Chapter 2
Differential Equations

Abstract When, for example, we seek to model changes in population size, or the path of a projectile, or the escape of water down a plughole, or rowing across a river, we generally have information about the rate of change of one variable, y, with respect to another one, x; writing down that information often leads to an equation involving quantities such as $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$, the derivatives of y with respect to x. In this introduction to a vast topic, we consider only straightforward first or second order ordinary differential equations: we show how they can be set up from verbal information, and how particular types can be solved by standard methods. We look at linked systems, with applications to predator-prey equations, and models for the spread of epidemics or rumours, with Exercises on topics such as carbon dating, cooling of objects, evaporation of mothballs, mixing of liquids and Lanchester's Square Law about conflicts.

2.1 What They Are, How They Arise

Many problems in maths are of the form: "We have two quantities of interest, and some information on how they are related. How to use that information to say as much as possible, and as precisely as possible, what that relation is." A *differential equation* may arise when the relation between these quantities includes information about the *rate* at which one variable changes as the other one changes.

For example, let t represent time, and let y be the distance travelled as in Fig. 2.1. If, as t increases from t_1 to t_2, so y increases from y_1 to y_2, then the *average velocity* over that period is $(y_2 - y_1)/(t_2 - t_1)$, the slope of the line AB.

But what about the actual velocity at time t_1? Keeping t_1 fixed, move t_2 closer and closer to t_1; write $t_2 = t_1 + \delta t$, where δt is a tiny quantity, and suppose the corresponding distance moved is $y_2 = y_1 + \delta y$. Then the average velocity over the time from t_1 to $t_1 + \delta t$ is the ratio $\frac{\delta y}{\delta t}$. In the limit, as δt decreases down to zero, this is the actual velocity at the time t_1, and we write it $\frac{dy}{dt}$. It might be more meaningful to write it as

$$\frac{d}{dt}(y),$$

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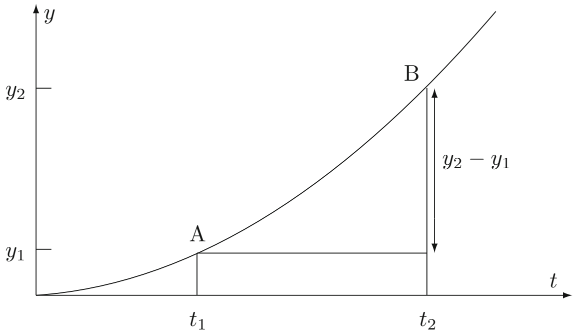

Fig. 2.1 Average velocity over a period of time

and read this as "the rate of change of distance y with time t", but we will stick to $\frac{dy}{dt}$. This will be the slope of the tangent to the curve at the point t_1.

Since $\frac{dy}{dt}$ represents velocity, so

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

is the rate of change of velocity with time, i.e. the *acceleration*. If a stone of mass $m\ \ (> 0)$ travels vertically (and we can ignore air resistance), then Newton's Law, that Mass $\times$ Acceleration = Force, says that

$$m\frac{d^2y}{dt^2} = mg, \quad \text{or simply} \quad \frac{d^2y}{dt^2} = g, \tag{2.1}$$

where g is the acceleration due to gravity. Our task would be to derive a formula for the distance y fallen at time t.

If air resistance is proportional to velocity, this changes to

$$\frac{d^2y}{dt^2} = g - c\frac{dy}{dt} \tag{2.2}$$

for some constant $c > 0$. Notice that, if we take $v = \frac{dy}{dt}$, this same equation becomes

$$\frac{dv}{dt} = g - cv. \tag{2.3}$$

For another example, suppose bacteria grow freely in a nutrient solution, with the population growth rate proportional to the current size. Here, with the natural notation that t is time, and x is population size, allowed to assume continuous values, we have

$$\frac{dx}{dt} = Kx, \tag{2.4}$$

$K > 0$ some constant. On the other hand, if there is competition for space or food, with a maximum possible size of N, the growth rate may be similar to (2.4) when x is small, but reduce as x increases: perhaps

$$\frac{dx}{dt} = Kx\left(1 - \frac{x}{N}\right) \tag{2.5}$$

is more realistic. Or maybe, as well as competition, the growth rate naturally decreases with time: perhaps K gets replaced by $K_0 \exp(-\lambda t)$. Then our relation becomes

$$\frac{dx}{dt} = K_0 \exp(-\lambda t) x \left(1 - \frac{x}{N}\right). \tag{2.6}$$

The general idea is to introduce symbols to represent relevant quantities, and then turn the physical description of what we believe is happening into an equation, or set of equations, intended to describe that belief.

Example 2.1 A snowplough, which clears snow at constant rate V/unit time sets off at time $t = 0$, and travels distance x_t by time $t > 0$. Snow began falling earlier, at time $t = -T$, and falls at a constant rate of s/unit time. Over the short time interval from time t to time $t + \delta t$, the amount of snow cleared is plainly $V\delta t$. But we can obtain another expression: as snow has been falling at rate s for a total time of $T + t$, the total amount of snow at any position is $s(T + t)$; and as the plough moves distance $x_{t+\delta t} - x_t$ over that short time interval, the amount cleared can also be written as $s(T + t)(x_{t+\delta t} - x_t)$.

Equating these, we have

$$s(T + t)(x_{t+\delta t} - x_t) = V\delta t.$$

Divide both sides by δt, pass to the limit as $\delta t \to 0$ to obtain

$$s(T + t)\frac{dx}{dt} = V.$$

Example 2.2 Consider a bath or sink with a small plughole with cross-sectional area A. At time t, take $V = V_t$ as the volume of water, x as the depth, and u as the escape velocity down the plughole. *Torricelli's Law* states that $u = \sqrt{2gx}$ (the same as the velocity of a body, initially at rest, falling distance x).

In the time interval from t to $t + \delta t$, the amount of water *lost* is $Au\delta t$, so

$$V_{t+\delta t} - V_t = -Au\delta t.$$

Divide by δt, let $\delta t \to 0$, to find

$$\frac{dV}{dt} = -Au = -A\sqrt{2gx}.$$

The volume V plainly depends on x, and the shape of the bath or sink. And since $\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = V'(x)\frac{dx}{dt}$, we can rewrite the relation as

$$\frac{dx}{dt} = -A\sqrt{2gx}/V'(x),$$

which, when solved, lets us find the time it takes for x to decrease from its initial level to zero.

Vocabulary. The *order* of a differential equation is the order of the highest derivative present—(2.1) and (2.2) above are second order, the others are first order. The *degree* is the highest power to which the derivative of highest order appears. All of the equations above are "first degree".

Thus the "simplest" format for a differential equation is

$$\frac{dy}{dx} = f(x, y) \tag{2.7}$$

for some function f; it is first order, first degree. You should not necessarily expect that such a differential equation, even if it only involves standard functions, is solvable with standard functions! For example, how might you set about solving $\frac{dy}{dx} = \sin(\exp(x) + 3y)$? But numerical methods could come to our rescue in such cases, and many differential equations that arise naturally fall into one of a small number of different types, each with its own method of solution. We look at some.

2.2 First Order Equations

(i). If the Eq. (2.7) can be put in the form $\frac{dy}{dx} = A(x).B(y)$, i.e. the function $f(x, y)$ is the product of some function of x alone with another involving y alone, we term it *variables separable*, and seek to solve it via

$$\int \frac{dy}{B(y)} = \int A(x)dx.$$

To complete the job, use standard techniques of integration to find both integrals, throw in a constant of integration, and use whatever other information we have—maybe the value of y when $x = 0$—to get the relevant solution.

Example 2.3 The Eq. (2.3) above has its variables separable: it can be turned into

$$\int \frac{dv}{g - cv} = \int dt,$$

hence $\frac{-1}{c}\log(g - cv) = t + K$, K a constant. If the stone starts from rest, so that $v = 0$ when $t = 0$, we find $v = \frac{g}{c}(1 - \exp(-ct))$.

Since $v = \frac{dx}{dt}$, this becomes $\frac{dx}{dt} = \frac{g}{c}(1 - \exp(-ct))$, which also has its variables separable, and leads to

$$\int dx = \frac{g}{c} \int \left(1 - e^{-ct}\right) dt,$$

easily integrable giving x as a function of t. See also Example 2.6 below.

(ii). There is a cunning trick available if the function f in (2.7) is *homogeneous*, i.e. if $f(tx, ty) = f(x, y)$ whenever t is non-zero. Change the variables from (x, y) to (x, v) by writing $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx},$$

and f becomes $f(x, vx)$; but $f(x, vx) = f(1, v)$ by the homogeneity property, so the original Eq. (2.7) becomes

$$v + x\frac{dv}{dx} = f(1, v).$$

This can be rewritten as $x\frac{dv}{dx} = f(1, v) - v = g(v)$ (say), clearly separable to

$$\int \frac{dv}{g(v)} = \int \frac{dx}{x}.$$

Remember the cunning trick, not this formula!

Example 2.4 The stream in a river c metres wide flows at the uniform rate a metres/second. A boat travelling at constant speed b metres/sec sets off to cross the river, always steering for the point directly opposite its initial point. Describe its path. For what values of a and b can it reach its goal?

Solution. Let the river banks be the parallel lines $x = 0$ (i.e. the y-axis) and $x = c$. The boat starts at $(c, 0)$, and always aims at $(0, 0)$. Suppose it is at position (x, y) as shown in Fig. 2.2.

Look separately at the components of the velocity, first across the river (i.e. x), and then upstream (i.e. y). Because the boat is aiming at the origin, we have

$$\frac{dx}{dt} = -b\cos(\theta).$$

Also, the influence of the current means that

$$\frac{dy}{dt} = -a + b\sin(\theta).$$

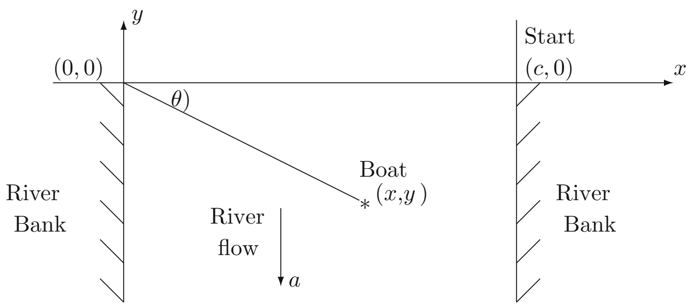

Fig. 2.2 The boat aims at its destination

Divide these out to get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a + b\sin(\theta)}{-b\cos(\theta)} = \frac{-a + b\left(-y/\sqrt{x^2+y^2}\right)}{-b\left(x/\sqrt{x^2+y^2}\right)},$$

which simplifies to

$$\frac{dy}{dx} = \frac{a\sqrt{x^2+y^2} + by}{bx}.$$

The right side is a homogeneous function, so put $y = vx$ to get

$$v + x\frac{dv}{dx} = \frac{a\sqrt{1+v^2} + bv}{b} = \frac{a}{b}\sqrt{1+v^2} + v.$$

Take $k = a/b$, the ratio of the speed of the river to the speed of the boat. Then $x\frac{dv}{dx} = k\sqrt{1+v^2}$, leading to

$$\int \frac{dv}{\sqrt{1+v^2}} = \int k\frac{dx}{x} = k\log(x) + Const.$$

To find the integral on the left side, if you have met the functions $\sinh(u)$ and $\cosh(u)$, you will be able to solve this by the substitution $v = \sinh(u)$, and some work with inverses. Otherwise, *define* $u = \log\left(v + \sqrt{1+v^2}\right)$, and see that

$$\frac{du}{dv} = \frac{1 + (1/2)\left(1+v^2\right)^{-1/2}.2v}{v + \sqrt{1+v^2}} = \frac{1}{\sqrt{1+v^2}}.$$

Hence, since differentiation and integration are inverse processes, the integral we seek is indeed $u = \log\left(v + \sqrt{1+v^2}\right)$.

Thus $\log\left(v + \sqrt{1 + v^2}\right) = k\log(x) + Const$. Initially, when $x = c$, so $y = 0$, hence also $v = 0$. So $0 = k\log(c) + Const$, giving

$$\log\left(v + \sqrt{1 + v^2}\right) = k\log(x) - k\log(c) = \log\left(\frac{x}{c}\right)^k.$$

Since $v = y/x$, this becomes

$$y + \sqrt{x^2 + y^2} = x^{k+1}/c^k \tag{2.8}$$

in our original notation.

This has "solved" the equation, but it is usual to try to end up with a solution in the form $y = A(x)$ for some expression $A(x)$. To this end, rewrite (2.8) as

$$\sqrt{x^2 + y^2} = \frac{x^{k+1}}{c^k} - y,$$

and square both sides. Cancel the term y^2 on each side, leading to

$$y = \frac{1}{2}\left(\frac{x^{k+1}}{c^k} - \frac{c^k}{x^{k-1}}\right).$$

Near $x = 0$, i.e. just before completing the crossing, the second term dominates. We look at the different cases.

(i) If $k > 1$, (the river flows faster than they can row), then as $x \to 0$, so $y \to -\infty$. The boat is swept downstream.

(ii) If $k = 1$, then $y = \left(x^2/c - c\right)/2$ so, as $x \to 0$ then $y \to -c/2$. The boat lands, but half a river's width downstream.

(iii) If $k < 1$, (they can row faster than the flow), then $y \to 0$ as $x \to 0$. The boat reaches its destination.

Let's check that these answers seem reasonable. If $k > 1$, when the boat is downstream of its target, and near its aim point, they are trying to row directly against the current, which is too strong.

When $k < 1$, they can indeed row faster than the opposing stream, so do reach their target.

When $k = 1$, they can row at exactly the stream speed. This case isn't important, as it will never happen than these speeds are EXACTLY equal. But even so, the answer, landing precisely c/2 downstream, is not easy to guess!

(iii) A *First Order Linear Equation* is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

and this too has a standard method of solution. Write $R(x) = \exp\left(\int P(x)dx\right)$, the so-called *integrating factor*: just multiply throughout by $R(x)$, leading to

$$R(x)\frac{dy}{dx} + R(x)P(x)y = R(x)Q(x).$$

This does represent progress *because* the left side is just $\frac{d}{dx}(R(x)y)$ (yes?), and so our equation becomes

$$\frac{d}{dx}(R(x)y) = R(x)Q(x).$$

Hence, integrating leads to

$$R(x)y = \int R(x)Q(x)dx.$$

Even if the function $R(x)Q(x)$ doesn't integrate to a standard form, numerical techniques can be used.

Again, do not remember this formula! Learn the method, as illustrated now.

Example 2.5 Solve

$$x\frac{dy}{dx} - 3y = x^4.$$

Solution.

(i) Rewrite it in the standard form for a first order linear equation,

$$\frac{dy}{dx} - \frac{3y}{x} = x^3.$$

(ii) Thus the integrating factor is $\exp\left(\int(-3/x)dx\right) = \exp(-3\log(x)) = x^{-3}$.

(iii) Multiplying through by this, we obtain

$$\frac{dy}{dx}x^{-3} - 3yx^{-4} = 1,$$

and so

$$\frac{d}{dx}\left(yx^{-3}\right) = 1.$$

(iv) Integrate to find $y/x^3 = x + c$, i.e. $y = x^4 + cx^3$, c an arbitrary constant.

(Check: given this y, differentiate to see that $\frac{dy}{dx} = 4x^3 + 3cx^2$, so that $x\frac{dy}{dx} - 3y = 4x^4 + 3cx^3 - 3\left(x^4 + cx^3\right) = x^4$, as we wanted.)

2.3 Second Order Equations with Constant Coefficients

There is a standard approach to solving

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{2.9}$$

when a, b and c are constants and $a \neq 0$. First, find the general solution to the corresponding *homogeneous equation*,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (HE)$$

by *assuming* a solution of the form $y = \exp(mx)$ for some unknown m. Then plainly (*HE*) leads to

$$\exp(mx)\left(am^2 + bm + c\right) = 0.$$

But, since $\exp(mx)$ is never zero, the only solutions come from solving the quadratic $am^2 + bm + c = 0$ (known as the *auxiliary equation*) in the standard way. There are three possibilities:

(a) two different real roots, m_1 and m_2;
(b) one repeated real root, m;
(c) two complex conjugate roots $u + iv$ and $u - iv$, u and v real.

All of these are dealt with in a similar way. Let A and B be arbitrary constants. In case (a), the general solution of (*HE*) is

$$Y_1(x) = A\exp(m_1x) + B\exp(m_2x);$$

in case (b), it is $Y_1 = (A + Bx)\exp(mx)$, while case in (c) it can be written as $Y_1 = \exp(ux)(A\cos(vx) + B\sin(vx))$.

The next step is to use your ingenuity to find, somehow or other, SOME solution of the original Eq. (2.9); you should pay close attention to the form of $f(x)$, and use trial and error. Let $Y_2(x)$ be some such solution. Then the general solution to (2.9) is

$$y = Y_1(x) + Y_2(x).$$

In case (a), this is $y = A\exp(m_1x) + B\exp(m_2x) + Y_2(x)$.

Finally, use other information, such as the values of y and $\frac{dy}{dx}$ when $x = 0$, or the values of y when $x = a$ and when $x = b$, to find the relevant values of A and B. Be clear that this last step is made only AFTER both Y_1 and Y_2 have been found!

Example 2.6 Equation (2.2) above, i.e.

$$\frac{d^2y}{dt^2} = g - c\frac{dy}{dt},$$

comes from Newton's Law. Here y is the distance travelled in time t when a stone moves vertically, subject to air resistance. Take $c > 0$, and assume that $y = 0$ and $\frac{dy}{dt} = v_0$ when $t = 0$.

Solution. Rewrite it as

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} = g,$$

so that the homogeneous version is

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} = 0.$$

This leads, using $y = \exp(mt)$, to the auxiliary equation $m^2 + cm = 0$, with two different solutions $m = 0$ and $m = -c$.

Thus the general solution of the homogeneous equation is

$$y = A + B\exp(-ct).$$

For a particular solution to

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} = g,$$

guess $y = Kt$, giving $cK = g$, i.e. $y = gt/c$. The general solution of the equation is the sum of these solutions, namely

$$y = A + B\exp(-ct) + gt/c.$$

Recall that $y = 0$ and $\frac{dy}{dt} = v_0$, when $t = 0$. Thus find A and B from $0 = A + B$ and $v_0 = -Bc + g/c$, leading to the solution

$$y = \frac{v_0}{c}(1 - \exp(-ct)) + \frac{g}{c^2}\left(\exp(-ct) + ct - 1\right).$$

This gives another excuse to use L'Hôpital's Rule. See what happens as $c \to 0$. The first term on the right side is

$$\frac{v_0(1 - \exp(-ct))}{c}.$$

Differentiate numerator and denominator (as functions of c) to get $v_0t\exp(-ct)$ and unity, whose ratio converges to v_0t. The second term is

$$\frac{g(\exp(-ct) + ct - 1)}{c^2},$$

and it will turn out that we need to differentiate numerator and denominator *twice*. Their respective second derivatives are $gt^2 \exp(-ct)$ and 2, so the second term converges to $gt^2/2$. The whole expression converges to $tv_0 + gt^2/2$.

This suggests that $y = tv_0 + gt^2/2$ solves the corresponding problem when $c = 0$, i.e. *without* air resistance. Exercise 2.14 ask you to check this, using the general solution to (HE) in the case (b) of equal roots of the quadratic.

2.4 Linked Systems

Sometimes, we have several interacting quantities, all also changing with time. We look first at the *Lotka-Volterra* model of a predator-prey system.

Rabbits on an island feed on an unlimited supply of clover, and foxes feed on rabbits. So let x be the number of rabbits at time t and let y be the number of foxes. (As we wish to differentiate, we shall treat x and y as continuous variables, even though they will be integers.) In the absence of foxes, rabbits breed happily, and we expect

$$\frac{dx}{dt} = ax$$

for some $a > 0$. With foxes, the rate at which foxes and rabbits meet is proportional to the *product* xy of their numbers, so, for some $b > 0$, we have

$$\frac{dx}{dt} = ax - bxy. \tag{2.10}$$

If there were no rabbits, the foxes would just die out, leading to

$$\frac{dy}{dt} = -cy$$

with $c > 0$. But eating rabbits enables breeding; and since the rate of meeting is proportional to xy, we see that

$$\frac{dy}{dt} = -cy + hxy \tag{2.11}$$

with $h > 0$. Setting the right sides of (2.10) and (2.11) to zero, i.e.

$$x(a - by) = 0 \quad \text{and} \quad y(hx - c) = 0$$

gives an *equilibrium*, as the rate of change of the size of both populations is then zero. With both species present, the only solution is $x = c/h$ and $y = a/b$.

Vito Volterra became interested in this sort of problem when his attention was drawn to the data on the proportion of predatory fish among all fish caught at the Italian port of Fiume. It rose from around 12 % just before the First World War in 1914 to around 30 % at the end of the war in 1918/9, before returning to its previous level. In the above model, the dramatic reduction in general fishing during the war changed the values of the parameters in Eqs. (2.10) and (2.11): the natural growth rate of prey, a, increased, the natural death rate of the predators, c, decreased, but both b and h, which apply to the interactions between the prey and predators, were not affected. Thus at the model's equilibrium point *(c/h, a/b)*, with little fishing, the first value dropped, the second increased—exactly what the observed data confirmed!

In 1910, Alfred Lotka had already developed the same system of equations, but in connection with the analysis of certain chemical reactions.

What happens if we are away from the equilibrium point? Dividing out (2.10) and (2.11), we get

$$\frac{dy}{dx} = \frac{y(hx - c)}{x(a - by)}$$

which has separable variables, hence

$$\int \frac{(a - by)dy}{y} = \int \frac{(hx - c)dx}{x}.$$

Integrate to find

$$a\log(y) - by = hx - c\log(x) + Const.$$

Take exponentials of both sides, rewrite as

$$y^a \exp(-by) = K\exp(hx)/x^c,$$

where K is a constant that can be found from the initial values of x and y.

This "solution" is a collection of closed curves round the equilibrium point—to a first approximation, they are distorted ellipses, as you will confirm when you solve Exercise 2.15. Figure 2.3 illustrates how the two populations change when disturbed from equilibrium—for this illustration, we have chosen $a = 30$, $b = 1$, $c = 50$ and $h = 0.5$, for which (100, 30) is the equilibrium. The arrows indicate the model's predictions for initial states either (120, 30) or (140, 30); in the former, the rabbit population ranges from 83 to 120, the fox population from 23 to 38, while in the latter, rabbit numbers run from 70 to 140, fox numbers from 18 to 46.

It is plain that the picture accords with our intuition. Starting with the equilibrium number of foxes, but excess rabbits (near point A, say), foxes are favoured: their numbers increase, rabbit numbers decrease, until there are too few rabbits to support the large excess of foxes (after B); both populations now decline, until rabbits are so scarce that foxes die off quickly (at C), enabling the rabbit population to recover

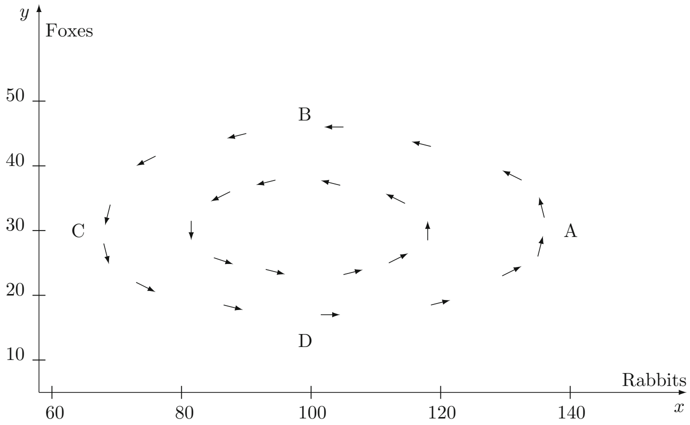

Fig. 2.3 How numbers change away from equilibrium

(along CD). With sufficiently many rabbits around (after D), fox numbers increase again, and the whole cycle repeats from A.

Another example of a linked system is the *Kermack-McKendrick* model for epidemics.

Imagine an isolated community of size n into which one person infected with an infectious disease is introduced. At time t, there are x Susceptibles (people who may catch the disease), y Infecteds (people with the disease, and spreading it) and z Removed cases (they have had the disease, and have recovered, or died, or been placed in isolation: they are no longer susceptible). Take $x + y + z = n + 1$, so that any two of x, y, z determine the third; and again treat then as *continuous* variables (even though they are not) so as to be able to differentiate them. Initially, $x = n$, $y = 1$ and $z = 0$.

When an Infected comes into contact with a Susceptible, the Susceptible *might* become an Infected (and so infectious), while as the disease runs its course, Infecteds become Removed. How might the disease spread, how many will escape infection?

The quantity xy is the number of possible encounters between a Susceptible and an Infected; the actual number of encounters in a given time period will depend on whether the population is tightly packed, or sparsely scattered, and whether the encounter leads to infection will depend on the infectiousness of the disease. For some constant $\beta > 0$, the quantity βxy describes the rate at which a Susceptible turns into an Infected; so we have

$$\frac{dx}{dt} = -\beta xy.$$

The right side also describes the rate at which the number of Infecteds increases; but Infecteds turn into Removed cases (by recovering, dying, being isolated, etc.) at a rate simply proportional to their number, so putting these two factors together we have

$$\frac{dy}{dt} = \beta xy - \gamma y$$

for some constant $\gamma > 0$. Finally, this last term also gives the rate at which Removed cases increase, so

$$\frac{dz}{dt} = \gamma y.$$

From the first two equations, divide out to find

$$\frac{dy}{dx} = \frac{\beta xy - \gamma y}{-\beta xy} = \frac{\rho - x}{x} \tag{2.12}$$

where $\rho = \gamma/\beta$. This has separable variables, so we integrate to find

$$\int dy = \int \frac{\rho - x}{x} dx,$$

which gives $y = Const - x + \rho \log(x)$. Since $y = 1$ when $x = n$, the general relationship is $y = n + 1 - x + \rho \log(x/n)$. The epidemic is over when $y = 0$, so we deduce that the number who escaped infection is x_1, found from the equation $x_1 = n + 1 + \rho \log(x_1/n)$.

Suppose first that $n > \rho$. Initially $x = n$, so (2.12) shows that $\frac{dy}{dx}$ is initially negative; and since x can only decrease, this negative slope means that y, the number infected, begins to increase, continues to do so while $x > \rho$, but then decreases until $y = 0$ and the epidemic ends, as in Fig. 2.4 (i). The formula for x_1 above shows that, to a first approximation, if there are $\rho + \delta$ Susceptibles initially, there are about $\rho - \delta$ when the epidemic is over. See Exercise 2.16.

By contrast, if $n < \rho$, (2.12) shows that the initial value of $\frac{dy}{dx}$ is positive, so y, already tiny, tends to decrease as x decreases, and the disease swiftly dies out, with no epidemic, as in Fig. 2.4 (ii).

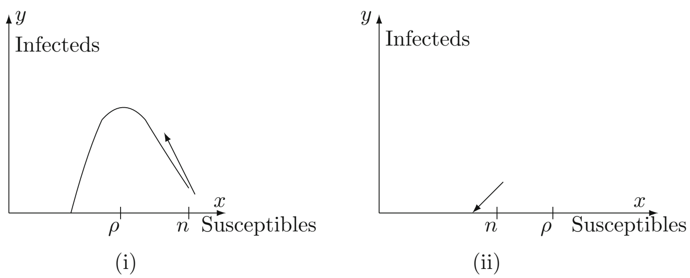

Fig. 2.4 How numbers of infecteds and susceptibles change

The crucial quantity $\rho = \gamma/\beta$ is termed the *threshold* for this system. If $n < \rho$, the disease is expected to die out quickly, whereas if $n > \rho$, an epidemic will occur. (Of course, this analysis is based on *averages* so, in practice, the size of any epidemic will be some random number, whose average we have found. In case (ii), this average is tiny, so there is no real chance of a large outbreak; in case (i), random chance MIGHT eliminate the disease before it has time to take hold, but if it does not die out quickly, there is a good possibility of a substantial epidemic.)

So overall, to prevent epidemics, we wish to ensure that $n < \rho$. One obvious way is vaccination, to reduce the value of n. Other ways are to increase $\rho = \gamma/\beta$: *either* by increasing γ—e.g. remove Infecteds quickly, or help them recover quickly (or cull them if this is a foot-and-mouth epidemic in cattle!); *or* by reducing β—perhaps disperse the population, or close schools, or postpone soccer matches to reduce the frequency of contact between Infecteds and Susceptibles. Even vaccination that is partly effective by reducing the infectiousness of a disease will also reduce β. Our mathematical model points to ways of reducing the frequency and severity of epidemics, and enables us to forecast how cost-effective possible measures to combat the disease might be.

2.5 Exercises

(Several of these Exercises, and the chapter examples, are adapted from material in the excellent book by George Simmons, that also contains interesting historical information on this subject and its pioneers.)

2.1 A model for the growth of bacteria subject to competition is

$$\frac{dx}{dt} = Kx\left(1 - \frac{x}{N}\right).$$

Here t represents time and x is the amount of bacteria. Suppose that, at time $t = 0$, then $x = N/4$, while at $t = 1$, then $x = N/2$. At what time will we find $x = 3N/4$?

2.2 The principle of radiocarbon dating is that air-breathing plants absorb carbon dioxide only while they are alive: radiocarbon is a radioactive isotope of carbon, and its proportion in the atmosphere has long been in equilibrium. It decays after the plant's death, so that comparing the proportion of radiocarbon in a piece of old wood with the current proportion in the atmosphere gives an estimate of the length of time the tree has been dead.

Suppose matter decays at a rate proportional to its quantity. Write down the form of the differential equation for $x(t)$, the amount of matter at time t, and deduce that $x(t) = x(0)\exp(-kt)$ for some constant $k > 0$.

It takes around 5600 years for half the initial amount to decay (hence the term *half-life*). Estimate the age of a wooden object whose proportion of radiocarbon is (i) 75 % (ii) 10 % of the current amount. (Round your answers sensibly.)

2.3 Snow has been falling steadily for some time when the snowplough begins to clear the road at noon. It removes snow at a constant rate, and has covered two miles by 1-00 pm, two further miles by 3-00 pm.
We have argued that, if snow began T hours before noon and fell at rate s/unit time, while the plough cleared V/unit time and had moved distance x in time t, then

$$s(T+t)\frac{dx}{dt} = V.$$

Solve this equation with the given conditions, and deduce the time that snow started to fall.

2.4 *Newton's Law of Cooling* states that the rate an object cools is proportional to the temperature difference between it and its surroundings. A rock is heated to 120 °C, and placed in a large room kept at a constant temperature at 20 °C. The temperature of the rock falls to 60 °C after an hour; how much longer does it take to cool to 30 °C?

2.5 A spherical mothball evaporates uniformly at a rate proportional to its surface area. Hence deduce a differential equation that links its radius with time. Given that the radius halves from its initial value in one month, how long will the mothball last?

2.6 We have shown that if the water level in a sink or bath is $x > 0$, then

$$\frac{dx}{dt} = -\frac{A\sqrt{2gx}}{V'(x)},$$

where V is the volume of water, V' is its derivative, t is time, and A is the cross-sectional area of the plughole.

(i) Suppose a bath can be taken as having a rectangular horizontal cross-section of fixed area B. How long will it take to empty from depth $D > 0$? (Your answer should just involve D, B, A and g.) Now suppose a blockage *halves* the cross-sectional area of the plughole, and you decide to have a really luxurious bath by doubling the amount of water: what difference does all that make to the time to empty the bath?

(ii) Take the plughole to be a circle, radius $r > 0$, and suppose a sink is a perfect hemisphere of radius $R > 0$. Show that the volume of water when the depth is $x > 0$ is $V = \pi x^2(3R - x)/3$. Deduce that the time for the bowl to empty from full is $\frac{14R^{5/2}}{15r^2\sqrt{2g}}$.

(iii) Let f be a smooth and strictly increasing function with $f(0) = 0$. A sink has the shape of the graph $y = f(x)$, for $0 \le x \le c$, being rotated round the y-axis. Give the form of f that corresponds to the water level falling at a *constant* rate.

(A *clepsydra* is a water clock consisting of a sink with a small hole to allow water to escape. It was used in ancient Greek and Roman courts to time the speeches of lawyers. It is plainly an advantage in such an implement to have the water level falling at a uniform rate.)

2.7 Find y as a function of x when $y = 1$ when $x = 1$, and

$$\frac{dy}{dx} = \frac{3xy + 2y^2}{x^2}.$$

2.8 Find y as a function of x for $x > 0$ when

$$\frac{dy}{dx} - \frac{5y}{x} = x,$$

and $y = 1$ when $x = 1$.

2.9 Suppose $\frac{dy}{dx} + P(x)y = Q(x)y^n$ for some n, with $n \neq 0$ and $n \neq 1$ (known as *Bernoulli's Equation*). Define $z = y^{1-n}$, and show that the resulting differential equation linking z and x is linear.
Hence solve $x\frac{dy}{dx} + y = x^4y^3$, given that $y = 1$ when $x = 1$.

2.10 In the model of rowing across a river described in this chapter, suppose that we can row twice as fast as the river flows. Use the result

$$y = \frac{1}{2}\left(\frac{x^{k+1}}{c^k} - \frac{c^k}{x^{k-1}}\right)$$

and the relation

$$\frac{dx}{dt} = \frac{-bx}{\sqrt{x^2 + y^2}}$$

to find an equation linking x and t. Deduce how long it takes to cross the river.

2.11 A 100-L tank is initially full of a mixture of 10 % alcohol and 90 % water. Simultaneously, a pump drains the tank at 4 L/s, while a mixture of 80 % alcohol and 20 % water is poured in at rate 3 L/s. Thus the tank will be empty after 100 seconds (yes?). Assume that the two liquids mix thoroughly, and let y litres be the amount of alcohol in the tank after t seconds; explain why the equation

$$\frac{dy}{dt} = 2.4 - \frac{4y}{100 - t}$$

holds for $0 \leq t < 100$. Find y as a function of t; hence deduce that the maximum amount of alcohol in the tank occurs after about 34 seconds, and is about 39.5 L.

2.12 25 grams (gm) of salt are dissolved in 50 L of water, and poured into tank A. From time $t = 0$, pure water is poured into this tank at the rate of 2 L/s, and simultaneously 2 L/s drain from the tank. Thus the tank always contains 50 L of liquid, thoroughly mixed. Let x (gm) denote the amount of salt in tank A at time $t \geq 0$; justify the equation

$$\frac{dx}{dt} = -x/25.$$

The liquid that drains from tank A pours into tank B, which initially has 50 L of pure water, and 2 L/s drain from tank B. Thus tank B also always contains 50 L of liquid, thoroughly mixed. Let y (gm) denote the amount of salt in tank B at time $t \geq 0$. Explain briefly why

$$\frac{dy}{dt} = x/25 - y/25.$$

Hence find x, then y. At what time is the amount of salt in tank B at its maximum?

2.13 A chain of length $L = 150$ cm has constant density ρ/cm. It rests on a smooth horizontal table with 30 cm hanging over the edge, and is released. Neglect friction. It slides off the table according to Newton's Law:

$$\text{Mass} \times \text{Acceleration} = \text{Force}.$$

The Mass is constant at $L\rho$, but the Force varies according to the amount of overhang; for an overhang of length x, the Force will be $x\rho g$. Set up the relevant differential equation, state the initial conditions, and find how long it takes for the chain to fall off the table. (Take $g = 10$ m/s/s.)

2.14 Solve the equation

$$\frac{d^2y}{dt^2} = g,$$

where y represents the distance travelled at time t when a stone is thrown vertically, with initial velocity v_0, in the absence of air resistance.

2.15 Using the notation in this chapter for the Lotka-Volterra model, write $x = c/h + X$ and $y = a/b + Y$, where X and Y are assumed *small* (i.e. (x, y) is close to the equilibrium point $(c/h, a/b)$). Find the exact equations for X and Y that correspond to (2.10) and (2.11), and then, neglecting the terms involving the product XY, divide out these equations to obtain dY/dX (approximately). Show that the exact solutions to this approximate equation form a family of ellipses centered at the equilibrium point; select a representative member of this family and show (by marking arrows at a selection of points) how the populations sizes are expected to change if the two populations are disturbed from the equilibrium point.

2.16 Suppose one infected individual enters a population of size 1100, when the relevant value of the threshold is $\rho = 1000$. According to the Kermack-McKendrick model, how many people will be infected before the epidemic dies out?

2.17 In a model of warfare, let the sizes of the opposing armies at time t be $x(t)$ and $y(t)$, and let a, b be positive constants. Then the equations

$$\frac{dx}{dt} = -by, \qquad \frac{dy}{dt} = -ax$$

might describe how the conflict develops; for example, b will be larger when the $y-$army has strong offensive capabilities and the $x-$army has weak defences, and vice versa for the interpretation of a.

Divide one equation by the other to eliminate t, and deduce *Lanchester's Square Law*, i.e. $a(x(0)^2 - x(t)^2) = b(y(0)^2 - y(t)^2)$.

Assume that fighting continues until one side is annihilated: show that the x-army wins whenever $ax(0)^2 > by(0)^2$, and that if this holds, the final size of the x-army is $\sqrt{x(0)^2 - (b/a)y(0)^2}$.

Now suppose the armies have equal initial sizes, and the same offensive and defensive capabilities so that $a = b$, but the x-army has split the other into two equal parts, and will fight them sequentially. What is the size of the x-army after the first battle? After the second?

Give a similar analysis for when the $y-$army has been split into three equal parts, tackled sequentially. Are your two answers consistent with the commonly stated principle of offensive warfare that "The more you use, the less you lose."?

2.18 As a parallel to the Kermack-McKendrick model for the spread of epidemics, Daryl Daley and David Kendall proposed a model for the spread of rumours. A homogeneously mixing community, initially contains one person (a Spreader) who knows a rumour, and N who do not (the Ignorants). When a Spreader meets an Ignorant, the Ignorant learns the rumour and becomes a Spreader; but if a Spreader attempts to tell the rumour to another person who already knows it, both of them believe the rumour to be "old hat", and decide to cease spreading it: people who know the rumour but no longer spread it are Stiflers.

At time t, let x be the number of Ignorants, y the number of Spreaders, and z the number of Stiflers; thus $x + y + z = N + 1$. By considering the outcomes of encounters between the different types of person, justify the equations

$$\frac{dx}{dt} = -Axy,$$
$$\frac{dy}{dt} = Axy - Ayz - Ay(y-1),$$

where $A > 0$ is a constant. Deduce an expression for $\frac{dy}{dx}$, and hence find y in terms of x. Show that, when the rumour dies out, the number of Ignorants, X, satisfies the equation

$$0 = 1 + N\log(X/N) + 2(N - X).$$

Hence show that, when N is large, this model predicts that about 80 % of the Ignorants will learn the rumour, and, in contrast to the epidemic model, there is no "threshold theorem".

References and Further Reading

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