Chapter 2
Price Dynamics of Leveraged ETFs

In this chapter, we investigate the empirical returns of LETFs and present models for their price dynamics. We highlight the effects of leverage ratios and holding horizon on returns. An series of empirical studies are conducted to examine the tracking performance of LETFs and evaluate various leverage replication strategies. The unique characteristics of LETF price evolution motivate us to construct and backtest static delta-neutral long-volatility strategies for LETF portfolios.

2.1 Returns of Leveraged ETFs

To examine the returns of leveraged ETFs, let us consider the following illustrative example. By design, an LETF seeks to provide a constant multiple of the daily returns of an underlying index or asset. Let β be the leverage ratio stated by the LETF, and R_j the daily return of the reference. Ideally, the LETF value on day n, denoted by L_n, is

$$L_n = L_0 \cdot \prod_{j=1}^{n} (1 + \beta\, R_j). \tag{2.1}$$

We call this the leveraged benchmark, and examine the empirical performance of various LETFs with respect to this benchmark.

For many investors, one appeal of LETFs is that leverage can amplify returns when the underlying is moving in the desired direction. Mathematically,

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we can see this as follows. Rearranging (2.1) and taking the derivative of the logarithm, we have

$$\frac{d}{d\beta}\left(\log\left(\frac{L_n}{L_0}\right)\right) = \sum_{j=1}^{n} \frac{R_j}{1+\beta\, R_j}. \tag{2.2}$$

With a positive leverage ratio $\beta > 0$, if $R_j > 0$ for all j, then $\log\left(\frac{L_n}{L_0}\right)$, or equivalently the value L_n, is increasing in β. In other words, when the reference asset is increasing in value, a larger, positive leverage ratio is preferred. On the other hand, if $R_j < 0$ for all j, and $\beta < 0$, a more negative β increases $\log\left(\frac{L_n}{L_0}\right)$ and thus L_n. This means that when the reference asset is decreasing in value, a more negative leverage ratio yields a higher return.

The example below illustrates the consequences of maintaining a constant leverage in an environment with nondirectional movements:

Day	ETF	%-change	+2x LETF	%-change	−2x LETF	%-change
0	100		100		100	
1	98	-2%	96	-4%	104	4%
2	99.96	2%	99.84	4%	99.84	-4%
3	97.96	-2%	95.85	-4%	103.83	4%
4	99.92	2%	99.68	4%	99.68	-4%
5	97.92	-2%	95.69	-4%	103.67	4%
6	99.88	2%	99.52	4%	99.52	-4%

Even though the ETF records a tiny loss of 0.12% after 6 days, the +2x LETF ends up with a loss of 0.48%, which is greater (in absolute value) than 2 times the return (−0.12%) of the ETF. We can see this to be the case on any day (e.g., not just the terminal date) except for day 1. For example, on day 3, the ETF has a net loss of 2.04% and the LETF has a net loss of 4.15%, which is greater (in absolute value) than 4.08% (twice the absolute value of the return of the ETF). Furthermore, it might be intuitive that the −2x LETF should have a positive return when the ETF and LETF have negative returns, this is not true. At the terminal date, both the long and short LETFs have recorded net losses of 0.48%. Again, this occurs throughout the period as well, not just the terminal date. In addition to day 6, both the long and short LETFs as well as the ETF itself are in the black. These results are consequences of volatility decay.

Although long and short LETFs are expected to move in opposite directions daily by design, it is often possible for both LETFs to have negative cumulative returns when held over a longer horizon. Figure 2.1 shows the historical cumulative returns of the ProShares leveraged gold ETFs, UGL (+2x) and GLL (−2x), which seek the corresponding multiples of the daily performance of the gold spot, over the period from July 2013 to July 2014. From trading day 124 (1/24/2014) onward, GLL has a negative cumulative return. There are points after trading date 124 where UGL also has a negative cumulative return. In fact, it starts in the black on this date and continues to have a net loss until trading date 146 (2/12/2014). This occurs again a few times, another long stretch where both have a net loss is trading date 210 (5/15/2014) through 233 (6/18/2014).This observation, though maybe counter-intuitive at first glance, is a consequence of daily replication of leveraged returns. The value erosion tends to accelerate during periods of nondirectional movements.

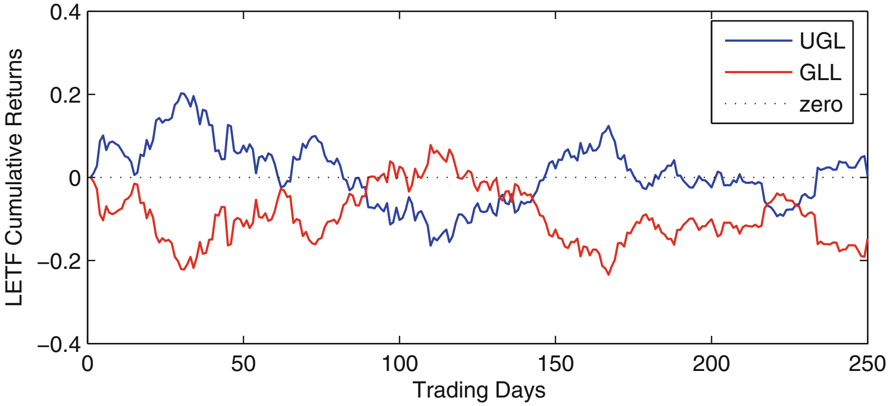


Fig. 2.1: UGL (+2x) and GLL (−2x) cumulative returns from July 2013 to July 2014. Observe that both UGL and GLL can give negative returns (below the dotted line of 0%) simultaneously over several periods in time.

A number of market observations suggest that LETFs exhibit value erosion as holding period increases, and it is more severe for highly leveraged ETFs. Let us illustrate by look at the empirical performance of LETFs over different holding horizons. Specifically, we compare the empirical returns of several major equity LETFs based on the S&P 500 index against multiples of the non-leveraged SPDR S&P 500 ETF (SPY).

In Figure 2.2, we present the returns of the ProShares Ultra S&P 500 ETF (SSO) with double long leverage ($\beta = +2$) and the ProShares Ultra-Short S&P 500 ETF (SDS) with double short leverage ($\beta = -2$), against ± 2 multiples of the SPY returns. We consider 1-day, 14-day, and 60-day rolling periods from September 29, 2010 to September 30, 2012. We observe from Figures 2.2(a)–2.2(b) that the returns fall along the straight line of slope 1. This reflects that both SSO and SDS are able to replicate, on a daily basis, the advertised multiple of the underlying ETF returns.

However, as the holding period lengthens to 14 days and 60 days, return discrepancies start to build (see Figures 2.2(c)–2.2(f)). In these cases, the LETF performance is often inferior to that of the underlying ETF, though the opposite could also happen, typically in a period with strong momentum. In general, a longer horizon also accumulates the erosion due to volatility drag. We shall investigate this more closely in subsequent sections.

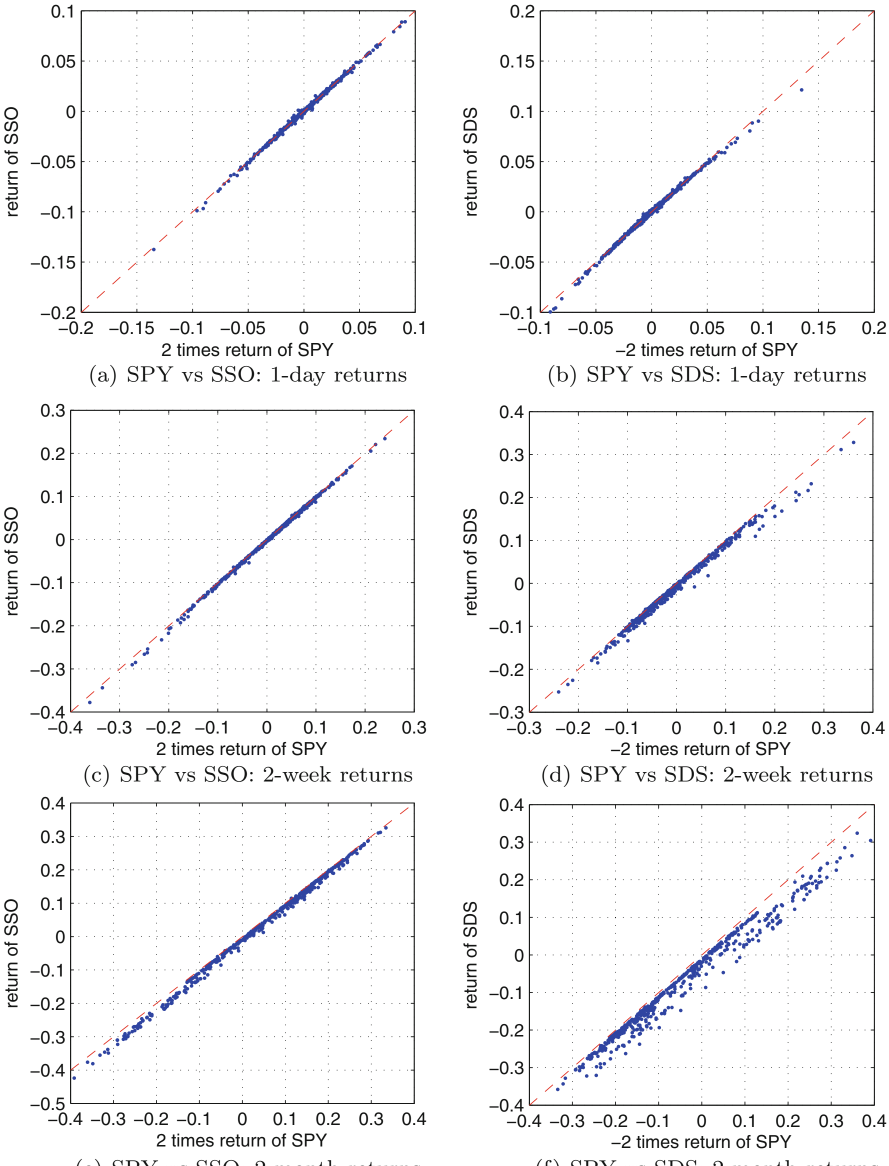
In Figure 2.3, we present the same analysis between SPY and the triple-leveraged ETFs, namely, UPRO and SPXU, with leverage ratios $\beta = +3$ and -3, respectively. As we can see, the one-day returns are matched very closely, but longer horizons again lead to higher discrepancies in returns between the triple LETFs and the underlying. Comparing across leverage ratios, the underperformance over a 60-day period is more pronounced for the triple than the double leverage ratios (see Figures 2.2(e)–2.2(f) and 2.3(e)–2.3(f)). Furthermore, short LETFs tend to fail to replicate the required returns more often than their long leveraged counterparts.

2.2 Continuous-Time Model for Leveraged ETFs

We model the evolution of the reference index $(S_t)_{t\geq 0}$ by the stochastic differential equation:

$$dS_t = S_t\,(\mu_t\,dt + \sigma_t\,dW_t), \tag{2.3}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under the historical measure $\mathbb{P}$. The stochastic drift $(\mu_t)_{t\geq 0}$ represents the ex-dividend annualized growth rate process, and $(\sigma_t)_{t\geq 0}$ is the stochastic volatility process. At this point, we do not specify a parametric stochastic volatility model, though many well-known models, such as the Heston model as well as other stochastic or local volatility models, also fit within the above framework.



(a) SPY vs SSO: 1-day returns

(b) SPY vs SDS: 1-day returns

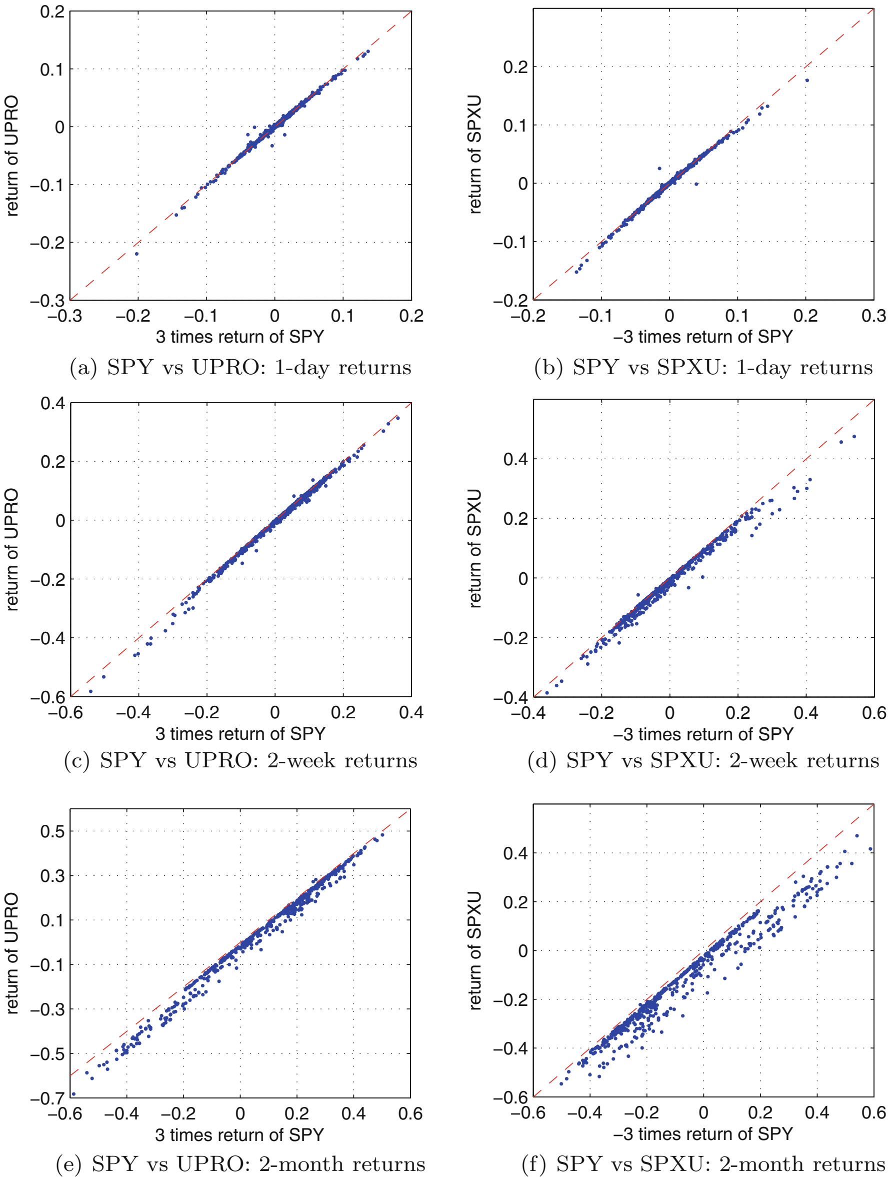
(c) SPY vs SSO: 2-week returns

(d) SPY vs SDS: 2-week returns

(e) SPY vs SSO: 2-month returns

(f) SPY vs SDS: 2-month returns

Fig. 2.2: 1-day (top), 2-week (center), and 2-month (bottom) returns of SPY against SSO (left) and SDS (right), in logarithmic scale. We considered 1-day, 2-week, and 2-month rolling periods from September 29, 2010 to September 30, 2012.



(a) SPY vs UPRO: 1-day returns

(b) SPY vs SPXU: 1-day returns

(c) SPY vs UPRO: 2-week returns

(d) SPY vs SPXU: 2-week returns

(e) SPY vs UPRO: 2-month returns

(f) SPY vs SPXU: 2-month returns

Fig. 2.3: 1-day (top), 2-week (center), and 2-month (bottom) returns of SPY against UPRO (left) and SPXU (right), in logarithmic scale. We considered 1-day, 2-week, and 2-month rolling periods from September 29, 2010 to September 30, 2012.

Based on the reference index S, a long leveraged ETF $(L_t)_{t\geq 0}$ with leverage ratio $\beta \geq 1$ is constructed by simultaneously investing the amount βL_t (β times the fund value) in the underlying S, and borrowing the amount $(\beta-1)L_t$ at the interest rate $r \geq 0$. This is essentially a constant proportion trading strategy. As is typical for all ETFs, a small expense rate $f \geq 0$ is incurred. As a result, the β-LETF value evolves according to

$$dL_t = L_t\,\beta\,\frac{dS_t}{S_t} - L_t((\beta-1)r + f)\,dt. \tag{2.4}$$

On the other hand, a leveraged fund with a negative leverage ratio $\beta \leq -1$ involves taking a short position of amount $|\beta L_t|$ in S and keeping $(1-\beta)L_t$ in the money market account. The fund value $(L_t)_{t\geq 0}$ also satisfies (2.4) with $\beta \leq -1$. For some short LETFs, it would be appropriate to incorporate the rate of borrowing $\lambda \geq 0$ for short selling S. This can be achieved by replacing μ_t with $\mu_t + \lambda$ in (2.4) with $\beta \leq -1$. Theoretically, one can also construct constant proportion portfolio with $\beta \in (-1, 1)$, but we do not discuss them since the most typical leverage ratios in practice are $\beta = 2, 3$ (long) and $-2, -3$ (short).

For both long and short LETFs, we recognize from (2.4) that L can be expressed in terms of the underlying index S:

$$L_t = L_0 \exp\left(\int_0^t (\beta\mu_u - (\beta-1)r - f - \frac{\beta^2\sigma_u^2}{2})\,du + \int_0^t \beta\sigma_u dW_u\right) \tag{2.5}$$

$$= L_0 \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left(-((\beta-1)r + f)t + \frac{1}{2}\beta(1-\beta)\int_0^t \sigma_u^2 du\right). \tag{2.6}$$

Taking (natural) log on both sides, we express the log-return of L in terms of that of S, namely,

$$\log\left(\frac{L_t}{L_0}\right) = \beta \log\left(\frac{S_t}{S_0}\right) - \big((\beta-1)r + f\big)t + \frac{1}{2}\beta(1-\beta)\int_0^t \sigma_u^2 du. \tag{2.7}$$

In view of the second term, the long and short LETFs possess asymmetric return characteristics and volatility exposure. First, we observe that

$$\frac{1}{2}\beta(1-\beta) < 0, \qquad \text{for} \qquad \beta \notin [0, 1],$$

so there is an erosion in log return proportional to the realized variance $\int_0^t \sigma_u^2 du$. Note that this effect is larger for a short LETF than its long leverage counterpart with the same magnitude. Since the realized variance is increasing in t, the value erosion, called *volatility decay*, is more significant over a longer holding horizon. Certainly, the expense fee also leads to decay in return. For more details on the derivation of (2.7) and its discrete-time analogue, we refer the reader to Avellaneda and Zhang (2010).

2.3 Empirical Leverage Ratio Estimation

Thus far the leverage ratio β has been taken as given. Indeed, it is advertised by the ETF provider as the target leverage ratio that the ETF seeks to achieve. In this section, we introduce a novel method to estimate the empirical leverage ratio realized by any given LETF.

In discrete time, with $\Delta t = 252^{-1}$, the k-day log-return of the LETF is given by

$$\log \frac{L_{t+k\Delta t}}{L_t} = \beta \log \frac{S_{t+k\Delta t}}{S_t} + \theta V_t^{(k)} + ((1-\beta)r - f)k\Delta t, \tag{2.8}$$

where the realized variance is computed by

$$V_t^{(k)} = \sum_{i=0}^{k-1} (R_{t+i\Delta t}^S - \bar{R}_t^S)^2, \quad \text{with} \quad \bar{R}_t^S = \frac{1}{n} \sum_{i=0}^{n-1} R_{t+i\Delta t}^S,$$

and R_t^S is the daily return of the reference index at time t.

The log-return equation (2.7), or its discretized version, immediately suggests a linear regression:

$$\log \frac{L_t}{L_0} = \hat{\beta} \log \frac{S_t}{S_0} + \hat{\theta} V_t + \hat{c} + \epsilon, \tag{2.9}$$

where $V_t = \int_0^t \sigma_u^2 du$ and $\epsilon \sim N(0, 1)$. Therefore, this results in a linear model, from which one can estimate from historical LETF and reference prices the constant coefficients $\hat{\beta}$, $\hat{\theta}$, and $\hat{c}$.

However, there is a significant collinearity issue due to the strong dependence of the two variables $\log \frac{S_t}{S_0}$ and V_t. Therefore, the coefficients estimated using this standard approach are not reliable. Guo and Leung (2015) have conducted regressions for 22 commodity LETFs and illustrated the issue of collinearity. For the purpose of estimating β, this approach also leads to another problem. Indeed, the first coefficient in (2.9) represents the estimated leverage ratio $\hat{\beta}$, but the coefficient $\hat{\theta}$ also has a theoretical value in terms of leverage ratio, i.e., $\beta(1-\beta)/2$. Thus, one can back out another estimated leverage ratio $\tilde{\beta}$ from $\hat{\theta} = \tilde{\beta}(1-\tilde{\beta})/2$, but there is no guarantee that the resulting leverage ratio will equal $\hat{\beta}$. In fact, as Guo and Leung (2015) have shown for many LETFs, the two estimated leverage ratios are most certainly different and deviate significantly from the theoretical leverage ratio β. This leads to an important dilemma - which estimate should we use? Is either estimate reliable?

Motivated by the above observations, we now discuss a new way to determine the realized leverage ratio. One of our main objectives is to give a single optimized estimate. To this end, we seek to find the leverage ratio β that minimizes the sum of squared differences between the realized LETF log-returns and the theoretical LETF log-return based on (2.7). As such, we solve the optimization problem:

$$\min_{\beta\in\mathbb{R}} \sum_{i=1}^{n}(y_i - f_i(\beta))^2$$

where $(y_i)_{i=1,\ldots,n}$ are the empirical log-returns of the LETF, and $(f_i(\beta))_{i=1,\ldots,n}$ are the theoretical returns given by

$$f_i(\beta) = \beta x_i - \frac{\beta(\beta-1)}{2}v_i + ((1-\beta)r - f)\Delta T \tag{2.10}$$

$$= \beta(x_i - r\Delta T) - \frac{\beta(\beta-1)}{2}v_i + (r-f)\Delta T, \tag{2.11}$$

where each $f_i(\beta)$ requires the log-return of the reference x_i, and the realized variance v_i over the same period of length ΔT (see (2.7)).

The optimal leverage ratio is found from the first-order optimality condition:

$$\sum_{i=1}^{n}(y_i - f_i(\beta))(x_i - r\Delta T - \beta v_i + \frac{1}{2}v_i) = 0.$$

We expand the left-hand side to get

$$
\begin{aligned}
&\sum_{i=1}^{n}(y_i - f_i(\beta))(x_i - r\Delta T - \beta v_i + \frac{1}{2}v_i)\\
&= \sum_{i=1}^{n}(y_i - \beta(x_i - r\Delta T) + \frac{\beta^2}{2}v_i - \frac{\beta}{2}v_i - (r-f)\Delta T)(x_i - r\Delta T - \beta v_i + \frac{1}{2}v_i)\\
&= \sum_{i=1}^{n}(y_i - (r-f)\Delta T - \beta(x_i - r\Delta T + \frac{v_i}{2}) + \frac{\beta^2}{2}v_i)(x_i - r\Delta T - \beta v_i + \frac{1}{2}v_i)\\
&= \left(-\sum_{i=1}^{n}\frac{v_i^2}{2}\right)\beta^3 + \left(\sum_{i=1}^{n}\frac{3}{2}(x_i - r\Delta T)v_i + v_i^2\right)\beta^2\\
&+ \left(\sum_{i=1}^{n} -((x_i - r\Delta T) + \frac{1}{2}v_i)^2 + v_i((r-f)\Delta T - y_i)\right)\beta\\
&+ \left(\sum_{i=1}^{n}(y_i - (r-f)\Delta T)((x_i - r\Delta T) + \frac{1}{2}v_i)\right).
\end{aligned}
$$

As a result, the optimality condition reduces to the cubic equation

$$
A\beta^3 + B\beta^2 + C\beta + D = 0, \tag{2.12}
$$

where the constant coefficients are given by

$$
\begin{aligned}
A &= -\sum_{i=1}^{n}\frac{v_i^2}{2},\\
B &= \sum_{i=1}^{n}\frac{3}{2}(x_i - r\Delta T)v_i + v_i^2,\\
C &= \sum_{i=1}^{n} -((x_i - r\Delta T) + \frac{1}{2}v_i)^2 + v_i((r-f)\Delta T - y_i,\\
D &= \sum_{i=1}^{n}(y_i - (r-f)\Delta T)((x_i - r\Delta T) + \frac{1}{2}v_i).
\end{aligned}
$$

Dividing by A in (2.12), we find the root of the equation:

$$
\beta^3 + b\beta^2 + c\beta + d = 0,
$$

with the obvious definitions for b, c, and d here. The discriminant of this cubic polynomial is

$$\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^3.$$

which tells us one of the following three cases:

1. If $\Delta > 0$, then the equation has 3 distinct real roots.
2. If $\Delta = 0$, then the equation has a multiple root and all its roots are real.
3. If $\Delta < 0$, then the equation has one real root and two complex conjugate roots.

By the well-known Cardano's method for cubic polynomials, the explicit solutions when $\Delta < 0$ are given by

$$\beta_1 = u_0 + u_1 - \frac{b}{3}, \quad \beta_{2,3} = -\frac{1}{2}(u_0 + u_1) \pm \frac{i\sqrt{3}}{2}(u_0 - u_1) - \frac{b}{3},$$

where

$$u_i = \sqrt[3]{-\frac{p}{2} + (-1)^i\sqrt{\frac{p^2}{4} + \frac{q^3}{27}}}, \quad i = 0, 1,$$
$$p = \frac{2b^3 - 9bc + 27d}{27}, \qquad q = \frac{3c - b^2}{3}.$$

If $\Delta = 0$ (iff $\frac{p^2}{4} + \frac{q^3}{27} = 0$), then the roots are real and at least two are the same:

$$\begin{array}{rrrl} -2\sqrt{-\frac{q}{3}} - \frac{b}{3}, & \sqrt{-\frac{q}{3}} - \frac{b}{3}, & \sqrt{-\frac{q}{3}} - \frac{b}{3} & \text{if } p > 0, \\ 2\sqrt{-\frac{q}{3}} - \frac{b}{3}, & -\sqrt{-\frac{q}{3}} - \frac{b}{3}, & -\sqrt{-\frac{q}{3}} - \frac{b}{3} & \text{if } p < 0, \\ 0, & 0, & 0 & \text{if } p = 0. \end{array}$$

If $\Delta > 0$ (iff $\frac{p^2}{4} + \frac{q^3}{27} < 0$), then the roots are real and can be expressed as

$$\beta_n = 2\sqrt{-\frac{q}{3}} \cos\left(\frac{\gamma}{3} + \frac{2n\pi}{3}\right), \quad n = 0, 1, 2$$

where

$$\gamma = \cos^{-1}\sqrt{\frac{p^2/4}{-q^3/27}}.$$

Alternatively, one can obtain numerical solutions using the root finding methods in a commercial computational software.

In Table 2.1, we apply our estimation method to six S&P500 based LETFs using the returns observed during 1/1/2013 to 5/31/2015. We can see that the leverage ratio β_{cub} estimated from our optimization method is extremely close to the target multiple β. The leverage ratio β_{reg} estimated from linear regression is also close to β. However, the associated realized volatility coefficient θ_{reg} from regression is not close to its theoretical value $\theta = (\beta - \beta^2)/2$, especially for the three LETFs: SSO, UPRO, and SH, where the absolute percentage errors are 69%, 14%, and 46%, respectively.

LETFs	β	θ	β_{cub}	β_{reg}	θ_{cub}	θ_{reg}
SPY	1	0	1.0004	1.0009	−0.0001	−0.1091
SSO	2	−1	2.0001	1.9910	−1.0001	−1.6918
UPRO	3	−3	3.0110	3.0003	−3.0275	−3.4178
SH	−1	−1	−1.0012	−0.9920	−1.0018	−0.5439
SDS	−2	−3	−2.0062	−1.9906	−3.0156	−2.9451
SPXU	−3	−6	−2.9965	−2.9708	−5.9879	−5.9619

Table 2.1: Estimated parameters using cubic root-finding ($\beta_{cub}, \theta_{cub}$) and linear regression ($\beta_{reg}, \theta_{reg}$) for six S&P500 based LETFs. Recall that theoretically $\theta = (\beta - \beta^2)/2$. Returns are computed based on 5-day holding periods during 01/01/2013 to 05/31/2015.

The leverage ratio estimation requires not only the returns of the LETFs but also the empirical variance of the reference index. Therefore, we need to partition the entire period into subintervals of n-days and compute the realized variance for each n-day horizon. If the sample period is short, e.g., a year or less, then this will yield a small number of data points and cause problems for the linear regression approach. This issue may arise for many LETFs that were introduced to the market only recently. Even for LETFs with longer histories, it would be useful to compare the estimated leverage ratio over different periods, for example, quarter by quarter. Again, this implies partitioning into short periods with low number of data points.

A major strength of our method, compared to linear regression, is that we do not need to work with a long sample period. This is mainly because our method involves finding a single variable β_{cub} by minimizing a univariate quadratic function. Once the estimated leverage ratio β_{cub} is obtained, the other coefficient θ_{cub} is instantly computed by $\theta_{cub} = (\beta_{cub} - \beta_{cub}^2)/2$ and is thus guaranteed to be consistent. In contrast, the linear regression involves finding the optimal pair $(\beta_{reg}, \theta_{reg})$ simultaneously from data, without constraining them to satisfy the known relationship.

Table 2.2 summarizes the estimated leverage ratios for six S&P500 LETFs from four quarters in 2014. With just over 60 trading days in each quarter, the linear regression method fails to return accurate or stable estimates for the realized variance coefficient θ_{reg}. In contrast, the cubic root-finding method generates a series of stable θ_{cub} that are very close to the theoretical value, over the quarters for each of the six LETFs.

Leveraged ETFs are commonly advertised to generate a prespecified multiple of the reference index return on a daily basis, regardless of the movements of the reference or market conditions. As is well known, the accuracy of the promised return replication varies across leveraged ETFs. Even if we focus on a single leveraged ETF, its deviation from the stated objective may change over time.

We want to measure the empirical leverage ratio conditioned on the *sign* of the returns of the reference index. To this end, we apply our cubic root-finding method for the periods during which the reference index experiences positive returns, and separately, for the periods with negative reference returns.

The results are quite surprising. Figure 2.4(a) displays the empirical leverage ratios estimated from 1/1/2013 to 5/31/2015, with different sampling subintervals (in days). When the reference returns are positive, the leverage ratio tends to be higher than the stated multiple (+2) for the double-long leveraged ETF, SSO. In contrast, this LETF realizes a leverage ratio less than +2 when the reference returns are negative.

If we turn to Figure 2.5(a) for the double-short leveraged ETF, SDS, we observe that the deviation from the stated leverage ratio −2 is more significant. When the reference returns are positive (resp. negative), the leverage ratio tends to be more (resp. less) negative than −2. This can be interpreted as the two LETFs tend to *over-leverage* when the reference index

LETFs	Qtr	β_{cub}	β_{reg}	θ_{cub}	θ_{reg}
SPY	1	0.9956	0.9938	0.0022	−0.4319
($\beta = +1$)	2	0.9924	1.0102	0.0038	2.3251
($\theta = 0$)	3	1.0080	1.0079	−0.0040	−0.0163
	4	0.9670	0.9634	0.0159	−1.1210
SSO	1	1.9941	1.9761	−0.9911	−4.4384
($\beta = +2$)	2	1.9999	1.9599	−0.9998	−5.1184
($\theta = -1$)	3	1.9947	1.9774	−0.9921	−4.1630
	4	1.9818	1.9975	−0.9729	−1.6769
UPRO	1	2.9801	2.9583	−2.9505	−7.0032
($\beta = +3$)	2	3.0236	2.9723	−3.0594	−8.1769
($\theta = -3$)	3	3.0047	2.9850	−3.0118	−6.7569
	4	2.9801	2.9728	−2.9504	−3.5249
SH	1	−0.9927	−0.9914	−0.9891	−1.3642
($\beta = -1$)	2	−1.0166	−1.0110	−1.0251	−1.3263
($\theta = -1$)	3	−0.9906	−0.9852	−0.9859	−0.0514
	4	−1.0060	−1.0025	−1.0090	−1.7533
SDS	1	−1.9821	−1.9741	−2.9555	−2.5680
($\beta = -2$)	2	−2.0148	−1.9757	−3.0380	−0.1355
($\theta = -3$)	3	−1.9775	−1.9711	−2.9440	−1.9718
	4	−2.0113	−2.0009	−3.0284	−3.2447
SPXU	1	−2.9693	−2.9497	−5.8932	−3.6919
($\beta = -3$)	2	−3.0198	−2.9583	−6.0697	−1.3943
($\theta = -6$)	3	−2.9601	−2.9384	−5.8610	−2.3462
	4	−3.0161	−2.9974	−6.0564	−6.2966

Table 2.2: Estimated parameters using cubic root-finding ($\beta_{cub}, \theta_{cub}$) and linear regression ($\beta_{reg}, \theta_{reg}$) for six S&P500 based LETFs. Recall that theoretically $\theta = (\beta - \beta^2)/2$. Returns are computed based on 3-day holding periods over the 4 quarters in 2014.

experiences positive returns and *under-leverage* when the reference loses value. Moreover, if we compare our results to those using linear regression in Figures 2.4(b) and 2.5(b), the latter approach fails to generate a clear pattern. This illustrates another useful application of our method and its advantage over linear regression.

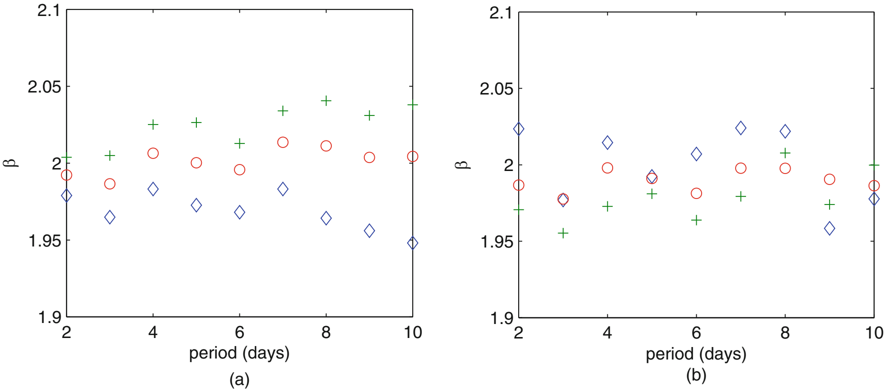


Fig. 2.4: Empirical leverage ratios for the double-long ($\beta = +2$) LETF, SSO, estimated using (a) the cubic root-finding approach, and (b) linear regression. Returns during 01/01/13 to 05/31/15 are used.

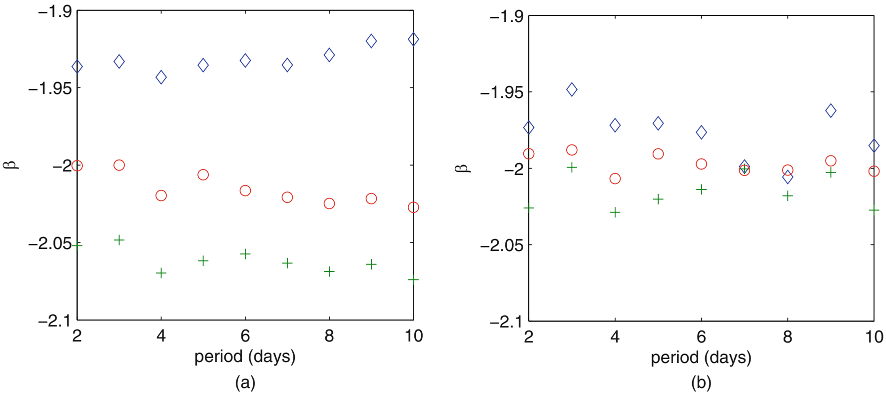


Fig. 2.5: Empirical leverage ratios for the double-short ($\beta = -2$) LETF, SDS, estimated using (a) the cubic root-finding approach, and (b) linear regression. Returns during 01/01/13 to 05/31/15 are used.

2.4 Dynamic Leveraged Futures Portfolio

For some ETFs, the reference asset or index may be very illiquid or not even traded. This can be a reason that the leveraged ETF price evolution may deviate from the dynamics of the leveraged portfolio described in (2.4). For LETFs that track commodity spot prices, one can alternatively trade the commodity futures to generate the required leverage.

In this section we analyze the returns and tracking performances of various leveraged ETFs. From historical prices of each LETF, we conduct an estimation of the leverage ratio and investigate the potential deviation from the target leverage ratio. Moreover, we construct a number of static portfolios with futures contracts to seek replication of some leveraged benchmarks. However, the static portfolios fail to effectively track the leveraged benchmarks. This motivates us to consider a dynamic portfolio with futures, which turns out to have a much better tracking performance.

First, we conduct a regression analysis and the results are given in Table 2.3. Each slope is approximately equal to the LETF's target leverage ratio. In principle, if each (L)ETF is able to generate the desired multiple of daily returns, the slopes of the regression should be equal to the various leverage ratios. In this table, we give an additional two columns for the t-statistic and p-value for testing the hypothesis: $\{H_0 : \text{slope} = \beta\}$ vs. $\{H_1 : \text{slope} \neq \beta\}$. Here, β is the target leverage ratio. We can see that each p-value is larger than 0.05 and therefore conclude that statistically, each (L)ETF does not differ from its target leverage ratio. This demonstrates us that the (L)ETFs are performing exactly as desired, at least on a daily basis.

(L)ETF	Slope	Intercept	t-stat	p-value	R^2	$RMSE$
GLD	1.0054	$-1.64 \cdot 10^{-5}$	1.4269	0.1538	0.9806	0.0016
UGL	2.0057	$-1.31 \cdot 10^{-4}$	0.7332	0.4636	0.9793	0.0034
GLL	-2.0056	$-1.06 \cdot 10^{-4}$	0.6709	0.5024	0.9767	0.0036
UGLD	2.9936	$-1.99 \cdot 10^{-4}$	0.4521	0.6513	0.9848	0.0042
DGLD	-2.9753	$-1.69 \cdot 10^{-5}$	0.9744	0.3302	0.9526	0.0075

Table 2.3: A summary of the regression coefficients and measures of goodness of fit for regressing one-day returns of (L)ETFs versus spot gold. We include 2 additional columns for the t-statistic and p-value for testing the hypothesis that the slope equals the leverage ratio in each case.

We see also that the R^2 values for each regression are quite high, all above 95%. Next, we compare the long and short LETFs for a fixed $|\beta| \in \{2, 3\}$. The short LETF tends to have a higher RMSE and lower R^2 value. Finally, we see in general that as the leverage ratio increases in absolute value, there is a higher RMSE. One possible explanation is that the benchmark is leveraged, and this could magnify the tracking error.

Let us analyze the effects of changing the holding period. In Table 2.4 we give the slopes and intercepts for the regressions of each (L)ETF's return versus the spot return while varying the holding period between 1 and 5 days. Our computations show the R^2 values are all above 95%. We can see that all the slopes are approximately equal to the target leverage ratio of the (L)ETF. However we notice that in general the intercepts get more negative as the holding period is lengthened. Although they are still quite small, they become more significant as the holding period increases. Our calculations show that the p-values for testing the hypothesis: $\{H_0 : \text{intercept} = 0\}$ vs. $\{H_1 : \text{intercept} \neq 0\}$ generally tend to decrease for each (L)ETF. In fact, for UGL the intercepts turn out to be statistically different from 0 (at the 5% level) for holding periods of 3, 4, and 5 days with p-values of 1.37%, 0.57%, and 0.33%, respectively. This is consistent with the volatility decay discussed above. We saw there an example where over shorter periods, the LETF tracks its leverage ratio well, but over a longer period it tends to lose money when there is high volatility. The intercepts being different from 0 is akin to the volatility decay in the following sense. Over longer periods, the regressions show that we require more information than just the gold return to predict the LETF return.

To compare the performance of the LETF versus the target multiple of the spot return, we also report in Table 2.4 the average return differential defined by

$$\overline{RD} = \frac{1}{m} \sum_{j=1}^{m} \left(R_j^{(L)} - \beta \cdot R_j^{(G)} \right), \tag{2.13}$$

where m is the number of the periods, $R_j^{(L)}$ is the LETF's return over the holding period, and $R_j^{(G)}$ is the spot's return over the holding period. We find this to be increasing (in absolute value) with the holding period length. That is, as we hold the LETF longer, it tends to increasingly underperform with respect to the multiple of the underlying return, on average. This is exactly the same notion described above, since over time, the volatility of the underlying causes the LETF to erode in value.

	Days	UGL	GLL	UGLD	DGLD	GLD
Slope	1	2.0057	-2.0056	2.9916	-2.9636	1.0054
	2	2.0083	-2.0040	2.9202	-3.0243	1.0048
	3	1.9777	-1.9966	2.9452	-3.0669	0.9919
	4	2.0007	-2.0091	2.9785	-3.0431	1.0021
	5	2.0208	-2.0327	2.8948	-3.0727	1.0104
Intercept $(\cdot 10^{-4})$	1	-1.3107	-1.0551	-1.9861	-0.1691	-0.1643
	2	-2.7328	-2.2881	-4.2312	-1.8155	-0.3339
	3	-4.0642	-3.9663	-4.9799	-1.0563	-0.4084
	4	-5.4443	-4.6217	-8.0848	-2.6997	-0.6613
	5	-6.8161	-4.8281	-8.0342	-0.3077	-0.9236
$\overline{RD}$ $(\cdot 10^{-3})$	1	-0.1289	-0.1076	-0.1967	-0.0241	-0.0144
	2	-0.2668	-0.2319	-0.4400	-0.1908	-0.0296
	3	-0.4325	-0.3927	-0.5351	-0.0846	-0.0503
	4	-0.5433	-0.4765	-0.8169	-0.2785	-0.0629
	5	-0.6408	-0.5471	-0.7918	0.0717	-0.0720

Table 2.4: A summary of the slopes and intercepts from the regressions of LETF returns versus gold returns, as well as the average return differential ($\overline{RD}$) over different holding periods.

2.4.1 Static Leverage Replication

We consider the problem of static leverage replication and seek an optimal static portfolio of futures which minimizes SSE. Let k be the number of futures contracts and $\mathbf{w} := (w_0, \ldots, w_k)$ be the real-valued vector of portfolio weights. As before, w_0 represents the weight given to the money market account. We seek the weights which minimize SSE over the 5-year period 12/22/2008 through 12/22/2013. Thus, we are led to the same constrained least squares optimization problem:

$$\begin{aligned} \min_{\mathbf{w}\in\mathbb{R}^{k+1}} \quad & \|\mathbf{C}\mathbf{w} - \mathbf{L}\|^2 \\ \text{s.t.} \quad & \sum_{j=0}^{k} w_j = 1 \end{aligned} \tag{2.14}$$

Again, the matrix **C** contains as columns, the historical prices of the various futures contracts and the money market account. Here, the vector **L** contains the historical prices of the leveraged benchmark in (2.1). Without loss of generality, we normalize the prices by $1000 so that our solution will give us a set of weights on each instrument.

To compare the tracking error of our optimized portfolios to that of investments in the LETFs, we will perform an out of sample analysis over the period 12/23/2013 through 7/14/2014 and see how $1000 invested in the LETFs and $1000 invested in our optimal portfolios perform. In order to quantify the performance we use the same root mean square error

$$RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(V_j - L_j)^2}, \tag{2.15}$$

where V_j is the dollar value of the portfolio on trading day j, while L_j is the dollar value of the leveraged benchmark on trading day j. Now, we present the results for the optimization and in sample/out of sample RMSE.

UGL(+2x)	Futures	w_0	w_1	w_2	$RMSE$ (in)	$RMSE$ (out)
1 Futures	1-m	-1.4816	2.4816	-	153.2015	41.4853
	2-m	-1.5760	2.5760	-	140.4125	49.5663
	6-m	-1.5811	2.5811	-	138.6312	47.4595
	12-m	-1.6263	2.6263	-	134.5068	48.2985
2 Futures	1-m, 2-m	-1.9450	-9.8944	12.8394	114.3095	81.3697
	1-m, 6-m	-1.9118	-8.4345	11.3463	113.6758	67.3943
	1-m, 12-m	-2.0222	-6.9279	9.9501	108.2990	67.1578
	2-m, 6-m	-1.6454	-34.2620	36.9073	126.6207	22.6495
	2-m, 12-m	-1.9910	-18.9915	21.9825	111.7505	40.9285
	6-m, 12-m	-2.2434	-35.6692	38.9126	102.8628	62.5383

Table 2.5: A summary of the weights and in/out of sample RMSEs for portfolios of 1 and 2 futures contracts which attempt to replicate a leveraged benchmark with $\beta = 2$. By comparison, the +2x LETF, UGL has an out of sample RMSE of only 5.5249.

GLL(−2x)	Futures	w_0	w_1	w_2	$RMSE$ (in)	$RMSE$ (out)
1 Futures	1-m	1.9754	-0.9754	-	152.3335	76.1438
	2-m	2.0107	-1.0107	-	155.7615	73.1470
	6-m	2.0123	-1.0123	-	156.4393	74.0060
	12-m	2.0293	-1.0293	-	158.0679	73.7924
2 Futures	1-m, 2-m	1.6951	-8.4629	7.7678	139.2744	100.1009
	1-m, 6-m	1.7051	-7.8346	7.1295	137.9877	92.2112
	1-m, 12-m	1.6077	-7.3754	6.7677	133.3171	93.1121
	2-m, 6-m	1.9390	-39.0864	38.1474	142.5732	43.4071
	2-m, 12-m	1.5768	-23.5616	22.9848	127.9063	62.1042
	6-m, 12-m	1.2789	-43.3691	-43.0902	117.8184	88.3599

Table 2.6: A summary of the weights and in/out of sample RMSEs for portfolios of 1 and 2 futures contracts which attempt to replicate a leveraged benchmark with $\beta = -2$. By comparison, the −2x LETF, GLL has an out of sample RMSE of only 4.7627.

UGLD(+3x)	Futures	w_0	w_1	w_2	$RMSE$ (in)	$RMSE$ (out)
1 Futures	1-m	-3.3370	4.3370	-	555.4915	111.5113
	2-m	-3.5063	4.5063	-	529.9086	125.9864
	6-m	-3.5156	4.5156	-	526.5845	122.3318
	12-m	-3.5959	4.5959	-	518.9690	123.8302
2 Futures	1-m, 2-m	-5.1815	-44.9250	51.1065	379.1168	270.5829
	1-m, 6-m	-5.0268	-38.5351	44.5619	381.9211	213.6217
	1-m, 12-m	-5.4197	-31.9108	38.3305	366.4976	210.9125
	2-m, 6-m	-3.7808	-141.3154	146.0961	472.3290	48.0413
	2-m, 12-m	-5.1256	-79.6610	85.7866	413.1968	98.3373
	6-m, 12-m	-6.1401	-147.0441	154.1842	376.4005	185.8717

Table 2.7: A summary of the weights and in/out of sample RMSEs for portfolios of 1 and 2 futures contracts which attempt to replicate a leveraged benchmark with $\beta = 3$. By comparison, the +3x LETF, UGLD has an out of sample RMSE of only 6.0813.

DGLD(−3x)	Futures	w_0	w_1	w_2	$RMSE$ (in)	$RMSE$ (out)
1 Futures	1-m	2.1474	-1.1474	-	222.5908	135.6776
	2-m	2.1889	-1.1889	-	225.7671	132.1496
	6-m	2.1908	-1.1908	-	226.4291	133.1623
	12-m	2.2106	-1.2106	-	228.1371	132.9166
2 Futures	1-m, 2-m	1.8265	-9.7161	8.8896	211.0905	163.0473
	1-m, 6-m	1.8353	-9.0649	8.2296	209.7556	154.1627
	1-m, 12-m	1.7078	-8.7977	8.0899	204.4129	155.8254
	2-m, 6-m	2.1026	-46.9910	45.8884	212.7849	95.2484
	2-m, 12-m	1.6398	-29.7260	29.0862	195.7478	117.3542
	6-m, 12-m	1.2446	-55.8313	55.5867	183.4220	150.5826

Table 2.8: A summary of the weights and in/out of sample RMSEs for portfolios of 1 and 2 futures contracts which attempt to replicate a leveraged benchmark with $\beta = -3$. By comparison, the −3x LETF, DGLD has an out of sample RMSE of only 4.4372.

The static portfolios do not replicate the leveraged benchmark well here. In Tables 2.5, 2.6, 2.7, and 2.8, the RMSE values are quite large for all the portfolios. The minimum RMSE for any portfolio of futures trying to replicate any leveraged benchmark is 22.6495 (achieved by a portfolio of 2-month and 6-month futures attempting to replicate a +2x investment in gold) and by comparison the maximum RMSE for any LETF trying to replicate its respective leveraged investment is 6.0813. (This is achieved by UGLD, which tracks a +3x investment in gold.) Unlike the unleveraged investment, the money market account is extensively used throughout the various portfolios. This is interesting but also logical. Indeed, in order to create leverage, the portfolio must either borrow if $\beta > 0$ or invest in the money market account if $\beta < 0$.

Furthermore, the optimal weights tend to lead to over/under-leveraging. Since we are considering an investment in gold, the sum of the weights on the futures (which are instruments for investment in gold) can be interpreted as the leverage on the portfolio. Since all the weights sum to 1, we can compute the approximate leverage as $1 - w_0$. For the +2x and +3x investments, these values are all larger than 2 and 3, respectively. For the −2x and −3x investments, these values are all smaller (in absolute value) than −2 and −3, respectively. Thus we see that the long portfolios tend to be over-leveraged, while the short portfolios tend to be under-leveraged.

The optimization procedure has led to some rather uneven portfolio weights. For example, the optimal portfolio of 6-month and 12-month futures that attempts to replicate a +3x investment in spot gold requires the following transactions at inception: borrow \$6,140.11 from the money market account, short \$147,044.08 in 6-month futures and long \$154,184.19 in 12-month futures. In practice this would not be possible in the marketplace due to position limits that may be in place.

2.4.2 Dynamic Leverage Replication

To improve upon the replication in Section 2.4.1, we now consider an example in which we construct a dynamic portfolio with the front-month futures contract and cash. Let P_t be our portfolio value at time t. At every point in time, the portfolio invests β times the value of the fund in the futures contract in order to achieve the required leverage. As a result, the value of our portfolio is similar to (2.4) but the reference price S (gold spot in this example) is replaced by the front-month futures price.[1]

To quantify our portfolio's replicating ability we will use the same root mean squared error:

$$RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(P_j - L_j)^2}, \tag{2.16}$$

where L_j is the value of a leveraged investment in gold and P_j is the value of our portfolio, each at trading day j. For this dynamic portfolio, there is no sample from which we will need to draw our weights or train our model in any way. Therefore, we can look at any conceivable time period and compare how the LETF (L) or leveraged portfolio (P) performs using the metric in (2.16).

For this tracking metric, we consider the period 1/3/2012 (first trading day of 2012) to 7/14/2014 for gold leveraged ETFs listed in Table 2.9. The results are shown in Table 2.10. The portfolio RMSEs range between 0.687%

[1] Leung and Ward (2015) show that the front-month futures is empirically most effective in replicating the spot gold price.

LETF	Reference	Underlying	Issuer	β	Fee	Inception
GLD	GOLDLNPM	Gold Bullion	iShares	1	0.40%	11/18/2004
UGL	GOLDLNPM	Gold Bullion	ProShares	2	0.95%	12/01/2008
GLL	GOLDLNPM	Gold Bullion	ProShares	−2	0.95%	12/01/2008
UGLD	SPGSGCP	Gold Bullion	VelocityShares	3	1.35%	10/17/2011
DGLD	SPGSGCP	Gold Bullion	VelocityShares	−3	1.35%	10/17/2011

Table 2.9: A summary of the gold LETFs, along with the non-leveraged ETF (GLD). The LETFs with higher absolute leverage ratios, $|\beta| \in \{2, 3\}$, tend to have higher expense fees.

and 3.291%, which are smaller than LETF RMSEs which range between 1.87% and 4.338%. Overall, we see that the portfolio RMSEs are lower than the LETF RMSEs. Indeed, we see that our dynamic portfolio is able to track the target leveraged index quite well according to the RMSE values for $\beta \in \{2, -2, 3\}$. However the tracking is not as strong for $\beta = -3$. Nonetheless, the value is quite small and not that far off from the LETF RMSE.

In Figure 2.6, we see the time evolution for both the dynamic portfolio and GLL compared to the −2x benchmark. It is visible that the LETF tends to underperform the benchmark and the difference worsens over time. On the other hand, the portfolio tends to stay close to the benchmark over the entire period. Though not reported here, we observe similar patterns for other gold LETFs.

In Table 2.10, we also give the annual returns for each asset for the years 2011, 2012, and 2013. For UGLD and DGLD we do not have data for the full year of 2011 (its issue date was 10/17/2011) so we do not have annual returns for these LETFs in 2011. The dynamic portfolio returns range between -69.22% and 107.54% while the LETF returns range between -69.90% and 106.16%. Comparing each year and leverage ratio pair, we find that, except for $\beta = -3$ in 2012, our portfolio outperforms the LETF in each year. Thus, we have shown that in general a dynamic portfolio consisting of just one futures contract can not only more closely track the target leveraged index, but it also outperforms the respective LETF.

			Annual Return (%)		
β	Asset	$RMSE$	2011	2012	2013
+2x	UGL	30.33	12.90	2.81	-52.31
	Portfolio	6.87	15.23	6.29	-51.83
−2x	GLL	40.12	-29.43	-16.40	67.82
	Portfolio	15.87	-27.06	-14.89	70.57
+3x	UGLD	43.38	-	0.41	-69.90
	Portfolio	12.55	-	5.29	-69.22
−3x	DGLD	18.70	-	-23.57	106.16
	Portfolio	32.91	-	-24.51	107.54

Table 2.10: A summary of the annual returns (over the periods: 1/3/2011 to 12/31/2011, 1/3/2012 to 12/31/2012, and 1/2/2013 to 12/31/2013) and RMSE for each LETF and a dynamic portfolio of front-month futures and cash. RMSE values are calculated over the period 1/3/2012 to 7/14/2014.

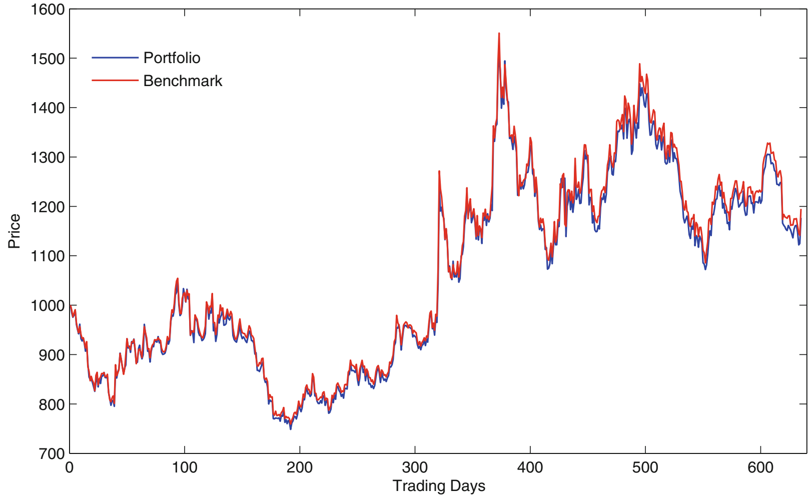


Fig. 2.6: Time evolution of our dynamic portfolio of front-month futures and cash (top) compared to the −2x benchmark and GLL (bottom) compared to the −2x benchmark. Time period displayed is 1/3/2012 to 7/14/2014.

2.5 Static Delta-Neutral Long-Volatility LETF Portfolios

The advent of ETFs has facilitated pairs trading in industry since many ETFs are designed to track identical or similar indexes and assets. For instance, Triantafyllopoulos and Montana (2009) study the mean-reverting dynamics of the spreads between commodity ETFs, and Leung and Li (2015b) derive the optimal timing strategies to trade an ETF pair.

LETFs can also be used in combination to construct various portfolios. Some strategies are designed accounting for the value erosion due to volatility decay associated with LETFs. In this section, we discuss how to construct LETF portfolios that are insensitive to the changes of the underlying (delta-neutral) but has a long volatility exposure.

For our analysis in this section, the reference index S is assumed to follow a general diffusion price dynamics described in (2.3), and the LETF value is given by

$$L_t = L_0 \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left((-(\beta-1)r - f)\,t - \frac{1}{2}\beta(\beta-1)V_t\right), \tag{2.17}$$

where $V_t = \int_0^t \sigma_u^2\,du$ is the realized variance of S up to time t.

Taking advantage of the volatility decay, a well-known trading strategy used by practitioners involves shorting a $\pm\beta$ pair of LETFs with the same reference, as discussed in Leung and Santoli (2012); Mackintosh and Lin (2010); Mason et al. (2010). Since the LETFs have opposite daily returns on the same reference index, the portfolio has little exposure to the reference index as long as the holding period is sufficiently short. With this strategy, the volatility decay can in fact help generate profit. However, the portfolio is exposed to risk during periods of low volatility and high trending even if we assume that the LETFs are tracking perfectly. We now describe an extension of this trading strategy by allowing the positive and negative leverage ratios to differ. As we determine the portfolio weights to eliminate the dependence on the reference, we show that the resulting portfolio is delta-neutral and long volatility.

We construct a weighted portfolio which is *short* the LETF with leverage ratio $\beta_+ > 0$ and *short* another LETF with leverage ratio $\beta_- < 0$. Let us emphasize that both LETFs have the same reference index, but that β_+ and $|\beta_-|$ may differ. A fraction $\omega \in (0,1)$ of the portfolio is short in the β_+-LETF and $(1-\omega)$ of the portfolio in the β_--LETF. At time T, the return from this strategy is given by

$$\mathcal{R}_T = 1 - \omega \frac{L_T^+}{L_0^+} - (1-\omega)\frac{L_T^-}{L_0^-}. \tag{2.18}$$

Applying (2.17), $\mathcal{R}_T$ can be expressed as an explicit function of the return and realized variance of the reference index. That is,

$$\mathcal{R}_T = 1 - \omega \left(\frac{S_T}{S_0}\right)^{\beta_+} \exp\left(\Gamma_T^+\right) - (1-\omega)\left(\frac{S_T}{S_0}\right)^{\beta_-} \exp\left(\Gamma_T^-\right), \tag{2.19}$$

where

$$\Gamma_T^{\pm} = \frac{\beta_{\pm} - \beta_{\pm}^2}{2} V_T + ((1-\beta_{\pm})r - f_{\pm})T, \tag{2.20}$$

Here, $\beta_{\pm}$ and $f_{\pm}$ are the respective leverage ratios and fees of the two LETFs in the portfolio defined in (2.18). Note that the return $\mathcal{R}_T$ over a short holding period such that $\frac{L_T}{L_0} \approx 1$, one can pick an appropriate weight ω^* to approximately remove the dependence of $\mathcal{R}_T$ on S_T.

If we select the portfolio weight

$$\omega^* = \frac{-\beta_-}{\beta_+ - \beta_-}, \tag{2.21}$$

then the return from this strategy is given by

$$\boxed{\mathcal{R}_T \approx \frac{-\beta_-\beta_+}{2} V_T - \frac{\beta_-}{\beta_+ - \beta_-}(f_+ - f_-)T + (f_- - r)T.} \tag{2.22}$$

To see this, we substitute for $\frac{L_T}{L_0}$ with $\log \frac{L_T}{L_0} + 1$, which is valid when $\frac{L_T}{L_0} \approx 1$. Then, setting $\omega = \frac{-\beta_-}{\beta_+ - \beta_-}$ and applying (2.17), we arrive at (2.22).

The static portfolio return corresponding to the weight ω^* in (2.21) reflects a linear dependence on the realized variance. In particular, the coefficient $\frac{-\beta_-\beta_+}{2}$ in (2.22) is strictly positive, so the strategy is effectively long realized variance. Thus, we call this strategy long volatility. Also, as it does not depend on S_T, the ω^* portfolio is Δ-*neutral* as long as the reference does not move significantly.

In Table 2.11, we summarize the coefficient of V_T and the (short) portfolio weights $(\omega^*, 1-\omega^*)$ for different combinations of leverage ratios. Note that as long as $\beta_+ = -\beta_-$, we end up with the portfolio weight $\omega^* = 0.5$, which means we short both $\pm\beta$ LETFs of the same cash amount. All ω^*'s in the table are between 0 and 1, so a short position is taken in both the long and short LETFs. Also, the coefficient $\frac{-\beta_-\beta_+}{2}$ exceeds or equals to 1 except for the pair $(\beta_+, \beta_-) = (1, -1)$. The pair $(\beta_+, \beta_-) = (3, -3)$ corresponds to the most long-volatility portfolio with a coefficient of 4.5.

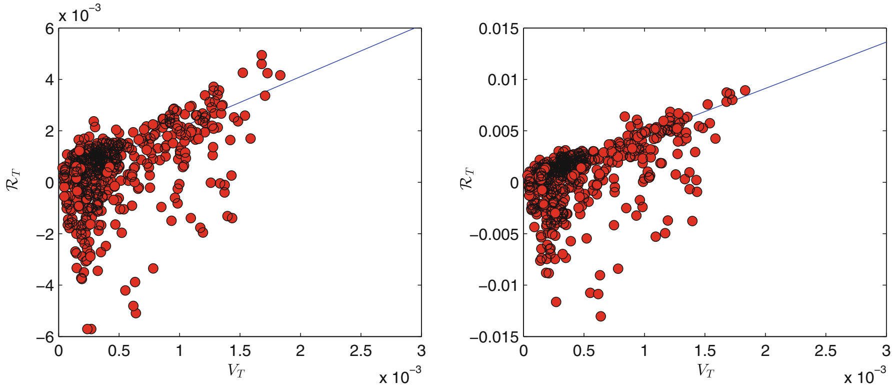
(β_+, β_-)	ω^*	$\frac{-\beta_-\beta_+}{2}$
$(1, -1)$	$1/2$	$1/2$
$(1, -2)$	$2/3$	1
$(1, -3)$	$3/4$	$3/2$
$(2, -1)$	$1/3$	1
$(2, -2)$	$1/2$	2
$(2, -3)$	$3/5$	3
$(3, -1)$	$1/4$	$3/2$
$(3, -2)$	$2/5$	3
$(3, -3)$	$1/2$	$9/2$

Table 2.11: Table of leverage ratios pairing (β_+, β_-), the static Δ-neutral portfolio weight ω^* (short position for the β_+-LETF), and realized variance coefficient $\frac{-\beta_-\beta_+}{2}$. Note that the (short) weight for the β_--LETF is $(1-\omega^*)$.

We now backtest the Δ-neutral strategy. For each LETF pair, we short \$0.5 of the β_+-LETF and \$0.5 of the β_-LETF with $\beta_+ = -\beta_- = 2$ and hold the position for 10 days. The return $\mathcal{R}_T$ depends on the relative weights on the long/short-LETFs but not the absolute cash amounts. Dividing the price data from the trading days during 2013–2015 into 10-day rolling periods, we calculate the returns from the strategy over each period. For every 10-day return, we compare it against the realized variance over the same time window.

This is illustrated in Figure 2.7 for the S&P-based LETFs, namely, SSO and SDS with ± 2 leverage ratios, and UPRO and SPXU with ± 3 leverage ratios. As a theoretical benchmark, we also plot $\mathcal{R}_T$ in (2.22) as a linear function. The 10-day returns are recorded over rolling periods, so they are not independent. In view of this, we emphasize that the straight lines in Figure 2.7 are not generated by regression but taken directly from (2.22). We choose (2.22) as a benchmark because it is expected to hold *pathwise* as long as $\frac{L_T}{L_0} \approx 1$ with negligible tracking error.

We observe from Figure 2.7 that the returns exhibit the predicted positive dependence on the realized variance (V_T) of the reference index. This confirms our intuition that the proposed strategy captures the volatility decay of LETF



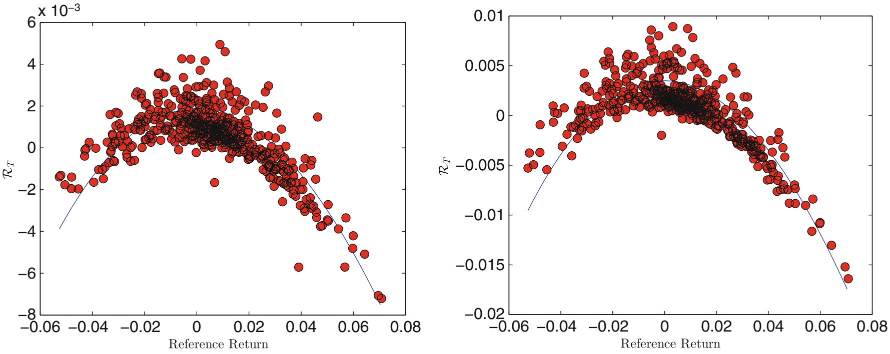
(a) SSO (+2) & SDS(-2) (b) UPRO (+3) & SPXU (-3)

Fig. 2.7: Plot of trading returns vs realized variance for a double short strategy over 5-day rolling holding periods, with $\beta_{\pm} = \pm 2$ for each LETF pair. We compare with the empirical returns (circle) from the ω^* strategy with the predicted return (solid line) in Prop. (2.22).

as profit. Comparing between the ± 2 LETF pair (SSO & SDS) and ± 3 pair (UPRO & SPXU), the latter is more sensitive to the realized variance. This is because the ± 3 pair has a larger coefficient of V_T in (2.22) than the ± 2 pair (4.5 vs 2 as seen in Table 2.11).

Nevertheless, there is also a visible amount of noise in the returns, especially for the ± 3 pair. The deviation from the straight line is asymmetric in both cases and is particularly skewed to the negative side for the UPRO-SPXU pair. This can be explained by the larger tracking errors commonly experienced by highly leveraged ETFs, such as the ± 3 pair.

While the ω^* portfolio is expected to be Δ-neutral for small movements in the reference index, the strategy is also short-Γ (with respect to the reference index). One way to see this is through Figure 2.8 that plots the returns against the reference index returns. Common to both LETF pairs, when the reference return is either very positive or negative, the return of the ω^*-strategy tends to be negative as a result of the short-Γ property. As a theoretical benchmark, we also plot the normalized return equation (2.19) which does not involve any approximation and applies even for large reference movements under the general diffusion model for the reference index. We see that the empirical returns follow the theoretical benchmark closely and show no directional trend when the reference index return is around zero as a consequence of the portfolio's Δ-neutrality.



(a) SSO (+2) & SDS(-2) (b) UPRO (+3) & SPXU (-3)

Fig. 2.8: Plot of trading returns vs realized variance for a double short strategy over 5-day rolling holding periods, with $\beta_{\pm} = \pm 2$ for each LETF pair. We compare with the empirical returns (circle) from the ω^* strategy with the predicted return (solid line) in (2.22).

More generally, one can also test strategies with other leverage ratios $\beta_{\pm}$ and the corresponding ω^*, or for other non-equity LETFs. For example, we refer to Guo and Leung (2015) for the implementation of this strategy for commodity LETFs.

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