Tropical Dominating Sets
in Vertex-Coloured Graphs

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Abstract. Given a vertex-coloured graph, a dominating set is said to
be tropical if every colour of the graph appears at least once in the set.
Here, we study minimum tropical dominating sets from structural and
algorithmic points of view. First, we prove that the tropical dominating
set problem is NP-complete even when restricted to a simple path. Last,
we give approximability and inapproximability results for general and
restricted classes of graphs, and establish a FPT algorithm for interval
graphs.

Keywords: Dominating set · Vertex-coloured graph · Approximation

1 Introduction

Vertex-coloured graphs are useful in various situations. For instance, the Web
graph may be considered as a vertex-coloured graph where the colour of a vertex
represents the content of the corresponding page (red for mathematics, yellow for
physics, etc.). Given a vertex-coloured graph G^c, a subgraph H^c (not necessarily
induced) of G^c is said to be tropical if and only if each colour of G^c appears
at least once in H^c. Potentially, any kind of usual structural problems (paths,
cycles, independent and dominating sets, vertex covers, connected components,
etc.) could be studied in their tropical version. This new tropical concept is

close to, but quite different from, the colourful concept used for paths in vertex-coloured graphs [1,15,16]. It is also related to (but again different from) the concept of *colour patterns* used in bio-informatics [11]. Here, we study minimum tropical dominating sets in vertex-coloured graphs. A general overview on the classical dominating set problem can be found in [13].

Throughout the paper let $G = (V, E)$ denote a simple undirected non-coloured graph. Let $n = |V|$ and $m = |E|$. Given a set of colours $\mathcal{C} = \{1, ..., c\}$, $G^c = (V^c, E)$ denotes a vertex-coloured graph where each vertex has precisely one colour from $\mathcal{C}$ and each colour of $\mathcal{C}$ appears on at least one vertex. The colour of a vertex x is denoted by $c(x)$. A subset $S \subseteq V$ is a *dominating set* of G^c (or of G), if every vertex either belongs to S or has a neighbour in S. The *domination number* $\gamma(G^c)$ $(\gamma(G))$ is the size of a smallest dominating set of G^c (G). A dominating set S of G^c is said to be *tropical* if each of the c colours appears at least once among the vertices of S. The *tropical domination number* $\gamma^t(G^c)$ is the size of a smallest tropical dominating set of G^c. A *rainbow dominating set* of G^c is a tropical dominating set with exactly c vertices. More generally, a c-element set with precisely one vertex from each colour is said to be a *rainbow set*. We let $\delta(G^c)$ (respectively $\Delta(G^c)$) denote the minimum (maximum) degree of G^c. When no confusion arises, we write γ, γ^t, δ and Δ instead of $\gamma(G)$, $\gamma^t(G^c)$, $\delta(G^c)$ and $\Delta(G^c)$, respectively. We use the standard notation $N(v)$ for the (open) neighbourhood of vertex v, that is the set of vertices adjacent to v, and write $N[v] = N(v) \cup \{v\}$ for its closed neighbourhood. The set and the number of neighbours of v inside a subgraph H is denoted by $N_H(v)$ and by $d_H(v)$, independently of whether v is in H or in $V(G^c) - V(H)$. Although less standard, we shall also write sometimes $v \in G^c$ to abbreviate $v \in V(G^c)$.

Note that tropical domination in a vertex-coloured graph G^c can also be interpreted as "simultaneous domination" in two graphs which have a common vertex set. One of the two graphs is the non-coloured G itself, the other one is the union of c vertex-disjoint cliques each of which corresponds to a colour class in G^c. The notion of simultaneous dominating set[1] was introduced by Sampathkumar [17] and independently by Brigham and Dutton [5]. It was investigated recently by Caro and Henning [6] and also by further authors. Remark that $\delta \geq 1$ is regularly assumed for each factor graph in the results of these papers that is not the case in the present manuscript, as we do not forbid the presence of one-element colour classes.

The Tropical Dominating Set problem (TDS) is defined as follows.

Problem 1. TDS
Input: A vertex-coloured graph G^c and an integer $k \geq c$.
Question: Is there a tropical dominating set of size at most k?

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[1] Also known under the names 'factor dominating set' and 'global dominating set' in the literature.

The Rainbow Dominating Set problem (RDS) is defined as follows.

Problem 2. RDS
Input: A vertex-coloured graph G^c.
Question: Is there a rainbow dominating set?

The paper is organized as follows. In Sect. 2 we give approximability and inapproximability results for TDS. We also show that the problem is FPT (fixed-parameter tractable) on interval graphs when parametrized by the number of colours.

2   Approximability and Fixed Parameter Tractability

We begin this section noting that the problem is intractable even for paths.

Theorem 1. *The RDS problem is NP-complete, even when the input is restricted to vertex-coloured paths.*

In the sequel, we assume familiarity with the complexity classes NPO and PO which are optimisation analogues of NP and P. A minimisation problem in NPO is said to be *approximable* within a constant $r \geq 1$ if there exists an algorithm A which, for every instance I, outputs a solution of measure $A(I)$ such that $A(I)/\mathsf{Opt}(I) \leq r$, where $\mathsf{Opt}(I)$ stands for the measure of an optimal solution. An NPO problem is in the class APX if it is approximable within *some* constant factor $r \geq 1$. An NPO problem is in the class PTAS if it is approximable within r for *every* constant factor $r > 1$. An APX-hard problem cannot be in PTAS unless P = NP. We use two types of reductions, L-reductions to prove APX-hardness, and PTAS-reductions to demonstrate inclusion in PTAS. In the Appendix we give a slightly more formal introduction and a description of reduction methods related to approximability. For more on these issues we refer to Ausiello et al. [3] and Crescenzi [8].

A problem is said to be *fixed parameter tractable* (FPT) with parameter $k \in \mathbb{N}$ if it has an algorithm that runs in time $f(k) \, |I|^{\mathcal{O}(1)}$ for any instance (I, k), where f is an arbitrary function that depends only on k.

In this section, we study the complexity of approximating and solving TDS conditioned on various restrictions on the input graphs and on the number of colours. First, we show that TDS is equivalent to MDS (Minimum Dominating Set) under L-reductions. In particular, this implies that the general problem lies outside APX. We then attempt to restrict the input graphs and observe that if MDS is in APX on some family of graphs, then so is TDS. However, there is also an immediate lower bound: TDS on any family of graphs that contains all paths is APX-hard. We proceed by adding an upper bound on the number of colours. We see that if MDS is in PTAS for some family of graphs with bounded degree, then so is TDS when restricted to $n^{1-\epsilon}$ colours for some $\epsilon > 0$. Finally, we show that TDS on interval graphs is FPT with the parameter being the number of colours and that the problem is in PO when the number of colours is logarithmic.

Proposition 1. *TDS is equivalent to MDS under L-reductions. It is approximable within* $\ln n + \Theta(1)$ *but NP-hard to approximate within* $(1 - \epsilon) \ln n$.

Proof. MDS is clearly a special case of TDS. For the opposite direction, we reduce an instance of TDS to an instance I of the Set Cover problem which is known to be equivalent to MDS under L-reductions [14]. In the Set Cover problem, we are given a ground set U and a collection of subsets $F_i \subseteq U$ such that $\bigcup_i F_i = U$. The goal is to cover U with the smallest possible number of sets F_i. Our reduction goes as follows. Given a vertex-coloured graph $G^c = (V^c, E)$, with the set of colours C, the ground set of I is $U = V^c \cup C$. Each vertex v of V gives rise to a set $F_v = N[v] \cup \{c(v)\}$, a subset of U. Every solution to I must cover every vertex $v \in V$ either by including a set that corresponds to v or by including a set that corresponds to a neighbour of v. Furthermore, every solution to I must include at least one vertex of every colour in C. It follows that every set cover can be translated back to a tropical dominating set of the same size. This shows that our reduction is an L-reduction.

The approximation guarantee follows from that of the standard greedy algorithm for Set Cover. The lower bound follows from the NP-hardness reduction to Set Cover in [9] in which the constructed Set Cover instances contain $o(N)$ sets, where N is the size of the ground set.

When the input graphs are restricted to some family of graphs, then membership in APX for MDS carries over to TDS.

Lemma 1. *Let $\mathcal{G}$ be a family of graphs. If MDS restricted to $\mathcal{G}$ is in APX, then TDS restricted to $\mathcal{G}$ is in APX.*

Proof. Assume that MDS restricted to $\mathcal{G}$ is approximable within r for some $r \geq 1$. Let G^c be an instance of TDS. We can find a dominating set of the uncoloured graph G of size at most $r\gamma(G)$ in polynomial time, and then add one vertex of each colour that is not yet present in the dominating set. This set is of size at most $r\gamma(G) + c - 1$. The size of an optimal solution of G^c is at least $\gamma(G)$ and at least c. Hence, the computed set will be at most $r + 1$ times the size of the optimal solution of G^c.

For $\Delta \geq 2$, let Δ-TDS denote the problem of minimising a tropical dominating set on graphs of degree bounded by Δ. The problem MDS is in APX for bounded-degree graphs, hence Δ-TDS is in APX by Lemma 1. The same lemma also implies that TDS restricted to paths is in APX. Next, we give explicit approximation ratios for these problems.

Proposition 2. *TDS restricted to paths can be approximated within* $5/3$.

Proof. Let $P^c = v_1, v_2, \ldots, v_n$ be a vertex-coloured path. For $i = 1, 2, 3$ let $\sigma_i = \{v_j \mid j \equiv i \pmod 3, \ 1 \leq j \leq n\}$. Select any subset σ_i' of V that contains precisely one vertex of each colour missing from σ_i. Let $S_i = \sigma_i \cup \sigma_i'$. By definition, S_i is a tropical set.

Taking into account that each colour must appear in a tropical dominating set, moreover any vertex can dominate at most two others, we see the following easy lower bounds:

$$n \leq 3\gamma^t(P^c),$$
$$2c \leq 2\gamma^t(P^c),$$
$$\frac{1}{5}(n + 2c) \leq \gamma^t(P^c).$$

Suppose for the moment that each of S_1, S_2, S_3 dominates G^c. Then, since each colour occurs in at most two of the σ_i', we have $|S_1| + |S_2| + |S_3| \leq n + 2c$ and therefore

$$\gamma^t(P^c) \leq \min(|S_1|, |S_2|, |S_3|) \leq \frac{1}{3}(n + 2c).$$

Comparing the lower and upper bounds, we obtain that the smallest set S_i provides a 5/3-approximation. It is also clear that this solution can be constructed in linear time.

The little technical problem here is that the set S_i does not dominate vertex v_1 if $i = 3$, and it does not dominate v_n if $i \equiv n - 2 \pmod 3$. We can overcome this inconvenience as follows.

The set S_3 surely will dominate v_1 if we extend S_3 with either of v_1 and v_2. This means no extra element if we have the option to select e.g. v_1 into σ_3'. We cannot do this only if $c(v_1)$ is already present in σ_3. But then this colour is common in σ_1 and σ_3; that is, although we take an extra element for S_3, we can subtract 1 from the term $2c$ when estimating $|\sigma_1'| + |\sigma_2'| + |\sigma_3'|$. The same principle applies to the colour of v_n, too.

Even this improved computation fails by 1 when $n \equiv 2 \pmod 3$ and $c(v_1) = c(v_n)$, as we can then write just $2c - 1$ instead of $2c - 2$ for $|\sigma_1'| + |\sigma_2'| + |\sigma_3'|$. Now, instead of taking the vertex pair $\{v_1, v_n\}$ into S_3, we complete S_3 with v_2 and v_n. This yields the required improvement to $2c - 2$, unless $c(v_2)$, too, is present in σ_3. But then $c(v_2)$ is a common colour of σ_2 and σ_3, while $c(v_1)$ is a common colour of σ_1 and σ_3. Thus $|\sigma_1'| + |\sigma_2'| + |\sigma_3'| \leq 2c - 2$, and $|S_1| + |S_2| + |S_3| \leq n + 2c$ holds also in this case.

Remark 1. In an analogous way — which does not even need the particular discussion of unfavourable cases — one can prove that the square grid $P_n \square P_n$ admits an asymptotic 9/5-approximation. (This extends also to $P_n \square P_m$ where $m = m(n)$ tends to infinity as n gets large.) A more precise estimate on grids, however, may require a careful and tedious analysis.

Proposition 3. *Δ-TDS is approximable within $\ln(\Delta + 2) + \frac{1}{2}$. Moreover, there are absolute constants $C > 0$ and $\Delta_0 \geq 3$ such that for every $\Delta \geq \Delta_0$, it is NP-hard to approximate Δ-TDS within $\ln \Delta - C \ln \ln \Delta$.*

Proof. The second assertion follows from [7, Theorem 3]. For the first part, we apply reduction from Set Cover, similarly as in the proof of Proposition 1. So, for $G^c = (V^c, E)$ we define $U = V^c \cup C$ and consider the sets $F_v = N[v] \cup \{c(v)\}$ for the vertices $v \in V^c$. Every set cover in this set system corresponds to a tropical dominating set in G^c. Moreover, the Set Cover problem is approximable within $\sum_{i=1}^k \frac{1}{i} - \frac{1}{2} < \ln k + \frac{1}{2}$ [10], where k is an upper bound on the cardinality of any set of I. In our case, we have $k = \Delta + 2$ since $|N(v)| \leq \Delta$ for all v. Hence, TDS is approximable within $\ln(\Delta + 2) + \frac{1}{2}$.

We now show that TDS for paths is APX-complete.

Theorem 2. *TDS restricted to paths is* APX-*hard.*

Proof. We apply an L-reduction from the Vertex Cover problem (VC): Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ of minimum cardinality such that, for every edge $uv \in E$, at least one of $u \in S$ and $v \in S$ holds. We write 3-VC for the vertex cover problem restricted to graphs of maximum degree three (subcubic graphs). The problem 3-VC is known to be APX-complete [2]. For a graph G, we write $\mathsf{Opt}_{VC}(G)$ for the minimum size of a vertex cover of G.

Let $G = (V, E)$ be a non-empty instance of 3-VC, with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Assume that G has no isolated vertices. The reduction sends G to an instance $\phi(G)$ of TDS which will have $m + n + 1$ colours: B (for black), E_i with $1 \leq i \leq m$ (for the ith edge), and S_j with $1 \leq j \leq n$ (for the jth vertex). The path has $9n + 3$ vertices altogether, starting with three black vertices of Fig. 1(a), we call this triplet V_0. Afterwards blocks of 6 and 3 vertices alternate, we call the latter $V_1, \ldots, V_n$, representing the vertices of G. Each V_j (other than V_0) is coloured as shown in Fig. 1(c). Assuming that v_j $(1 \leq j \leq n)$ is incident to the edges e_{j_1}, e_{j_2}, and e_{j_3}, the two parts V_{j-1} and V_j are joined by a path representing these three incidences, and coloured as in Fig. 1(b). If v_j has degree less than 3, then the vertex in place of E_{j_3} is black; and if $d(v_j) = 1$, then also E_{j_2} is black.
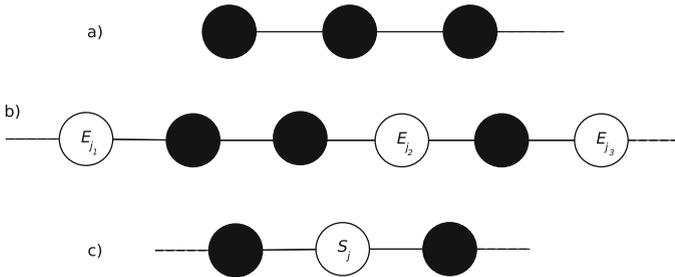

Fig. 1. Gadgets for the reduction of Theorem 2

Let $\sigma \subseteq V$ be an arbitrary solution to $\phi(G)$. First, we construct a solution σ' from σ with more structure, and with a measure at most that of σ. For every j, σ contains the vertex coloured S_j. Let σ' contain these as well. At least one of the first two vertices coloured B must also be in σ. Let σ' contain the second vertex coloured B. Now, if any V_j $(0 \leq j \leq n)$ has a further (first or third) vertex which is an element of σ, then we can replace it with its predecessor or successor, achieving that they dominate more vertices in the path. This modification does not lose any colour because the first and third vertices of any V_j are black, and B is already represented in $\sigma \cap V_0$.

Now we turn to the 6-element blocks connecting a V_{j-1} with V_j. Since the third vertex of V_{j-1} and the first vertex of V_j are surely not in the modified σ, which still dominates the path, it has to contain at least two vertices of the 6-element block. And if it contains only two, then those necessarily are the second and fifth, both being black. Should this be the case, we keep them in σ'. Otherwise, if the modified σ contains more than two vertices of the 6-element block, then let σ' contain precisely E_{j_1}, E_{j_2}, and E_{j_3}. Since σ is a tropical dominating set, the same holds for σ'. It is also clear that $|\sigma'| \leq |\sigma|$.

Next, we create a solution $\psi(G, \sigma)$ to the vertex cover problem on G, using σ'. Let $v_j \in \psi(G, \sigma)$ if and only if $\{E_{j_1}, E_{j_2}, E_{j_3}\} \subseteq \sigma'$. Then, $|\psi(G, \sigma)| = |\sigma'| - 1 - 3n \leq |\sigma| - 1 - 3n$, and when σ is optimal, we have the equality $\mathsf{Opt}_{VC}(G) = \gamma^t(\phi(G)) - 1 - 3n$. Therefore,

$$|\psi(G, \sigma)| - \mathsf{Opt}_{VC}(G) \leq |\sigma| - \gamma^t(\phi(G)). \tag{1}$$

We may assume that G does not contain any isolated vertices. Under this assumption, we prove the lower bound $\mathsf{Opt}_{VC}(G) \geq n/4$ by induction, as follows: The bound clearly holds for an empty graph. Suppose that the bound holds for all graphs without isolated vertices with fewer than n vertices. Let σ^* be a minimal vertex cover of G and let $v \in V \setminus \sigma^*$. Then, all of v's neighbours are in σ^*. Let G' be the graph G with $N[v]$ removed as well as any isolated vertices resulting from this removal. Let n' be the number of vertices in G'. If v has $1 \leq n_v \leq 3$ neighbours, then $0 \leq n_i \leq 2n_v$ vertices become isolated when $N[v]$ is removed, so $\mathsf{Opt}_{VC}(G) = n_v + \mathsf{Opt}_{VC}(G') \geq n_v + n'/4 = n_v + (n - 1 - n_v - n_i)/4 \geq n_v + (n - 1 - 3n_v)/4 \geq n/4$.

This allows us to upper-bound the optimum of $\phi(G)$:

$$\begin{aligned} \gamma^t(\phi(G)) &= \mathsf{Opt}_{VC}(G) + 1 + 3n \\ &\leq \mathsf{Opt}_{VC}(G) + 1 + 12 \cdot \mathsf{Opt}_{VC}(G) \leq 14 \cdot \mathsf{Opt}_{VC}(G). \end{aligned} \tag{2}$$

It follows from (1) and (2) that ϕ and ψ constitute an L-reduction.

Corollary 1. *Fix $0 < \epsilon \leq 1$, and let $\mathcal{P}$ be the family of all vertex-coloured paths with at most n^ϵ colours, where n is the number of vertices. Then TDS restricted to $\mathcal{P}$ is NP-hard.*

Proof. We reduce from TDS on paths with an unrestricted number of colours which is NP-hard by Theorem 2. Let P^c be a vertex-coloured path on n vertices with $c \leq n$ colours. Let $Q^{c'}$ be the instance obtained by adding a path $v_1, v_2, \ldots, v_N$ with $N = \lceil (n+2)^{1/\epsilon} \rceil$ vertices to the end of P^c (this is a polynomial-time reduction for any fixed constant $\epsilon > 0$). Let A and B be two new colours. In the added path $v_1, v_2, \ldots, v_N$, let v_2 have colour A and all the other vertices have colour B. The instance $Q^{c'}$ has $n' = n + N$ vertices and $c' = c + 2 \leq n + 2 \leq N^\epsilon \leq (n')^\epsilon$ colours, so $Q^{c'} \in \mathcal{P}$.

Given a minimum tropical dominating set σ of $Q^{c'}$, we see that v_2 must be in σ to account for the colour A. We may further assume that v_1 is not in σ. If it were, then we could modify σ by removing v_1 and adding the last vertex of

P^c instead. It is now clear that taking σ restricted to $\{v_1, v_2, \ldots, v_N\}$ together with a tropical dominating set of P^c yields a tropical dominating set of $Q^{c'}$ and that σ restricted to P^c is a tropical dominating set of P^c. Hence, σ restricted to P^c is a minimum tropical dominating set of P^c.

We have seen that restricting the input to any graph family that contains at least the paths can take us into APX but not further. To find more tractable restrictions, we now introduce an additional restriction on the number of colours. The following lemma says that if the domination number grows asymptotically faster than the number of colours, then we can lift PTAS-inclusion of MDS to TDS.

Lemma 2. *Let $\mathcal{G}$ be a family of vertex-coloured graphs. Assume that there exists a computable function $f \colon \mathbb{Q} \cap (0, \infty) \to \mathbb{N}$ such that for every $r > 0$, $\gamma(G) > c/r$ whenever $G^c \in \mathcal{G}$ and $n(G^c) \geq f(r)$. Then, TDS restricted to $\mathcal{G}$ PTAS-reduces to MDS restricted to $\mathcal{G}$.*

Proof. To design a polynomial-time $(1+\varepsilon)$-approximation for any rational $\varepsilon > 0$, we pick $r = \varepsilon/2$; hence let $n_0 = f(\varepsilon/2)$. Let $G^c \in \mathcal{G}$ be a vertex-coloured graph. The reduction sends G^c to $\phi(G^c) = G$, the instance of MDS obtained from G^c by simply forgetting the colours. Let σ be any dominating set in G. Assuming that σ is a good approximation to $\gamma(G)$, we need to compute a good approximation $\psi(G^c, \sigma)$ to $\gamma^t(G^c)$. If $n(G^c) < n_0$, then we let $\psi(G^c, \sigma)$ be an optimal tropical dominating set of G^c. Otherwise, let $\psi(G^c, \sigma)$ be σ plus a vertex for each remaining non-covered colour. Since n_0 depends on ε but not on G^c or σ, it follows that ψ can be computed in time that is polynomial in $|V(G^c)|$ and $|\sigma|$.

We claim that ϕ and ψ provide a PTAS-reduction. This is clear if $n(G^c) < n_0$ since ψ then computes an optimal solution to G^c. Otherwise, assume that $n(G^c) \geq n_0$ and that $|\sigma| / \gamma(G) \leq 1 + \varepsilon/2$, i.e., σ is a good approximation. Then,

$$\frac{|\psi(G^c, \sigma)|}{\gamma^t(G^c)} \leq \frac{|\sigma| + c}{\gamma(G)} \leq \frac{2 + \varepsilon}{2} + \frac{c}{\gamma(G)} < 1 + \varepsilon,$$

where the last inequality follows from $n(G) \geq n_0$ and the definition of f.

Example 1. The problem MDS is in PTAS for planar graphs [4], but NP-hard even for planar subcubic graphs [12]. Let $\mathcal{G}$ be the family of planar graphs of maximum degree Δ, for any fixed $\Delta \geq 3$, and with a number of colours $c < n^{1-\epsilon}$ for some fixed $\epsilon > 0$. Let $f(r) = \lceil (\frac{\Delta+1}{r})^{1/\epsilon} \rceil$ and note that $\gamma(G) \geq n/(\Delta + 1) > cn^\epsilon/(\Delta + 1) \geq cf(r)^\epsilon/(\Delta + 1) \geq c/r$ whenever $n \geq f(r)$. It then follows from Lemma 2 that TDS is in PTAS when restricted to planar graphs of fixed maximum degree.

Example 2. As a second example, we observe how the complexity of TDS on a path varies when we restrict the number of colours. For an arbitrary number of colours, it is APX-complete by Lemma 1 and Theorem 2. If the number of colours is $\mathcal{O}(n^{1-\epsilon})$ for some $\epsilon > 0$, then it is in PTAS by Lemma 2, but NP-hard

by Corollary 1. Finally, if the number of colours is $\mathcal{O}(\log n)$, then it can be shown to be in PO by a simple dynamic programming algorithm.

In the rest of this section, we look at the restriction where we consider the number of colours as a fixed parameter. We prove the following result.

Theorem 3. *There is an algorithm for TDS restricted to interval graphs that runs in time $\mathcal{O}(2^c n^2)$.*

This shows that TDS for interval graphs is FPT and, furthermore, that if $c = \mathcal{O}(\log n)$, then TDS is in PO.

Let G^c be a vertex-coloured interval graph with vertex set $V = \{1, \ldots, n\}$ and colour set C, and fix some interval representation $I_i = [l_i, r_i]$ for each vertex $1 \le i \le n$. Assume that the vertices are ordered non-decreasingly with respect to r_i. For $a, b \in V$, we use (closed) intervals $[a, b] = \{i \in V \mid a \le i \le b\}$ to denote subsets of vertices with respect to this order.

Define an *i-prefix dominating set* as a subset $U \subseteq V$ of vertices that contains i and dominates $[1, i]$ in G^c. We say that U is *proper* if, for every $i, j \in U$, we have neither $I_i \subseteq I_j$ nor $I_j \subseteq I_i$.

Let $f \colon \mathcal{P}(C) \times [0, n] \to \mathbb{N} \cup \{\infty\}$ be the function defined so that, given a subset $S \subseteq C$ of colours and a vertex $i \in V$, $f(S, i)$ is the least number of vertices in a proper i-prefix dominating set that covers precisely the colours in S, or ∞ if there is no such set. The value of $f(S, 0)$ is defined to be 0 when $S = \emptyset$ and ∞ otherwise. Our proof is based on a recursive definition of f (Lemma 5) and the fact that f determines γ^t (Lemma 4). First, we need a technical lemma.

Lemma 3. *Let $U \subseteq V$ and let i be the largest element in U. If U is i-prefix dominating, then it dominates precisely the same vertices as $[1, i]$. In particular, U dominates G if and only if $[1, i]$ does.*

Proof. Assume to the contrary that there is a $j \in [1, i] - U$ that dominates some $k > i$, and that k is not dominated by U. This means that j is connected to k in G, so $l_k \le r_j$. But then we have $l_k \le r_j \le r_i \le r_k$, so $[l_i, r_i] \cap [l_k, r_k] \ne \emptyset$, hence $i \in U$ dominates k, a contradiction.

Lemma 4. *For every interval graph G^c, we have*

$$\gamma^t(G^c) = \min\{f(S, i) + |C - S| \mid S \subseteq C, i \in V, [1, i] \text{ dominates } G^c\}.$$

Proof. $f(S, i)$ is the size of some set $U \subseteq V$ that covers the colours S and that, by Lemma 3, dominates G^c. We obtain a tropical dominating set by adding a vertex of each missing colour in $C - S$. Therefore, each expression $f(S, i) + |C - S|$ on the right-hand side corresponds to the size of a tropical dominating set, so $\gamma^t(G^c)$ is at most the minimum of these.

For the opposite inequality, let U be a minimum tropical dominating set of G^c. Remove from U all vertices i for which there is some $j \in U$ with $I_i \subseteq I_j$, and call the resulting set U'. By construction U' still dominates G^c. Let S be the set of colours covered by U'. Then U' is a minimum set with these properties, so by

the definition of f, $|U'| = f(S, i)$, where i is the greatest element in U'. Since $U' \subseteq [1, i]$, it follows that $[1, i]$ dominates G^c. Therefore, the right-hand side is at most $f(S, i) + |C - S| = |U'| + |C - S| \leq |U| = \gamma^t(G^c)$.

The following lemma gives a recursive definition of the function f that permits us to compute it efficiently when the number of colours in C grows at most logarithmically.

Lemma 5. *For every interval graph G^c, the function f satisfies the following recursion:*

$$f(S, 0) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \infty & \text{otherwise;} \end{cases}$$

$$f(S, i) = 1 + \min\{f(S', j) \mid S' \cup \{c(i)\} = S, j \in P_i\}, \qquad \text{for } i \in V,$$

where $j \in P_i$ if and only if either $j = 0$ and $\{i\}$ is i-prefix dominating, or $j \in V$, $j < i$, $[1, j] \cup \{i\}$ is i-prefix dominating, and $I_i \not\subseteq I_j$, $I_j \not\subseteq I_i$.

Proof. The proof is by induction on i. The base case $i = 0$ holds by definition. Assume that the lemma holds for all $0 \leq i \leq k - 1$ and all $S \subseteq C$.

Let U be a minimum proper k-prefix dominating set that covers precisely the colours in S. We want to show that $|U| = f(S, k)$. If $U = \{k\}$, then $S = \{c(k)\}$, and it follows immediately that $f(S, k) = 1$. Otherwise, $U - \{k\}$ is non-empty. Let $j < k$ be the greatest vertex in $U - \{k\}$. Assume that $U - \{k\}$ is not j-prefix dominating. Then, there is some $i < j$ that is not dominated by j but that is dominated by k, hence $l(k) \leq r(i) < l(j)$. Therefore $I_j \subseteq I_k$, so U is not proper, a contradiction. Hence, $U - \{k\}$ is a proper j-prefix dominating set. By induction, $|U - \{k\}| \geq \min\{f(S', j) \mid S' \cup \{c(k)\} = S\}$. This shows the inequality $|U| \geq f(S, k)$.

For the opposite inequality, it suffices to show that if $[1, j] \cup \{k\}$ is k-prefix dominating, U' is any proper j-prefix dominating set, and $I_k \not\subseteq I_j, I_j \not\subseteq I_k$, then $U' \cup \{k\}$ is a proper k-prefix dominating set. It follows from Lemma 3 that $U' \cup \{k\}$ is k-prefix dominating. Since $I_k \not\subseteq I_j$, we must have $r_i \leq r_j < r_k$ for all $i < j$, hence $I_k \not\subseteq I_i$. Assume that $I_i \subseteq I_k$ for some $i < j$. Then, since $I_j \not\subseteq I_k$, we have $l_j < l_k \leq l_i \leq r_i \leq r_j$, which contradicts U' being proper. It follows that $U' \cup \{k\}$ is proper.

Proof of Theorem 3. The sets P_i for $i \in V$ in Lemma 5 can be computed in time $\mathcal{O}(n^2)$ as follows. Let $a_i \in V$ be the least vertex such that i dominates $[a_i, i]$, and let $b_j \in V$ be the least vertex such that $[1, j]$ does not dominate b_j, or ∞ if $[1, j]$ dominates G. Note that i does not dominate any vertex strictly smaller than a_i since the vertices are ordered non-decreasingly with respect to the right endpoints of their intervals. Therefore, $P_i = \{j < i \mid a_i \leq b_j, I_i \not\subseteq I_j, I_j \not\subseteq I_i\}$. The vectors a_i and b_j are straightforward to compute in time $\mathcal{O}(n^2)$, hence P_i can be computed in time $\mathcal{O}(n^2)$ using this alternative definition.

When P_i is computed for all $i \in V$, the recursive definition of f in Lemma 5 can be used to compute all values of f in time $\mathcal{O}(2^c n^2)$, and it can easily be

modified to compute, for each S and i, some specific i-prefix dominating set of size $f(S, i)$, also in time $\mathcal{O}(2^c n^2)$. Therefore, by Lemma 4, one can find a minimum tropical dominating set in time $\mathcal{O}(2^c n^2)$.   □

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