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One of the most exciting developments in mathematics in the latter half of the twentieth century was the introduction of what is nowadays called microlocal techniques into the theory of partial differential equations. This development began in the early 1960s with the invention of pseudodifferential operators by Calderón, Kohn-Nirenberg, A. and J. Unterberger and Hörmander; a tool which enabled one to replace abstract existence theorems for differential equations by concrete methods for actually solving them. Louis Boutet de Monvel gave early on two very important extensions of microlocal analysis.

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## 1.1 Boundary Value Problems

In his thesis and in the papers [B66a, B66b, B69] and [B71] Boutet describes how to treat boundary value problems for partial differential operators from a microlocal point of view and how to make this theory applicable to pseudodifferential operators; and the techniques he developed in these

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papers have, in the intervening years, become standard tools in the theory of elliptic boundary problems. (See the commentary of Gerd Grubb for more details.)

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## 1.2 Analytic Pseudodifferential Operators

Boutet's work with Paul Krée [BKr67] on elliptic regularity for pseudodifferential operators with Gevrey coefficients is his first foray into the world of real analytic analysis, an area that, with his concurrent interests in  $C^\infty$  and complex analysis, he continued to explore in much of his later work. The work [BKr67] now appears as fundamental in all later developments of analytic microlocal analysis. (For some of the directions in which this exploration led him, see the commentary of Kawai.)

At the end of the 1960s, Sato and Hörmander showed how to refine greatly the study of singularities of solutions of differential equations and to show that these singularities can be described as subsets of the phase space of the physical system that the equations are describing, not just, as had previously been the case, as subsets of the configuration space of this system. In the  $C^\infty$  version of Hörmander this is done by using pseudodifferential operators as “cutoff functions” in the same way that  $C_0^\infty$ -functions are used in the definition of the singular

support of a distribution. This changed the perspectives in this subject quite radically, and among the mathematicians involved in these changes, Boutet de Monvel played a central role.

In particular it had been noticed early on, as an amusing curiosity, that there exist generalized functions whose singularities are concentrated on a single ray in phase space, but Boutet was one of the first persons to realize the importance of this fact and in his paper, Hypoelliptic operators with double characteristics [B74], to make a systematic study of these Hermite distributions.

This leads us to

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### 1.3 Operators with Multiple Characteristics

We cited this paper above as the paper in which Boutet introduced the notion of Hermite distribution and began to explore the ideas that were to become the focus of much of his later research. The developments in microlocal analysis in the 1950s and 1960s that were described above made it possible, as we pointed out above, to prove concrete existence theorems for a large class of linear differential operators. However, for operators with multiple characteristics analogues of these theorems had to await the refinements in pseudodifferential operator theory. The work [B74] was one of the first steps in this direction. A second related paper, with Grigis and Helffer, [BGrHe76] develops further the theory of these (so called) “exotic” pseudodifferential operators and applies it to the construction of parametrices in the case when the multiply characteristic variety has variable symplectic rank. In [B75b] Boutet gives a pioneering result about propagation of singularities for Schrödinger operators, for which the usual conical point of view has to be abandoned and replaced by a parabolic quasihomogeneous point of view. (The commentary of Helffer describes in more detail what this involved.)

Boutet de Monvel went on to apply these ideas to the theory of Toeplitz operators in the papers [B79, B81, B85], and in the book [BGu81], and also to apply them to problems in several complex variables in [BSj76], to problems in index theory in [B79] and to problems in geometric quantization in [B99] and [B02]. (For a more detailed account of Boutet’s work in this area see the commentaries below of Malgrange, Epstein, Guillemin and Weinstein.)

Other areas in which Boutet made enduring contributions are

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## 1.4 Index Theory

Boutet’s paper [B79] provides a unified approach to index theory that encompasses index theory in the  $C^\infty$  domain (Atiyah-Singer) and index theory in the algebraic domain (Kawai, Kashiwara et al.) The first of these is a consequence of the fact that a pseudodifferential operator can be viewed as a Toeplitz operator and the second a consequence of the fact that if  $X$  is the boundary of a strictly pseudoconvex domain and  $H^2$  the Hardy space of  $X$  (i.e. the  $L^2$  closure of the space of boundary values of holomorphic functions), then for a pseudodifferential operator,  $Q$ , the contraction of  $Q$  to  $H^2$  is a Toeplitz operator. Thus to obtain an index theory unifying these two theories, it suffices to extend index theory to the Toeplitz setting, and the technical details involved in doing this are the theme of this paper.

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## 1.5 The CR-Embedding Theorem

In the article [B75a] Boutet shows that every strictly pseudoconvex compact CR manifold can be imbedded in  $\mathbb{C}^N$ , a result which, combined with results of Harvey-Lawson proves: A CR manifold,  $X$ , with the properties above is the boundary of a Stein manifold. (See Epstein’s commentary for implications of this result.)

## 1.6 Zoll Manifolds

A compact Riemannian manifold,  $X$ , is Zoll if all its geodesics are closed. For such manifolds a celebrated theorem of Colin de Verdière asserts that the Weyl function of  $X$ , the function that, for every  $\lambda$ , counts the number of eigenvalues of the Laplace operator that are less than  $\lambda$ , is a polynomial function of  $\lambda$ . However, Colin's result left unanswered the question: What is this polynomial? In [BGu81] and [B85] Boutet settled this question by discovering a remarkable analogy between this polynomial and the Hilbert polynomial of an algebraic variety. More explicitly he generalized Colin's theorem to the Toeplitz setting and observed that in the Toeplitz world there is a much larger class of operators having the Zoll property: the property that the Hamiltonian flow associated with the symbol of the operator is periodic. For example, for a projective variety,  $X$ , the operator associated with the action of  $S^1$  on sections of the canonical line bundle of  $X$  is such an operator and in this case Colin's polynomial becomes the Hilbert polynomial of  $X$ .

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## 1.7 Bergman Kernels

The paper [BSj76] on Bergman and Szegő kernels represents a new perspective in the study of holomorphic functions on strictly pseudoconvex domains. For  $X$  the boundary of such a domain, the projection of  $L^2$  onto  $H^2$  in item 3 above is a Toeplitz operator, and alternatively a Fourier integral operator with complex phase. This fact allowed the authors to simplify the proof of a much cited result of C. Fefferman, and to include Szegő kernels and off-diagonal behaviour in this theory.

Boutet de Monvel and Sjöstrand [BSj76] also provided much of the source material for the monograph [BGu81]. Moreover, Boutet continued in later articles to use the techniques of this paper to investigate Bergman kernel issues. In particular, [BSj76] left open the question of the existence of a log term in the asymptotic expansion of the Bergman kernel, and in the paper

[B88] he made a deep and detailed study of the Bergman kernel in 2 dimensions and completely settled this log term question.

Finally the note [B78] characterizes series of eigenfunctions of the Laplacian on an analytic manifold whose sums extend to certain tubular neighborhoods of the real domain, and relates complex analysis to spectral theory in a beautiful way.

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## 1.8 Star Products

The monograph [BGu81] contains two very important inadvertent results. One is the proof of the conjecture that for circle actions the quantization commutes with reduction. A decade later this became, for arbitrary actions of Lie groups, a much investigated conjecture in the wake of Witten's revolutionary work on quantum field theory, and was settled in full generality, independently by Meinrenken and Vergne, in 1995.

The second inadvertent result involved the calculus of Toeplitz operators. At the symbol level this calculus can be viewed as a way of equipping a symplectic manifold with a star product operation, however the authors of [BGu81] neglected to draw attention to this fact, and the first official proof of it appeared a couple years later in a paper of De Wilde and Lecomte.

As concerns non-inadvertent results Boutet made major contributions to the theory of star products and deformation quantization in two later papers of his: the papers, [B99] and [B02]. (For more on this see the commentary by Alan Weinstein.)

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## 1.9 Conclusion

The purpose of this volume of Boutet de Monvel's selected works is to make easily available the papers cited above and other major publications of his, many of which have been, up until now, only available in obscure, hard-to-access journals. Also, to make this volume more readable it was decided not to republish these papers in chronological order, but instead

to organize them into relevant sections and to supply each of these sections with a commentary. (This organization of the material above was in fact suggested to us by Boutet himself in two internal texts [summary, synopsis].) Needless to say, the insightful observations of the commentators have been essential for the success of this scheme and we would like to express to them our gratitude.

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## References

- [summary] L. Boutet de Monvel, *lbm-selecta-summary*, 2012.
- [synopsis] L. Boutet de Monvel, *lbm-selecta-synopsis*, 2012.
- [B66a] Louis Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord. I. La propriété de transmission*, J. Analyse Math. **17** (1966), 241–253.
- [B66b] L. Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord. II. Pseudo-noyaux de Poisson*, J. Analyse Math. **17** (1966), 255–304.
- [BKr67] Louis Boutet de Monvel and Paul Krée, *Pseudo-differential operators and Gevrey classes*, Ann. Inst. Fourier (Grenoble) **17** (1967), no. 1, 295–323.
- [B69] L. Boutet de Monvel, *Opérateurs pseudo-différentiels analytiques et problèmes aux limites elliptiques*, Ann. Inst. Fourier (Grenoble) **19** (1969), no. 2, 169–268.
- [B71] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), no. 1–2, 11–51.
- [B74] L. Boutet de Monvel, *Hypoelliptic operators with double characteristics and related pseudo-differential operators*, Comm. Pure Appl. Math. **27** (1974), 585–639.
- [B75a] L. Boutet de Monvel, *Intégration des équations de Cauchy-Riemann induites formelles*, Séminaire Goulaouic-Lions-Schwartz 1974–1975; Équations aux dérivées partielles linéaires et non linéaires, Centre Math., École Polytech., Paris, 1975, Exp. No. 9.
- [B75b] L. Boutet de Monvel, *Propagation des singularités des solutions d'équations analogues à l'équation de Schrödinger*, Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 1–14. Lecture Notes in Math., Vol. 459.
- [BGrHe76] Louis Boutet de Monvel, Alain Grigis, and Bernard Helffer, *Parametrixes d'opérateurs pseudo-différentiels à caractéristiques multiples*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Soc. Math. France, Paris, 1976, pp. 93–121. Astérisque, No. 34–35.
- [BSj76] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Soc. Math. France, Paris, 1976, pp. 123–164. Astérisque, No. 34–35.
- [B78] L. Boutet de Monvel, *Convergence dans le domaine complexe des séries de fonctions propres*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), no. 13, A855–A856.
- [B79] L. Boutet de Monvel, *On the index of Toeplitz operators of several complex variables*, Invent. Math. **50** (1978/79), no. 3, 249–272.
- [BGu81] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981.
- [B81] L. Boutet de Monvel, *Opérateurs à coefficients polynomiaux, espace de Bargmann, et opérateurs de Toeplitz*, Goulaouic-Meyer-Schwartz Seminar, 1980–1981, École Polytech., Palaiseau, 1981, Exp. No. II bis.
- [B85] L. Boutet de Monvel, *Opérateurs à bicaractéristiques périodiques*, Bony-Sjöstrand-Meyer seminar, 1984–1985, École Polytech., Palaiseau, 1985, Exp. No. 20.
- [B88] L. Boutet de Monvel, *Le noyau de Bergman en dimension 2*, Séminaire sur les Équations aux Dérivées Partielles 1987–1988, École Polytech., Palaiseau, 1988, Exp. No. XXII.
- [B99] L. Boutet de Monvel, *Complex star algebras*, Math. Phys. Anal. Geom. **2** (1999), no. 2, 113–139.
- [B02] L. Boutet de Monvel, *Related semi-classical and Toeplitz algebras*, Deformation quantization (Strasbourg, 2001), IRMA Lect. Math. Theor. Phys., vol. 1, de Gruyter, Berlin, 2002, pp. 163–190.<sup>1</sup>

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<sup>1</sup>Not included in this volume.



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