laplacian on these spaces, by combining the real Laplacian on segments and the combinatorial Laplacian at branch points (and endpoints). This Laplacian allows us to understand both semipositive singular metrics and the Monge–Ampère operator.

Namely, fix a model metric ϕ_0 on L^{an}. It is represented by a $\mathbf{Q}$-line bundle on some SNC model $\mathcal{X}^{(0)}$. The measure $\omega_0 := dd^c\phi_0$ is supported on the vertices of $\Delta_{\mathcal{X}^{(0)}}$. Now, a singular metric ϕ is semipositive iff for every SNC model $\mathcal{X}$ dominating $\mathcal{X}^{(0)}$, the restriction of the function $\phi - \phi_0$ to $\Delta_{\mathcal{X}}$ is a ω_0-subharmonic

[3] Indeed, Thuillier [47] systematically develops a potential theory on Berkovich curves in a very general setting.

[4] This means that some distances may be infinite.

function in the sense that $\Delta((\phi - \phi_0)|_{\Delta_{\mathcal{X}}}) = \mu_{\mathcal{X}} - \omega_0$, where $\mu_{\mathcal{X}}$ is a positive measure on $\Delta_{\mathcal{X}}$ of mass d_L. In this case, there further exists a unique measure μ on X^{an} of mass d_L such that $(r_{\mathcal{X}})_*\mu = \mu_{\mathcal{X}}$ for all $\mathcal{X}$, and we have $\mathrm{MA}(\phi) = \mu$.

To solve the equation $\mathrm{MA}(\phi) = \mu$ for a positive measure μ of mass d_L, it suffices by linearity to treat the case $\mu = d_L\delta_x$ for a point $x \in X^{\mathrm{an}}$. In this case, the function $\varphi := \phi - \phi_0$ will be locally constant outside the convex hull of $\{x\} \cup \Delta_{\mathcal{X}^{(0)}}$. The latter is essentially a finite metric graph on which we need to find a function whose Laplacian is equal to $d_L\delta_x - \omega_0$. This can be done in a quite elementary way.

An interesting example of semipositive metrics, both in the complex and non-Archimedean case, comes from dynamics [51]. Suppose $f : (X, L) \circlearrowleft$ is a polarized endomorphism of degree $\lambda > 1$. In other words, $f : X \to X$ is an endomorphism and f^*L is linearly equivalent to λL. Then there exists a unique *canonical metric* ϕ_{can} on L^{an}, satisfying $f^*\phi = \lambda\phi$. This metric is continuous and semipositive but usually not a model metric.

As a special case, suppose X is an elliptic curve and that f is the map given by multiplication by λ. In the complex case, $X^{\mathrm{an}} \simeq \mathbf{C}/\Lambda$ is a torus and $\mu_{\mathrm{can}} := \mathrm{MA}(\phi_{\mathrm{can}})$ is given by a multiple of Haar measure on X^{an}. In the non-Archimedean case, there are two possibilities. If X has good reduction over $\mathrm{Spec}\, R$, then μ_{can} is a point mass. Otherwise, μ_{can} is proportional to Lebesgue measure on the *skeleton* $\mathrm{Sk}(X^{\mathrm{an}})$, a subset homeomorphic to a circle. A similar description of the measure μ_{can} in the case of higher-dimensional abelian varieties is given in [29].

11 Toric Varieties

For general facts about toric varieties, see [18, 27, 36]. In this section we briefly describe how the complex and non-Archimedean points of view elegantly come together in the toric setting and translate into statements about convex functions and the real Monge–Ampère operator. As before, we only consider the non-Archimedean field $K = k((t))$ with $\mathrm{char}\, k = 0$; however, most of what we say here should be true in a more general context: see [30].

Let $M \simeq \mathbf{Z}^n$ be a free abelian group, N its dual, and let $T = \mathrm{Spec}\, K[M]$ be the corresponding split K-torus. A polarized toric variety (X, L) is then determined by a rational polytope $\Delta \subset M_{\mathbf{R}}$. The variety X is described by the normal fan to Δ in $N_{\mathbf{R}}$ and the points of $M \cap \Delta$ are in 1–1 correspondence with equivariant sections of L; we write χ^u for the section of L associated with $u \in M$. This description is completely general and holds over any field as well as over $\mathbf{Z}$.

There is also a "tropical" space X^{trop} associated with X. As a topological space, it is compact and contains $N_{\mathbf{R}}$ as an open dense subset.[5] For any valued field K, there

[5]In our setting, X^{trop} can be identified with the (moment) polytope Δ in such a way that $N_{\mathbf{R}}$ corresponds to the interior of Δ, but this identification does not preserve the affine structure on $N_{\mathbf{R}}$.

is a tropicalization map trop : $X^{\mathrm{an}} \to X^{\mathrm{trop}}$, where X^{an} refers to the analytification with respect to the norm on K. The inverse image of $N_{\mathbf{R}}$ is the torus T^{an}.

There is a natural correspondence between equivariant metrics on L^{an} and functions on $N_{\mathbf{R}}$. Let ϕ is an equivariant metric on L^{an}. For every $u \in M$, χ^u is a nonvanishing section of L on T so $\phi - \log|\chi^u|$ defines a function on T^{an} that is constant on the fibers of the tropicalization map. In particular, picking $u = 0$, we can write

$$\phi - \log|\chi^0| = g \circ \mathrm{trop} \tag{5}$$

for some function g on $N_{\mathbf{R}}$. Conversely, given a function g on $N_{\mathbf{R}}$, (5) defines an equivariant metric on the restriction of L^{an} to T^{an}.

We now go from the torus T to the polarized variety (X, L). After replacing L by a multiple, we may assume that all the vertices of Δ belong to M. Set

$$\phi_\Delta := \max_{u \in \Delta} \log|\chi^u|.$$

This is a semipositive, equivariant model metric on L^{an}. Its restriction to T^{an} corresponds to the homogeneous, nonnegative, convex function

$$g_\Delta := \max_{u \in \Delta} u$$

on $N_{\mathbf{R}}$. In general, an equivariant singular metric ϕ on L^{an} corresponds to a convex function g on $N_{\mathbf{R}}$ such that $g \leq g_\Delta + O(1)$. It is bounded iff $g - g_\Delta$ is bounded on $N_{\mathbf{R}}$.

The real Monge–Ampère measure of any convex function g on $N_{\mathbf{R}}$ is a well-defined positive measure $\mathrm{MA}_{\mathbf{R}}(g)$ on $N_{\mathbf{R}}$ (see, e.g., [46]). When $g = g_\Delta + O(1)$, its total mass is given by

$$\int_{N_{\mathbf{R}}} \mathrm{MA}_{\mathbf{R}}(g) = \mathrm{Vol}(\Delta) = \frac{(L^n)}{n!},$$

where the last equality follows from [27, p. 111].

We now wish to relate the real Monge–Ampère measure of g and the Monge–Ampère measure of the corresponding semipositive metric ϕ on L^{an}.

First consider the non-Archimedean case, in which there is a natural embedding $j : N_{\mathbf{R}} \to T^{\mathrm{an}} \subset X^{\mathrm{an}}$ given by monomial valuations that sends $v \in N_{\mathbf{R}}$ to the norm

$$\sum_{u \in M} a_u u \in K[M] \mapsto \max_{u \in M}\{|a_u| \exp(-\langle u, v\rangle)\}.$$

In particular, $j(0) = x_G$, the Gauss point of the open T-orbit.

If g is a convex function on $N_{\mathbf{R}}$ with $g = g_\Delta + O(1)$, and if ϕ is the corresponding continuous semipositive metric on L, then [18, Theorem 4.7.4] asserts that

$$\mathrm{MA}(\phi) = n!\, j_* \mathrm{MA}_{\mathbf{R}}(g).$$

For a compactly supported positive measure ν on $N_{\mathbf{R}}$ of mass (L^n), solving the Monge–Ampère equation $\mathrm{MA}(\phi) = j_*(\nu)$ therefore amounts to solving the real Monge–Ampère equation $\mathrm{MA}_{\mathbf{R}}(g) = \nu/n!$. This can be done explicitly when ν is a point mass, say, supported at $v_0 \in N_{\mathbf{R}}$. Indeed, the function $g_{v_0} : N \to \mathbf{R}$ defined by $g = g_\Delta(\cdot - v_0)$ is convex and satisfies $g = g_\Delta + O(1)$. Further, for every point $v \neq v_0$ there exists a line segment in $N_{\mathbf{R}}$ containing v in its interior and on which g is affine. This implies that $\mathrm{MA}_{\mathbf{R}}(g)$ is supported at v_0. As a consequence, the corresponding continuous metric ϕ on L^{an} satisfies $\mathrm{MA}_{\mathbf{R}}(\phi) = (L^n)\delta_{j(u_0)}$.

This solution can be shown to tie in well with the construction at the end of Sect. 9, but is of course much more explicit. For example, when $u_0 \in N_{\mathbf{Q}}$, so that $j(u_0) \in X^{\mathrm{an}}$ is divisorial, the function g_{u_0} is $\mathbf{Q}$-piecewise linear so that the corresponding metric ϕ is a model metric.

Finally we consider the complex case. In this case we cannot embed $N_{\mathbf{R}}$ in T^{an}. However, the preimage of any point $v \in N_{\mathbf{R}}$ under the tropicalization is a real torus of dimension n in T^{an} on which the multiplicative group $(S^1)^n$ acts transitively. To any compactly supported positive measure ν on $N_{\mathbf{R}}$ of mass $(L^n)/n!$ we can therefore associate a unique measure μ on T^{an}, still denoted $\mu := j_*\nu$, that is invariant under the action of $(S^1)^n$ and satisfies $\mathrm{trop}_*\,\mu = \nu$.

If ϕ is an equivariant semipositive metric on L^{an}, corresponding to a convex function g on $N_{\mathbf{R}}$, we then have

$$\mathrm{MA}(\phi) = n!\,j_*\,\mathrm{MA}_{\mathbf{R}}(g).$$

For $(S^1)^n$-invariant measures μ on L^{an} of mass (L^n), solving the complex Monge–Ampère equation $\mathrm{MA}(\phi) = \mu$ thus reduces to solving the real Monge–Ampère equation $\mathrm{MA}_{\mathbf{R}}(g) = \frac{1}{n!}\,\mathrm{trop}_*\,\mu$.

12 Outlook

In this final section we indicate some possible extensions of our work and make a few general remarks.

First of all, it would be nice to have a *local* theory for semipositive singular metrics. Indeed, while the global approach in [13, 14] works well for the Calabi–Yau problem, it has some unsatisfactory features. For example, it is not completely trivial to prove that the Monge–Ampère operator is local in the sense that if ϕ_1, ϕ_2 are two (say) continuous semipositive metrics that agree on an open subset $U \subset X^{\mathrm{an}}$, then $\mathrm{MA}(\phi_1) = \mathrm{MA}(\phi_2)$ on U. We prove this in [13] using the Monge–Ampère capacity. Still, it would be desirable to say that the restriction of a semipositive metric to (say) an open subset of X^{an} remains semipositive!

In contrast, in the complex case, the classical approach is local in nature. Namely, one first defines and studies psh functions on open subsets of $\mathbf{C}^n$ and then defines singular semipositive metrics as global analogues. By construction, the Monge–Ampère operator is a local (differential) operator.[6]

In a general non-Archimedean setting, Chambert-Loir and Ducros [22] (see also [31, 32]) define psh functions as continuous functions φ such that $d'd''\varphi$ is a positive closed current (in their sense), for suitable operators d' and d'' analogous to their complex counterparts and modeled on notions due to Lagerberg [41]. While this leads to a very nice theory, that moreover works for general Berkovich spaces, the crucial compactness and regularization results are so far missing. At any rate, the tropical charts used in [22] may be a good substitute for dual complexes of SNC models.

Going back to the projective setting, there are several open questions and possible extensions, even in the case of a discretely valued ground field of residue characteristic zero.

First, when solving the Monge–Ampère equation, we needed to assume that the variety X was obtained by base change from a variety over a k-curve. This assumption was made in order to use the orthogonality result in [12], but is presumably redundant.

Second, one should be able to solve the Monge–Ampère equation $\mathrm{MA}(\phi) = \mu$ for more general measures μ. In the complex setting, this is done in [24, 34] for non-pluripolar measures μ. The analogous result should be valid in the non-Archimedean setting, too, although some countability issues seem to require careful attention. Having such a general result would allow for a nice Legendre duality, as explored in [4, 6] in the complex case.

Third, one could try to get more specific information about the solution. We already mentioned at the end of Sect. 9 that we don't know whether the solution to the equation $\mathrm{MA}(\phi) = d_L\,\mu_x$ is a model function for x a divisorial point (and $d_L = (L^n)$). In a different direction, one could consider the case when X is a Calabi–Yau variety, in the sense that $K_X \simeq \mathcal{O}_X$. Then there exists a canonical subset $\mathrm{Sk(X)} \subset X^{\mathrm{an}}$, the *Kontsevich–Soibelman skeleton*, see [39, 44, 45]. It is a subcomplex of the dual complex of any SNC model and comes equipped with an integral affine structure, inducing a volume form on each face. One can solve $\mathrm{MA}(\phi) = \mu$, for linear combinations of these volume forms, viewed as measures on X^{an}. Can we say anything concrete about the solution ϕ, as in the case of maximally degenerate abelian varieties considered in [43]?

It would obviously be interesting to work over other types of non-Archimedean fields, such as $\mathbf{Q}_p$. Here there are several challenges. First, we systematically use SNC models, which are only known to exist in residue characteristic zero (except in low dimensions). It is possible that the tool of SNC models can, with some additional effort, be replaced by alterations, tropical charts, or other methods. However, we also crucially use the assumption of residue characteristic

[6] However, one also needs to verify that the Monge–Ampère operator is local for the *plurifine* topology. This is nontrivial in both the complex and non-Archimedean case.

zero when applying the vanishing theorems that underlie the regularization theorem for singular semipositive metrics. Here some new ideas are needed.

A simpler situation to handle is that of a *trivially* valued field. This is explored in [17] and can be briefly explained as follows. Let k be any field of characteristic zero, equipped with the trivial norm. Let (X, L) be a polarized variety over k. In this setting, the notion of model metrics and model functions seemingly does not take us very far, as the only model of X is X itself! Instead, the idea is to use a non-Archimedean field extension. Set $K = k((t))$, $X_K := X \otimes_k K$, etc. The multiplicative group $G := \mathbf{G}_{m,k}$ acts on X_K^{an} and X^{an} can be identified with the set of G-equivariant points in X_K^{an}. Similarly, singular semipositive metrics on L^{an} are defined as G-invariant singular semipositive metrics on L_K^{an}. In this way, the main results about $\mathrm{PSH}(L^{\mathrm{an}})$ follow from the corresponding results about $\mathrm{PSH}(L_K^{\mathrm{an}})$ and the same is true for the solution of the Monge–Ampère equation.

A primary motivation for studying the trivially valued case, at least in the case $k = \mathbf{C}$, is that the space of singular semipositive metrics on L^{an} naturally sits "at the boundary" of the space of positive (Kähler) metrics on the holomorphic line bundle L. As such, it can be used to study questions on K-stability and may be useful for the study of the existence of constant scalar curvature metrics, see [15, 16]. A different scenario where a complex situation degenerates to a non-Archimedean one occurs in [35].

In yet another direction, one could try to consider line bundles that are not necessarily ample, but rather big and nef, or simply big. In the complex case this was done in [7, 25]. One motivation for such a generalization is that it is invariant under birational maps and would hence allow us to study singular varieties.

Finally, it would be interesting to have transcendental analogues. Indeed, in the complex case, one often starts with a Kähler manifold X together with a Kähler class ω, rather than a polarized pair (X, ω). A notion of Kähler metric is proposed in [40, 49], but it is not clear whether or not this plays the role of a (possibly) transcendental Kähler metric.

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