Singularity Analysis of a Novel Minimally-Invasive-Surgery Hybrid Robot Using Geometric Algebra

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Abstract The paper presents an analysis of the singularities of a novel type of medical robot for minimally invasive surgery (MIS) using the language of the geometric algebra. The analysis focuses on the parallel manipulator, which is the key component of the robot. The proposed new parallel manipulator provides a remote centre of motion located at the incision point of the patient's body. The aim of the paper is to derive the geometric condition for singularity in terms of geometric algebra and thus to reveal the singular configurations in order to avoid them during the surgical procedure. The obtained geometric condition for singularity leads further to the derivation of the algebraic formulation of the singularity surface which is graphically presented.

Keywords Singularity · Parallel robot · Kinematics · Minimally invasive surgery · Geometric algebra

1 Introduction

Surgical robots enhance the capability of surgeons and allow the development of new operating procedures in the treatment of diseases. These techniques are less invasive and more precise, which leads to better clinical outcome, reduced hospital stay and decreased trauma for the patients. At the end of 20th century, the traditional open surgery evolved to minimally invasive surgery and further to robotised minimally invasive surgery. In the innovative minimally invasive surgery, long surgical instruments are inserted into the human body through small incisions (less than 10 mm in diameter) on the abdomen. The surgeon holds and manoeuvres the instruments thus operating remotely through them. The next logical step of development of the minimally invasive technique was to replace the surgeon's hands with robot arms guided by surgeons, who monitor remotely the internal movements of the instruments through video images delivered by an endoscopic camera.

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From kinematics point of view, the small abdominal incision restricts the motion of the instrument and acts as a pivoting point. Thus, only four degrees of freedom are possible, three rotations around three intersecting axes in the pivoting point and a translation along the longitudinal axis of the surgical tool. The term Remote Centre-of-Motion (RCM) [23] is used to describe this specific movement. RCM is a point where one or more rotations are centred and located outside the mechanism itself. Robots for minimally invasive surgery utilize several types of mechanisms which can enforce RCM. Zeus medical system applies an isocentre-based RCM mechanism [11], while da Vinci surgical system [5] and BlueDRAGON [18] use parallelograms for RMC mechanism. The spherical mechanism is preferably a RCM mechanism in many proposed devices for minimally invasive surgery [15, 24]. Parallel manipulators with spherical linkages have also been proposed especially for laparoscopic surgery [10, 13]. Some other parallel and hybrid manipulators have been suggested for application in laparoscopy [17, 19]. Obviously, the general type of parallel manipulators used in the robot-assisted minimally invasive surgery need to provide programmable RCM [2]. The mechanical RCMs are considered to be more suitable for the robot-assisted minimally invasive surgery due to the fewer degrees of freedom, the simpler control system and safer manipulation.

The singular configurations in parallel manipulators should be avoided during the robot motion since in such configurations the manipulator may have undesired behaviour and compromised performance. This is especially important for medical robots as such undesired behaviour could endanger both the outcome of the surgical procedure and the safety of the patient. That is why the singularities in parallel medical manipulators need to be well analysed and established and therefore should be avoided during surgical manipulations. Many researchers have studied the singularities of parallel manipulators using different mathematical tools. Geometric methods are applied by many researchers [6, 14, 25]. Some "non-traditional" methods, such as Grassmann geometry [16], Grassmann-Cayley algebra [1, 9] and geometric algebra [20, 22], have been used for singularity analysis of parallel manipulators.

This paper utilizes geometric algebra to geometrically characterize the singularities of a novel hybrid robot, especially the parallel manipulator of the robot. The proposed hybrid robot consists of a parallel manipulator and an additional translational joint attached to the moving platform. This robot, which was first proposed in [21], possesses the ability of the spherical linkage to provide remote centre of motion and overcomes some drawbacks of the parallel spherical manipulators, such as a possible clogging and a bulky structure.

2 Robot Design and Kinematics

The considered minimally-invasive-surgery hybrid robot consists of a parallel manipulator and an additional translational joint attached to the moving platform [21]. The novel parallel manipulator, which is a key component of the considered hybrid robot,

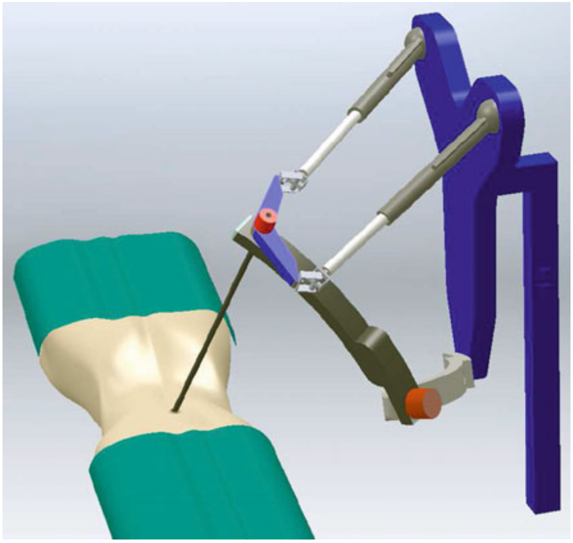

Fig. 1 A CAD model of the robot for minimally invasive surgery

has three limbs (legs) and 2SPU1RRR structure. A CAD model of the considered hybrid robot is shown in Fig. 1.

Two limbs (A_1B_1 and A_2B_2) have an identical SPU (spherical-prismatic-universal) joints structure and the third one (A_3B_3) is a RRR (R stands for revolute) spherical linkage, i.e. the axes of all three revolute joints (R) of this limb intersect in a single point (Fig. 2). An additional prismatic joint is attached to the moving platform, which provides a translation along the longitudinal axis of the surgical instrument (end-effector). The parallel manipulator provides RCM of the end-effector and the RCM is at the incision point of the patient's body. Thus, the parallel mechanism has three degrees of freedom and the hybrid robot has four degrees of freedom in total.

Each limb has one driven joint, respectively. The prismatic joints of the SPU limbs and the second (middle) revolute joint of the RRR limb are driven. In addition to these three active joints for the parallel mechanism, an active prismatic joint allowing translation of the end-effector along the line OB_3 is added. The axis of the revolute joint (at A_3), attached to the base platform, is perpendicular to the plane of the base platform ($A_1A_2A_3$) and the axis of the revolute joint (at B_3), attached to the moving platform, is perpendicular to the plane of the moving platform ($B_1B_2B_3$). The origins of the reference (base) coordinate system OXYZ and the coordinate systems $\{1\}$, $\{2\}$ and $\{3\}$ coincide with the intersection point of the three axes of the revolute joints of the spherical RRR limb. The Z_i-axes ($i = 1, 2, 3$) are along the axes of the three revolute joints, respectively. The $O_4X_4Y_4Z_4$ coordinate system is attached to the moving platform and Z_4-axis is along the direction of translation of the last prismatic joint.

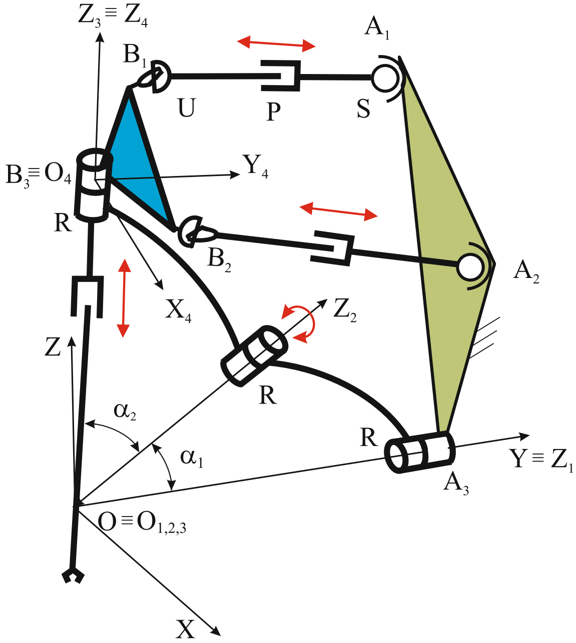

Fig. 2 Kinematic scheme of the robot

2.1 Robot Kinematics

The kinematics of the robot is presented very briefly here, as a more detailed description is given in [21]. The kinematic modelling of the parallel mechanism is parametrized by the three angles of rotation (θ_1, θ_2 and θ_3) about the three Z_i ($i = 1, 2, 3$) axes. Using the D-H (Denavit-Hartenberg) notations, the transformation of the coordinate systems can be written as a product of the following rotation (R) and translation (T) matrices

$$\mathbf{Q}_i = R(\alpha_{i-1})T(a_{i-1})R(\theta_i)T(d_i). \tag{1}$$

The D-H parameters are given in Table 1, where α_1, α_2, d_4 are constant design parameters and θ_1, θ_2, θ_3 are joint variables (α_i is angle between Z_i and Z_{i+1} axes measured about X_i, $i = 1, 2$, i.e., α_1 and α_2 are angles between joint axes of the RRR spherical linkage and are determined by the design of the links of this linkage; $d_4 = \|\mathbf{OO}_4\|$).

Then, the transformation matrix for the considered robot manipulator can be written as follows

$$\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4. \tag{2}$$

For the forward position problem, the following constrained equations can be written :

$$L_i = \|\mathbf{OA}_i - \mathbf{OB}_i\|, \; i = 1, 2, \tag{3}$$

Table 1 D-H parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$-\pi/2$	0	0	θ_1
2	α_1	0	0	θ_2
3	α_2	0	0	θ_3
4	0	0	d_4	0

where $\|...\|$ denotes the Euclidean norm; $\mathbf{OB}_i = \mathbf{Q} \cdot (\mathbf{O}_4\mathbf{B}_i)$; $\mathbf{O}_4\mathbf{B}_i$ is a vector given in the $O_4X_4Y_4Z_4$ coordinate system. The vectors $\mathbf{OA}_i$ and $\mathbf{O}_4\mathbf{B}_i$ are determined by the design of the manipulator.

The two leg lengths (L_1 and L_2) and the angle θ_2 are the given parameters for the forward problem of the parallel mechanism. Expanding Eq. 3, we get the following two equations for the unknown variables θ_1 and θ_3

$$p_{i1}c_1c_3 + p_{i2}c_1s_3 + p_{i3}s_1c_3 + p_{i4}s_1s_3 + p_{i5}c_1 + p_{i6}s_1 + p_{i7}c_3 + p_{i8}s_3 + p_{i9} = 0, \quad (4)$$

where $c_i = \cos\theta_i$, $s_i = \sin\theta_i$ (i = 1, 2); $p_{i1}, p_{i2}, \ldots, p_{i9}$ (i = 1, 2) are coefficients which are determined by the design and input parameters.

Applying tangent-half formulas in Eq. 4 and eliminating one of the two unknowns, an eight-order polynomial in one unknown is obtained [21].

3 Singularity Analysis of a General Parallel Robot with Limited Mobility Using Geometric Algebra

The geometric algebra approach for singularity analysis of parallel manipulators with limited mobility has been developed and presented by the author in two previous papers [20, 22]. A brief recollection of the approach is given here. Firstly, a very concise introduction to the geometric algebra is presented in this section, since extensive treatments can be found in [3, 4, 7].

3.1 Concise basics of the geometric algebra

Clifford algebra was created in the 19th century. In the second half of the 20th century this algebra has been rediscovered and further developed into a unified language named geometric algebra by Hestenes [7], Lasenby and Doran [3], Dorst, Fontijne and Mann [4], and some other authors. In geometric algebra, a single basic kind of multiplication called *geometric product* between two vectors is defined. The geometric product of two vectors **a** and **b** can be decomposed into symmetric and antisymmetric parts [7]. i.e.

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \tag{5}$$

where $\mathbf{a} \cdot \mathbf{b}$ is the *inner product* and $\mathbf{a} \wedge \mathbf{b}$ is the *outer product* of the two vectors.

The inner product $\mathbf{a} \cdot \mathbf{b}$ is a scalar-valued (grade 0). The result of the other product $\mathbf{a} \wedge \mathbf{b}$ is an entity called *bivector* (grade 2). Higher-grade elements can be constructed by introducing more vectors. The outer product of k vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ generates a new entity $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k$ called a k blade. The integer k is named grade. The geometric algebra G_n contains nonzero blades of maximum grade n which are called pseudoscalars of G_n. The unit pseudoscalar of G_3 of 3-D Euclidean metric space with the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ could be written as

$$I_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \tag{6}$$

The inverse of the unit pseudoscalar of G_3 is

$$I_3^{-1} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3. \tag{7}$$

The idea for a unit pseudoscalar and its inverse can be extended for an n-dimensional Euclidean metric space. The inverse of the unit pseudoscalar is used for obtaining the dual of a blade, which will be applied in the next sections.

An addition between elements of different grades results in an entity called a *multivector*. Inner and outer products are dual to one another and the following identities could be written [8]

$$(\mathbf{a} \cdot \mathbf{M})I_n = \mathbf{a} \wedge (\mathbf{M}I_n), \tag{8}$$

$$(\mathbf{a} \wedge \mathbf{M})I_n = \mathbf{a} \cdot (\mathbf{M}I_n), \tag{9}$$

where $\mathbf{a}$ is a vector; $\mathbf{M}$ is multivector and I_n is the unit pseudoscalar of n-dimensional space.

3.2 Geometric Algebra Approach to Singularity of Parallel Manipulators with Limited Mobility

We need to represent screws and joint screws in terms of the geometric algebra. Any oriented line $\mathbf{l}$ is uniquely determined by giving its direction $\mathbf{u}$ and its moment, thus, in the geometric algebra of 3-D vector space with the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ it can be written as [7]

$$\mathbf{l} = \mathbf{u} + \mathbf{r} \wedge \mathbf{u}, \tag{10}$$

where $\mathbf{r} = r_x\mathbf{e}_1 + r_y\mathbf{e}_2 + r_z\mathbf{e}_3$ is the position vector of a point on the line; $\mathbf{u} = u_x\mathbf{e}_1 + u_y\mathbf{e}_2 + u_z\mathbf{e}_3$ is a vector along the line.

Thus, in the geometric algebra of the 3-D vector space G_3, a line is expressed as a multivector composed from a vector part plus a bivector. An extension of the equation of the line (Eq. (10)), i.e. adding the moment corresponding to the pitch, leads to the equation of a general screw:

$$\mathbf{s} = \mathbf{u} + \mathbf{r} \wedge \mathbf{u} + h\mathbf{u}I_3 \equiv v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + b_1\mathbf{e}_2 \wedge \mathbf{e}_3 + b_2\mathbf{e}_3 \wedge \mathbf{e}_1 + b_3\mathbf{e}_1 \wedge \mathbf{e}_2, \quad (11)$$

where $v_i (i = 1, 2, 3)$ and $b_i (i = 1, 2, 3)$ are scalar coefficients; $I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit pseudoscalar of G_3; h is the pitch of the screw.

The joint screw of a rotational joint is a line (screw with a zero pitch) and it is given by Eq. (10), while the joint screw of a prismatic joint can be written as follows

$$\mathbf{s} = \mathbf{w}I_3 \equiv w_1\mathbf{e}_2 \wedge \mathbf{e}_3 + w_2\mathbf{e}_3 \wedge \mathbf{e}_1 + w_3\mathbf{e}_1 \wedge \mathbf{e}_2, \quad (12)$$

where $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ is a unit vector along the translation direction of the prismatic joint.

In Eq. (11), the screw is expressed as a multivector in G_3. It could also be expressed as a vector in the geometric algebra G_6. In the geometric algebra of 6-D vector space with the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6$, a screw can be written as a vector (grade 1), i.e.,

$$\mathbf{S} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + b_1\mathbf{e}_4 + b_2\mathbf{e}_5 + b_3\mathbf{e}_6, \quad (13)$$

where the coefficients are the same as in Eq. (11).

In this paper, the screws written in G_3 and G_6 are distinguished by the following notations: a lower case letter (**s**, **l**) denotes a screw written as a multivector in G_3 of 3-D space; an upper case letter (**S**, **L**) denotes a screw written as a vector in G_6 of 6-D space; letters with a tilde mark ($\tilde{\mathbf{s}}, \tilde{\mathbf{S}}$) denote the elliptic polars of the screws (**s** and **S**), given in G_3 and G_6 , respectively.

The relationship between twists of freedom and the wrenches of constraint, as well as the duality in geometric algebra, are employed for the development of the geometric algebra approach to singularity analysis of parallel manipulators with limited mobility. The twists of non-freedom (wrenches of non-constraint) and wrenches of constraint (twists of freedom) are elliptic polars; twists of freedom (wrenches of constraint) and twists of non-freedom (wrenches of non-constraint) are orthogonal complements which together span a six space [12]. In terms of the geometric algebra, the operation of transformation of a screw (Eq. (11)) into an elliptic polar screw can be written as

$$\mathbf{s} = I_3\langle\mathbf{s}\rangle_1 + I_3^{-1}\langle\mathbf{s}\rangle_2 \equiv b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + v_1\mathbf{e}_2 \wedge \mathbf{e}_3 + v_2\mathbf{e}_3 \wedge \mathbf{e}_1 + v_3\mathbf{e}_1 \wedge \mathbf{e}_2, \quad (14)$$

where $\langle\mathbf{s}\rangle_k$ denotes the k-vector part of **s**; I_3^{-1} is the inverse of the unit pseudoscalar I_3 for the geometric algebra G_3.

In case of a parallel manipulator with fewer than six degrees of freedom, some legs may not possess full mobility. A leg with full mobility and a leg with less than six dof could be treated in a similar way. In that case, we suppose that the remaining degrees of freedom are represented by dummy joints (or driven but locked joints) and the associated with them dummy screws. Taking the outer product of five screws of the jth leg, written in G_6, gives the following 5-blade

$$^{j}\mathbf{A}_k =^{j} \mathbf{S}_1 \wedge^{j} \mathbf{S}_2 \wedge \ldots \wedge^{j} \mathbf{S}_{k-1} \wedge^{j} \mathbf{S}_{k+1} \wedge \ldots \wedge^{j} \mathbf{S}_6, \tag{15}$$

where the subscript k denotes the active joint of the jth leg.

The 5-blade from Eq. (15) involves five screws (out of six with the exception of the $^{j}\mathbf{S}_k$ screw). The kth joint is active. In a non-degenerate space, the dual of a blade represents the orthogonal complement of the subspace represented by the blade. The dual of the above 5-blade $^{j}\mathbf{A}_k$ is given by the following formula

$$^{j}\mathbf{D}_k =^{j} \mathbf{A}_k I_6^{-1}, \tag{16}$$

where $I_6 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 e_5\mathbf{e}_6$ is the unit pseudoscalar of the G_6 and I_6^{-1} is its inverse.

Suppose that the parallel manipulator has n legs and m ($m = 6 - q$) degrees of freedom. We assume that the remaining q degrees of freedom are represented by dummy joints (or driven but locked joints) and the associated with them dummy screws. Thus, in non-singular configuration the driven joints and the geometry (or the dummy joints) of the manipulator sustain a general wrench applied to the moving platform. Therefore, the singular configuration can occur when all dual $^{j}\mathbf{D}_k$ or reciprocal $^{j}\mathbf{R}_k$ ($^{j}\mathbf{R}_k =^{j} \tilde{\mathbf{D}}_k$) screws, representing active and dummy joints, are linearly dependent. Using the language of the geometric algebra, the condition of singularity for the parallel manipulator with less than six degrees of freedom (but with dummy joints) can be expressed as

$$\mathbf{D}_{a_1} \wedge \ldots \wedge \mathbf{D}_{a_k} \wedge \mathbf{D}_{d_1} \wedge \ldots \wedge \mathbf{D}_{d_r} = 0, \tag{17}$$

where $k + r = 6$; k is the number of the active joints and r is the number of the dummy joints; $\mathbf{D}_{a_i}$ is a dual vector (grade 1-blade) associated to the ith active joint and $\mathbf{D}_{d_i}$ is a dual vector (grade 1-blade) associated to the ith dummy joint.

Equation (17) involves dummy screws, and therefore, they need to be eliminated. Applying some identities of the geometric algebra [8] and manipulating certain blades from Eq. (17) the dummy screws could be eliminated. Thus, the new equation resulting from Eq. (17) after elimination of the dummy screws will contain only passive joint screws, which was shown in [20, 22]. This process of elimination is explained in more details in the next section.

The effectiveness and advantages of the proposed approach for singularity analysis are summarized next. The approach allows manipulating the components of the equation for singularity in a basis-free manner and thus, establishing and interpreting the singular configurations. This property of the approach could be considered an advantage over the screw Jacobian approach, for example. This advantage is especially useful for parallel manipulators with limited mobility, which is shown in the next section. Although the proposed method utilizes the idea of dummy joints in case of parallel manipulators with limited mobility, there is no need to use a particular dummy joint since the dummy screws are represented only by the space of non-freedom. In addition, the approach can establish the singularities for parallel manipulators with both full and limited mobility as well as distinguish different types of singularities such as constraint ones. A further benefit which arises from the established geometrical condition for singularity is that the algebraic formulation can be obtained from the geometrical condition. This algebraic equation is equivalent to the determinant of the 6×6 screw Jacobian of the manipulator.

4 Singularity Analysis of the Minimally-Invasive-Surgery Parallel Manipulator

In this section, the singularity analysis of the novel robot for minimally invasive surgery is presented. Here, the parallel manipulator, which is the essential part of the robot, is the main subject of the analysis.

4.1 Geometric Condition for Singularity

The two SPU legs (leg 1 and leg 2) of the parallel manipulator have full mobility and therefore, have only dual vectors $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively, associated with the active joints. The third (RRR) leg has one dual vector ($\mathbf{D}_3$) associated with the active joint (middle R joint) and three dual vectors ($\mathbf{D}_4$, $\mathbf{D}_5$ and $\mathbf{D}_6$) associated with the three dummy joints. The dual vectors for the two SPU legs can be written as follows

$$\mathbf{D}_j \equiv^j \mathbf{D}_k = ({}^j\mathbf{S}_1 \wedge^j \mathbf{S}_2 \wedge^j \mathbf{S}_3 \wedge^j \mathbf{S}_5 \wedge^j \mathbf{S}_6) I_6^{-1}, \ (j = 1, 2), \tag{18}$$

where the fourth P (prismatic) joint is active and is not included in the formula.

These dual vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ are lines (zero pitch screws) and their elliptic polars $\mathbf{R}_1$ ($\mathbf{R}_1 = \tilde{\mathbf{D}}_1$) and $\mathbf{R}_2$ ($\mathbf{R}_2 = \tilde{\mathbf{D}}_2$) are lines along the SPU legs, respectively.

The dual vectors (one vector associated with the middle R joint and three vectors associated with the dummy joints) for the third RRR leg are as follows

$$\mathbf{D}_3 \equiv^3 \mathbf{D}_a = ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_1} \wedge^3 \mathbf{S}_{d_2} \wedge^3 \mathbf{S}_{d_3}) I_6^{-1}, \tag{19}$$

$$\mathbf{D}_4 \equiv^3 \mathbf{D}_{d_1} = ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_2 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_2} \wedge^3 \mathbf{S}_{d_3}) I_6^{-1}, \tag{20}$$

$$\mathbf{D}_5 \equiv^3 \mathbf{D}_{d_2} = ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_2 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_1} \wedge^3 \mathbf{S}_{d_3}) I_6^{-1}, \tag{21}$$

$$\mathbf{D}_6 \equiv^3 \mathbf{D}_{d_3} = ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_2 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_1} \wedge^3 \mathbf{S}_{d_2})I_6^{-1}, \tag{22}$$

where ${}^3\mathbf{S}_i$, $(i = 1, 2, 3)$ are joint screws of the R-joints (${}^3\mathbf{S}_1$ is a joint screw of the R-joint attached to the base platform, while ${}^3\mathbf{S}_3$ is a joint screw of the R-joint attached to the moving platform) and ${}^3\mathbf{S}_{d_i}$, $(i = 1, 2, 3)$ are screws of the dummy joints.

The condition for singularity (Eq. (17)) becomes

$$\mathbf{D}_1 \wedge \mathbf{D}_2 \wedge \mathbf{D}_3 \wedge \mathbf{D}_4 \wedge \mathbf{D}_5 \wedge \mathbf{D}_6 = 0. \tag{23}$$

The process of the elimination of the dummy vectors is explained below, where some identities (Eqs. (8) and (9)) of the geometric algebra are used. Let us consider the blade composed by the two dual vectors $\mathbf{D}_3$ and $\mathbf{D}_4$

$$\begin{aligned} \mathbf{D}_3 \wedge \mathbf{D}_4 = -\mathbf{D}_4 \wedge \mathbf{D}_3 &= -\mathbf{D}_4 \wedge [({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_1} \wedge^3 \mathbf{S}_{d_2} \wedge^3 \mathbf{S}_{d_3})I_6^{-1}] \\ &= -[\mathbf{D}_4 \cdot ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_1} \wedge^3 \mathbf{S}_{d_2} \wedge^3 \mathbf{S}_{d_3})]I_6^{-1} \\ &= -(\mathbf{D}_4 \cdot^3 \mathbf{S}_{d_1})({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_2} \wedge^3 \mathbf{S}_{d_3})I_6^{-1} \\ &= -c_1({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3 \wedge^3 \mathbf{S}_{d_2} \wedge^3 \mathbf{S}_{d_3})I_6^{-1} \end{aligned} \tag{24}$$

where $c_1 = \mathbf{D}_4 \cdot^3 \mathbf{S}_{d_1} \equiv^3 \mathbf{D}_{d_1} \cdot^3 \mathbf{S}_{d_1} \neq 0$ is a scalar, while referring to Eq. (20) it can be seen that the following inner products are zero: $\mathbf{D}_4 \cdot^3 \mathbf{S}_i = 0, (i = 1, 2, 3)$ and $\mathbf{D}_4 \cdot^3 \mathbf{S}_{d_i} = 0, (i = 2, 3)$.

Similarly, the above elimination procedure can be applied for the remaining outer products involving dummy vectors and the final result can be written as the following 4-blade

$$\mathbf{D}_3 \wedge \mathbf{D}_4 \wedge \mathbf{D}_5 \wedge \mathbf{D}_6 = -c_1c_2c_3({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3)I_6^{-1}, \tag{25}$$

where $c_1 =^3 \mathbf{D}_{d_1} \cdot^3 \mathbf{S}_{d_1}$, $c_2 =^3 \mathbf{D}_{d_2} \cdot^3 \mathbf{S}_{d_2}$, $c_3 =^3 \mathbf{D}_{d_3} \cdot^3 \mathbf{S}_{d_3}$; $c_i(i = 1, 2, 3)$ are scalars.

Thus, the condition for the singularity (Eq. (23)) becomes

$$\mathbf{D}_1 \wedge \mathbf{D}_2 \wedge ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3)I_6^{-1} = 0. \tag{26}$$

Keeping in mind Eq. (18), it could be seen that only passive joint screws are involved in the components of Eq. (26) and besides that all of the passive joint screws. Applying the identities of the geometric algebra, Eq. (26) further becomes

$$\mathbf{D}_1 \wedge [\mathbf{D}_2 \cdot ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3)]I_6^{-1} = \mathbf{D}_1 \wedge (\mathbf{V}I_6^{-1}) = (\mathbf{D}_1 \cdot \mathbf{V})I_6^{-1} = 0. \tag{27}$$

The inner product between a vector and a bivector (Eq. (27)) produces the vector $\mathbf{V} = \mathbf{D}_2 \cdot ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3)$. Therefore, $\mathbf{V}$ is perpendicular to $\mathbf{D}_2$ in 6-D space and $\mathbf{V}$ belongs to the bivector space ${}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3$. Since the screws ${}^3\mathbf{S}_1$ and ${}^3\mathbf{S}_3$ are lines passing through the origin of the coordinate system, i.e., the secondary parts of these screws are zero, therefore the vector $\mathbf{V}$ represents a line passing through the origin, too (Fig. 3). Thus, the line $\mathbf{V}$ lies in the plane defined by the lines ${}^3\mathbf{S}_1$ and ${}^3\mathbf{S}_3$.

The fact that the vectors $\mathbf{V}$ and $\mathbf{D}_2$ are mutually perpendicular in 6-D space implies that the elliptic polar $\mathbf{R}_2$ (line) of the screw $\mathbf{D}_2$ ($\mathbf{R}_2 = \tilde{\mathbf{D}}_2$) and the line $\mathbf{V}$ should intersect at a common point, i.e., the screws $\mathbf{V}$ and $\mathbf{R}_2$ are reciprocal (Fig. 3). From Eq. (27) it can be concluded that the manipulator is in singular configuration when the inner product of vectors $\mathbf{D}_1$ and $\mathbf{V}$ is zero ($\mathbf{D}_1 \cdot \mathbf{V} = 0$). This means that the elliptic polar $\mathbf{R}_1$ (line) of the screw $\mathbf{D}_1$ ($\mathbf{R}_1 = \tilde{\mathbf{D}}_1$) and the line $\mathbf{V}$ should intersect, i.e., the screws $\mathbf{V}$ and $\mathbf{R}_1$ are reciprocal (Fig. 3). Thus, the geometric condition for singularity could be stated as: the considered parallel manipulator is in singular configuration if the vector (line) $\mathbf{V}$ intersects both lines ($\mathbf{R}_1$ and $\mathbf{R}_2$) which are along the SPU legs, respectively. This condition could be restated as: the parallel manipulator is in singular configuration if the intersection points of the two lines along the SPU legs with the plane, defined by the axes of the first and the third revolute joints, and the pivoting point (the intersection point of axes of the three R-joints) lie in a single line (Fig. 4). The uncontrollable motion in this singular configuration is a pure rotation about the line $\mathbf{V}$ (Figs. 3 and 4). The intersection at infinity is included, too. For example, in a special case (a special geometry of the robot) the tree lines $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{V}$ could be parallel and in this case they intersect at infinity.

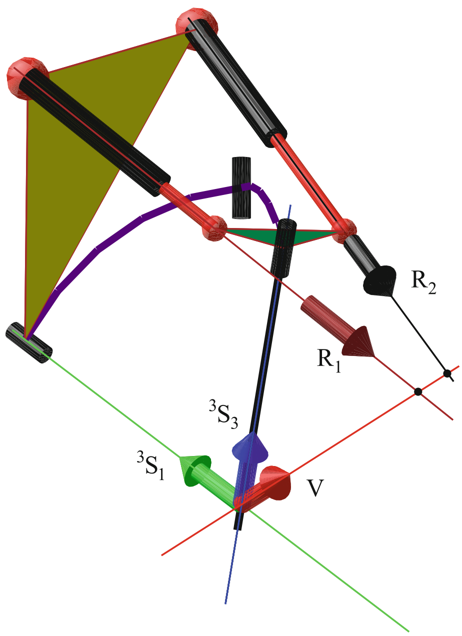

Fig. 3 A singular configuration of the robot

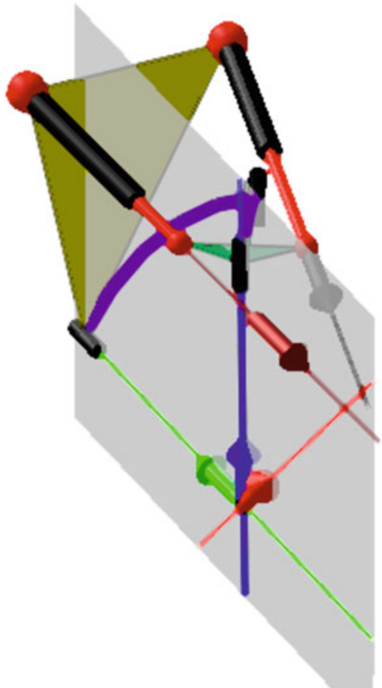

Fig. 4 A singular configuration of the robot and the plane defined by two joint axes ${}^3\mathbf{S}_1$ and ${}^3\mathbf{S}_3$

4.2 Singularity Surface

Algebraic formulation of the singular condition can be obtained from Eq. (26), i.e.,

$$B = [\mathbf{D}_1 \wedge \mathbf{D}_2 \wedge ({}^3\mathbf{S}_1 \wedge^3 \mathbf{S}_3)I_6^{-1}]I_6 = 0, \tag{28}$$

where B is a scalar-valued (grade 0) function of design and input parameters and it is parametrized by the three angles of rotation (θ_1, θ_2 and θ_3).

The singularity surface obtained from B (Eq. (28)) and the workspace of the manipulator are shown in Fig. 5 in terms of the three joint angles θ_1, θ_2 and θ_3.

The singularity surface and workspace are obtained for the following design parameters:
$\mathbf{OA}_1 = (-0.20, 0.52, 0.56)^T$; $\mathbf{OA}_2 = (0.20, 0.52, 0.56)^T$; $\mathbf{OA}_3 = (0, 0.52, 0)^T$; $\mathbf{O}_4\mathbf{B}_1 = (-0.1075, 0.05, 0)^T$; $\mathbf{O}_4\mathbf{B}_2 = (0.1075, 0.05, 0)^T$; $\mathbf{O}_4\mathbf{B}_3 = (0, 0, 0)^T$; $\alpha_1 = 55^{\circ}$; $\alpha_2 = 43^{\circ}$; $d_4 = 0.52$ m. Also, the following constraints on the motion are imposed in the workspace derivation: $L_{min} = 0.30$ m; $L_{max} = 0.58$ m (minimum and maximum lengths of the SPU limbs); minimum angle between each SPU limb and the plane of the base platform = 30°. A section of the singularity surface and workspace are shown in Fig. 6 for the angle $\theta_3 = 0$. The singularity point given in this figure corresponds to the robot configuration from Figs. 3 and 4. A more detailed analysis of the workspace of this robot is presented in [21].

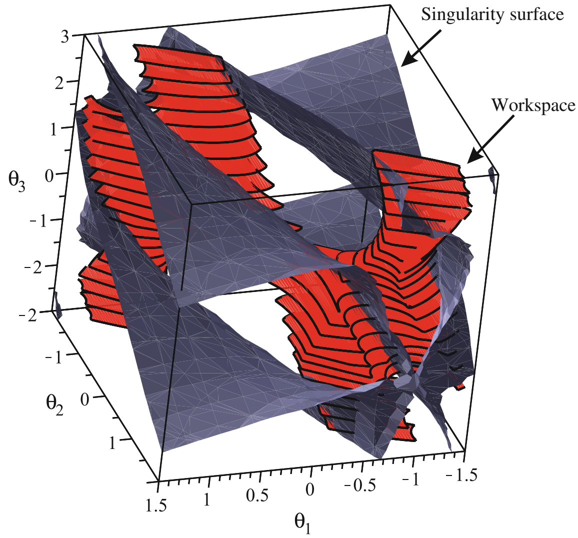

Fig. 5 The singularity surface and workspace boundary of the robot

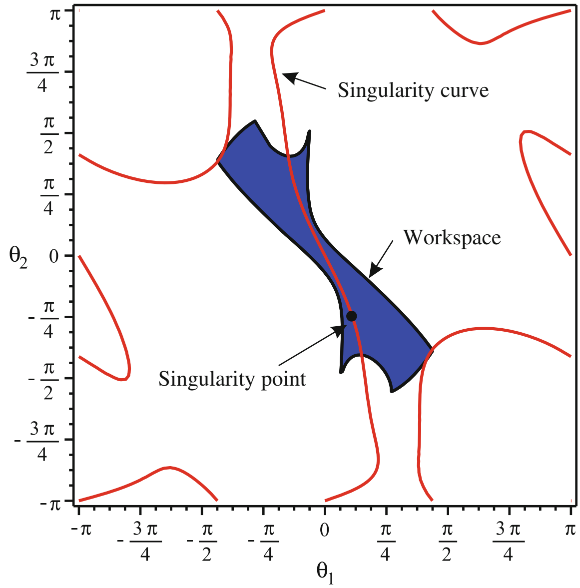

Fig. 6 The singularity curve and workspace boundary of the robot for $\theta_3 = 0$

5 Conclusions

The geometric algebra approach proves its ability of providing a good geometrical insight into the singularities of parallel manipulators with limited mobility as in the case of the proposed novel parallel manipulator for minimally invasive surgery. Using this approach, the geometrical condition for singular configuration of the proposed robot is obtained in a basis-free form. The obtained geometric condition for singularities allows to derive the algebraic formulation and to present the singularity surface in terms of three angles. The results, obtained in this paper, are essential for the robot path planning, where the singular configurations for the proposed novel type of minimally-invasive-surgery parallel robot should be avoided. This is especially important for the medical parallel manipulators from the point of view of the successful surgical outcome and the patient's safety. It could be concluded from the presented graphical results that adding a new degree of freedom (the fifth one), namely a redundant rotation around the surgical instrument (end-effector), will result in a more dexterous hybrid manipulator which will be able to easily avoid the singular configurations. This additional rotation would be a redundant one and could be realized by a modification either of the robot or the surgical instrument.

References

1. Ben-Horin, P., Shoham, M.: Singularity of a class of Gough-Stewart platforms with concurrent joints. In: Lenarčič, J., Roth, B. (eds.) Advances in Robot Kinematics, pp. 265–274. Springer, The Netherlands (2006)
2. Dalvand, M.M., Shirinzadeh, B.: Remote Centre-of-Motion control algorithms of 6-RRCRR parallel robot assisted surgery system (PRAMiSS). In: IEEE International Conference on Robotics and Automation (ICRA), pp. 3401–3406. Saint Paul, Minnesota, USA, 14–18 May 2012
3. Doran, C., Lasenby, A.: Geometric Algebra for Physicists. Cambridge University Press (2007)
4. Dorst, L., Fontijne, D., Mann, S.: Geometric Algebra for Computer Science, An Object Oriented Approach to Geometry. Morgan Kaufmann Publishers (2007)
5. Guthart, G., Salisbury, J.K.: The Intuitive telesurgery system: overview and applications. In: IEEE International Conference on Robotics and Automation, pp. 618–621 (2000)
6. Hao, F., McCarthy, J.: Conditions for line-based singularities in spatial platform manipulators. J. Robot. Syst. **15**(1), 43–55 (1998)
7. Hestenes, D.: New Foundations for Classical Mechanics, 2nd edn. Kluwer Academic Publishers, Dordrecht (1999)
8. Hestenes, D., Li, H., Rockwood, A.: New algebraic tools for classical geometry. In: Sommer, G. (ed.) Geometric Computing with Clifford Algebra. Springer, Berlin (1999)
9. Kanaan, D., Wenger, P., Chablat, D.: Singularity analysis of limited-DOF parallel manipulators using Grassmann-Cayley algebra. In: Lenarčič, J., Wenger, P. (eds.) Advances in Robot Kinematics, Analysis and Design, pp. 59–68. Springer (2008)
10. Khan, M.A., Zoppi, M., Molfino, R.: 4-DOF parallel architecture for laparoscopic surgery. In: Lenarčič, J., Wenger, P. (eds.) Advances in Robot Kinematics: Analysis and Design, pp. 119–126. Springer (2008)
11. Kuo, C-H., Dai, J.S.: Robotics for minimally invasive surgery: a historical review from the perspective of kinematics. In: International Symposium on History of Machines, pp. 337–354. Springer Science+Business Media B.V. (2009)

12. Lipkin, H., Duffy, J.: The elliptic polarity of screws. ASME J. Mech. Transm. Autom. Des. **107**, 377–387 (1985)
13. Li, T., Payandeh, S.: Design of spherical parallel mechanisms for application to laparoscopic surgery. Robotica **20**, 133–138 (2002)
14. Liu, G., Lou, Y., Li, Z.: Singularities of parallel manipulators: a geometric treatment. Trans. Robot. Automat **19**(4), 579–594 (2003)
15. Lum, M.J.H., et al.: Kinematic optimization of a spherical mechanism for a minimally invasive surgical robot. In: Proceedings of IEEE International Conference on Robotics and Automation, New Orleans, La, May 2004
16. Merlet, J.-P.: Singular configurations of parallel manipulators and Grassmann geometry. Int. J. Robot. Res. **8**(5), 45–56 (1989)
17. Pisla, D., et al.: Kinematic modelling of a 5-DOF hybrid parallel robot for laparoscopic surgery. Robotica **30**(07), 1095–1107 (2012)
18. Rosen, J., et al.: The BlueDRAGON—a system for measuring the kinematics and the dynamics of minimally invasive surgical tools in-vivo. In: Proceedings of IEEE International Conference on Robotics and Automation, Washington, USA, pp. 1876–1881 (2002)
19. Sima'an, N., Glozman, D., Shoham, M.: Design considerations of new six degrees-of-freedom parallel robots. In: Proceedings of IEEE International Conference on Robotics and Automation, pp. 327–1333. Leuven, Belgium (1998)
20. Tanev, T.K.: Geometric algebra approach to singularity of parallel manipulators with limited mobility. In: Lenarčič, J., Wenger, P. (eds.) Advances in Robot Kinematics, Analysis and Design. Springer (2008)
21. Tanev, T.K.: Minimally-invasive-surgery parallel robot with non-identical limbs. In: Proceedings of 10th IEEE/ASME International Conference on Mechatronic and Embedded Systems and Applications, (MESA2014), Senigallia, Italy, 10–12 Sept 2014. doi:10.1109/MESA.2014.6935558
22. Tanev, T.K.: Singularity analysis of a 4-dof parallel manipulator using geometric algebra. In: Lenarčič, J., Roth, B. (eds.) Advances in Robot Kinematics, Mechanism and Motion, pp. 275–284. Springer (2006)
23. Taylor, R.H., Stoianovici, D.: Medical robotics in computer-integrated surgery. IEEE Trans. Robot. Autom. **19**(5), 765–781 (2003)
24. Zemiti, N., et al.: Mechatronic design of a new robot for force control in minimally invasive surgery. IEEE/ASME Trans. Mechatron. **12**(2), 143–153 (2007)
25. Zlatanov, D., Bonev, I.A., Cosselin, C.M.: Constraint singularities of parallel mechanisms. In: Proceedings of IEEE International Conference on Robotics and Automation, vol. 1, pp. 496–502. Washington, DC (2002)