Chapter 2

Tools from Differential Geometry

In this chapter we introduce the necessary background in differential geometry of closed compact hypersurfaces in $\mathbb{R}^n$. We investigate the differential geometric properties of embedded hypersurfaces in $n$-dimensional Euclidean space, introducing the notions of Weingarten tensor, principal curvatures, mean curvature, tubular neighbourhood, surface gradient, surface divergence, and Laplace-Beltrami operator. The main emphasis lies in deriving representations of these quantities for hypersurfaces $\Gamma = \Gamma_\rho$ that are given as parameterized surfaces in normal direction of a fixed reference surface $\Sigma$ by means of a height function $\rho$. We derive all of the aforementioned geometric quantities for $\Gamma_\rho$ in terms of $\rho$ and $\Sigma$. It is also important to study the mapping properties of these quantities in dependence of $\rho$, and to derive expressions for their variations. For instance, we show that

$$\kappa'(0) = \text{tr} \, L^2_\Sigma + \Delta_\Sigma,$$

where $\kappa = \kappa(\rho)$ denotes the mean curvature of $\Gamma_\rho$, $L_\Sigma$ the Weingarten tensor of $\Sigma$, and $\Delta_\Sigma$ the Laplace-Beltrami operator on $\Sigma$. This is done in Section 2.

We also study the first and second variations of the area and volume functional, respectively. In Section 3 we show, among other things, that $C^2$-hypersurfaces can be approximated in a suitable topology by smooth (i.e., analytic) hypersurfaces. This leads, in particular, to the existence of parameterizations. In Section 4 we show that the class of compact embedded hypersurfaces in $\mathbb{R}^n$ gives rise to a new manifold (whose points are the compact embedded hypersurfaces). We also show that the class $\mathcal{M}^2(\Omega, r)$ of all compact embedded hypersurfaces contained in a bounded domain $\Omega \subset \mathbb{R}^n$, and satisfying a uniform ball condition with radius $r > 0$, can be identified with a subspace of $C^2(\bar{\Omega})$. This is important, as it allows us to derive compactness and embedding properties for $\mathcal{M}^2(\Omega, r)$. Finally, in Section 5 we consider moving hypersurfaces and prove various transport theorems.
2.1 Differential Geometry of Hypersurfaces

We consider a closed embedded hypersurface \( \Sigma \) of class \( C^k \), \( k \geq 3 \), enclosing a bounded domain \( \Omega \) in \( \mathbb{R}^n \). Thus for each point \( p \in \Sigma \) there is a ball \( B(p, r) \subset \mathbb{R}^n \) and a diffeomorphism \( \Phi : B(p, r) \to U \subset \mathbb{R}^n \) such that \( \Phi(p) = 0 \in U \) and
\[
\Phi^{-1}(U \cap (\mathbb{R}^{n-1} \times \{0\})) = B(p, r) \cap \Sigma.
\]
We may assume that \( \Sigma \) is connected; otherwise we would concentrate on one of its components. The points of \( \Sigma \) are denoted by \( p \), and \( \nu_\Sigma = \nu_\Sigma(p) \) means the outer unit normal of \( \Sigma \) at \( p \). Locally at \( p \in \Sigma \) we have the parameterization
\[
p = \phi(\theta) := \Phi^{-1}(\theta, 0),
\]
where \( \theta \) runs through an open parameter set \( \Theta \subset \mathbb{R}^{n-1} \). We denote the tangent vectors generated by this parameterization by
\[
\tau_i = \tau_i(p) = \frac{\partial}{\partial \theta_i} \phi(\theta) = \partial_i \phi, \quad i = 1, \ldots, n-1.
\]
These vectors \( \tau_i \) form a basis of the tangent space \( T_p \Sigma \) of \( \Sigma \) at \( p \). Note that \( (\tau_i|\nu_\Sigma) = 0 \) for all \( i \), where \( (\cdot|\cdot) := (\cdot|\cdot)_{\mathbb{R}^n} \) denotes the Euclidean inner product in \( \mathbb{R}^n \). Similarly, we set \( \tau_{ij} = \partial_i \partial_j \phi \), \( \tau_{ijk} = \partial_i \partial_j \partial_k \phi \), and so on. In the sequel we employ Einstein’s summation convention, which means that equal lower and upper indices are to be summed, and \( \delta_i^j \) are the entries of the unit matrix \( I \). For two vectors \( a, b \in \mathbb{R}^n \) the tensor product \( a \otimes b \in \mathcal{B}(\mathbb{R}^n) \) is defined by \( [a \otimes b](x) = (b|x)a \) for \( x \in \mathbb{R}^n \). If \( a \) belongs to the tangent space \( T_p \Sigma \), we may represent \( a \) as a linear combination of the basis vectors of \( T_p \Sigma \), i.e., \( a = a^i \tau_i \). The coefficients \( a^i \) are called the contravariant components of \( a \). On the other hand, this vector \( a \) is also uniquely...
characterized by its covariant components, \( a_i \) defined by \( a_i = (a|\tau_i) \), which means that the covariant components are the coefficients of the representation of \( a \) in the basis \( \{\tau^i\} \) dual to the basis \( \{\tau_j\} \), defined by the relations \( (\tau^i|\tau_j) = \delta^i_j \). Similarly, if \( K \in \mathcal{B}(T_p\Sigma) \) is a tensor we have the representations

\[
K = k^i_j \tau_i \otimes \tau_j = k^i_j \tau^i \otimes \tau^j = k^j_i \tau_i \otimes \tau_j = k^j_i \tau^i \otimes \tau_j,
\]

with e.g. \( k_{ij} = (\tau_i|K\tau_j) \) and \( k^i_j = (\tau^i|K\tau_j) \). Moreover, \( \text{tr} K \), the trace of \( K \), is given by

\[
\text{tr} K = (K\tau_i|\tau^i) = (K\tau^i|\tau_i). \quad (2.2)
\]

In particular, \( \text{tr} [a \otimes b] = (a|b) = a_i b^i = a^i b_i \).

### 1.1 The First Fundamental Form

Define

\[
g_{ij} = g_{ij}(p) = (\tau_i(p)|\tau_j(p)) = (\tau_i|\tau_j), \quad i, j = 1, \ldots, n - 1. \quad (2.3)
\]

The matrix \( G = [g_{ij}] \) is called the first fundamental form of \( \Sigma \). Note that \( G \) is symmetric and also positive definite, since

\[
(G\xi|\xi) = g_{ij}\xi^i\xi^j = (\xi^i\tau_i|\xi^j\tau_j) = |\xi^i\tau_i|^2 > 0, \quad \text{for all } \xi \in \mathbb{R}^{n-1}, \xi \neq 0.
\]

We let \( G^{-1} = [g^{ij}] \), hence \( g_{ik}g^{kj} = \delta^i_j \), and \( g^{ij}g_{ij} = \delta^i_j \). The determinant \( g := \det G \) is positive. Let \( a \) be a tangent vector. Then \( a = a^i \tau_i \) implies

\[
a_k = (a|\tau_k) = a^i(\tau_i|\tau_k) = a^i g_{ik} \quad \text{and} \quad a^i = g^{ik} a_k.
\]

Thus the fundamental form \( G \) allows for the passage from contra- to covariant components of a tangent vector and vice versa. If \( a, b \) are two tangent vectors, then

\[
(a|b) = a^i b^j (\tau_i|\tau_j) = g_{ij} a^i b^j = a_j b^i = a^i b_i = g^{ij} a_i b_j := (a|b)_\Sigma
\]

defines an inner product on \( T_p\Sigma \) in the canonical way, the Riemannian metric. By means of the identity

\[
(g^{ik} \tau_k|\tau_j) = g^{ik} g_{kj} = \delta^i_j
\]

we further see that

\[
\tau^i = g^{ij} \tau_j \quad \text{and} \quad \tau_j = g_{ij} \tau^i.
\]

This implies the relations

\[
k^i_j = g^{ir} k_{rj} = g_{jr} k^{ri}, \quad k^i_j = g^{ir} k^j_r, \quad k_{ij} = g_{ir} k^r_j,
\]

for any tensor \( K \in \mathcal{B}(T_p\Sigma) \). We set for the moment \( \mathcal{G} = g^{ij} \tau_i \otimes \tau_j \) and have equivalently

\[
\mathcal{G} = g^{ij} \tau_i \otimes \tau_j = g_{ij} \tau^i \otimes \tau^j = \tau_i \otimes \tau^i = \tau^j \otimes \tau_j.
\]

Let \( u = u^k \tau_k + (u|\nu\Sigma) \nu_{\Sigma} \) be an arbitrary vector in \( \mathbb{R}^n \). Then

\[
\mathcal{G} u = g^{ij} \tau_i (\tau_j|u) = g^{ij} \tau_i u^k g_{jk} = u^k \tau_k,
\]
i.e., \( G \) equals the orthogonal projection \( \mathcal{P}_\Sigma = I - \nu_\Sigma \otimes \nu_\Sigma \) of \( \mathbb{R}^n \) onto the tangent space \( T_p\Sigma \) at \( p \in \Sigma \). Therefore, we have the relation
\[
\mathcal{P}_\Sigma = I - \nu_\Sigma \otimes \nu_\Sigma = \tau_i \otimes \tau_i = \tau_i \otimes \tau_i,
\]
where \( I \) denotes the identity map on \( \mathbb{R}^n \). These properties explain the meaning of the first fundamental form \([g_{ij}]\).

1.2 The Second Fundamental Form

Define
\[
l_{ij} = l_{ij}(p) = (\tau_{ij}|\nu_\Sigma), \quad L = [l_{ij}].
\]  
(2.4)

\( L \) is called the second fundamental form of \( \Sigma \). Note that \( L \) is symmetric, and differentiating the relations \((\tau_i|\nu_\Sigma) = 0\) we derive
\[
l_{ij} = (\tau_{ij}|\nu_\Sigma) = -(\tau_i \partial_j \nu_\Sigma) = -(\tau_j \partial_i \nu_\Sigma).
\]  
(2.5)

The matrix \( K \) with entries \( l_{ij}^i \), defined by
\[
l_{ij}^i = g^{ir} l_{rj}, \quad K = G^{-1} L,
\]
is called the shape matrix of \( \Sigma \). The eigenvalues \( \kappa_i \) of \( K \) are called the principal curvatures of \( \Sigma \) at \( p \), and the corresponding eigenvectors \( \eta_i \) determine the principal curvature directions. Observe that \( K \eta_i = \kappa_i \eta_i \) is equivalent to \( L \eta_i = \kappa_i G \eta_i \), hence the relation
\[
(L \eta_i | \eta_i) = \kappa_i (G \eta_i | \eta_i)
\]
and symmetry of \( L \) and \( G \) show that the principal curvatures \( \kappa_i \) are real. Moreover,
\[
\kappa_i (G \eta_i | \eta_j) = (L \eta_i | \eta_j) = (\eta_i | L \eta_j) = \kappa_j (\eta_i | G \eta_j) = \kappa_j (G \eta_i | \eta_j)
\]
implies that principal directions corresponding to different principal curvatures are orthogonal with respect to the inner product \((G \cdot | \cdot)_{\mathbb{R}^{n-1}}\). We can always assume that eigenvectors associated to an eigenvalue \( \kappa_i \) are orthogonal w.r.t. \((G \cdot | \cdot)_{\mathbb{R}^{n-1}}\) in case \( \kappa_i \) has geometric multiplicity greater than one. The eigenvalues \( \kappa_i \) are semi-simple, i.e., \( N((\kappa_i - K)^2) = N(\kappa_i - K) \). In fact, suppose \( x \in N((\kappa_i - K)^2) \). Then
\[
(\kappa_i - K)x = \sum_{r=1}^{m_i} t_r \eta_{i,r},
\]
with \( t_r \in \mathbb{R} \), where \( \{\eta_{i,r} : 1 \leq r \leq m_i\} \) is an (orthogonal) basis of \( N(\kappa_i - K) \). Therefore,
\[
t_k (G \eta_{i,k} | \eta_{i,k}) = \left( \sum_{r=1}^{m_i} t_r G \eta_{i,r} | \eta_{i,k} \right) = (G(\kappa_i - K)x | \eta_{i,k}) = (x | (\kappa_i G - L) \eta_{i,k}) = 0
\]
for \( 1 \leq k \leq m_i \). Since \( G \) is positive definite, \( t_k = 0 \), and hence \( x \in N(\kappa_i - K) \). This shows that \( K \) is diagonalizable.
2.1. Differential Geometry of Hypersurfaces

The trace of $K$, i.e., the first invariant of $K$, is called the $(n-1)$-fold mean curvature $\kappa$ of $\Sigma$ at $p$, i.e., we have

$$\kappa_{\Sigma} := \text{tr} \ K = l_i^i = g^{ij} l_{ij} = \sum_{i=1}^{n-1} \kappa_i.$$  (2.6)

The Gaussian curvature $K_{\Sigma}$ is defined as the last invariant of $K$, i.e.,

$$K_{\Sigma} = \det K = g^{-1} \det L = \Pi_{i=1}^{n-1} \kappa_i.$$  

We define the Weingarten tensor $L_{\Sigma}$ by means of

$$L_{\Sigma} = L_{\Sigma}(p) = l^{ij} \tau_i \otimes \tau_j = l^i_j \tau_i \otimes \tau^j = l_{ij} \tau^i \otimes \tau^j.$$  (2.7)

$L_{\Sigma}$ is symmetric with respect to the inner product $(\cdot | \cdot)$ in $\mathbb{R}^n$. We note that $L_{\Sigma} \in B(\mathbb{R}^n)$ leaves the tangent space $T_p\Sigma$ invariant and, moreover, $L_{\Sigma} \nu_{\Sigma} = 0$. This shows that $L_{\Sigma}$ enjoys the decomposition

$$L_{\Sigma} = \begin{bmatrix} L_{\Sigma}|_{T_p\Sigma} & 0 \\ 0 & 0 \end{bmatrix} : T_p\Sigma \oplus \mathbb{R} \nu_{\Sigma} \to T_p\Sigma \oplus \mathbb{R} \nu_{\Sigma}.$$  (2.8)

In particular, we note

$$\text{tr} L_{\Sigma}(p) = \text{tr}[L_{\Sigma}|_{T_p\Sigma}], \quad \det[I + rL_{\Sigma}(p)] = \det[(I + rL_{\Sigma}(p))|_{T_p\Sigma}]$$  (2.9)

for $r \in \mathbb{R}$. We will in the following not distinguish between $L_{\Sigma}$ and its restriction to $T_p\Sigma$. Observe that

$$\text{tr} L_{\Sigma} = l_i^i = g^{ij} l_{ij} = \kappa_{\Sigma},$$  (2.10)

and the eigenvalues of $L_{\Sigma}$ in $T_p\Sigma$ are the principal curvatures, since

$$L_{\Sigma} \eta_k = l^{ij}(\tau^j | \eta_k) \tau_i = l^{ij}_j \eta^j_k \tau_i \tau^j = \kappa_k \eta_k \tau_i = \kappa_k \eta_k.$$  

The remaining eigenvalue of $L_{\Sigma}$ in $\mathbb{R}^n$ is 0 with eigenvector $\nu_{\Sigma}$.

1.3 The Third Fundamental Form

To obtain another property of the shape matrix $K$ we differentiate the identity $|\nu_{\Sigma}|^2 = 1$ to the result $(\partial_i \nu_{\Sigma} | \nu_{\Sigma}) = 0$. This means that $\partial_i \nu_{\Sigma}$ belongs to the tangent space, hence $\partial_i \nu_{\Sigma} = \gamma_i^k \tau_k$ for some numbers $\gamma_i^k$. Taking the inner product with $\tau_j$ we get

$$\gamma_i^k g_{kj} = \gamma_i^k (\tau_k | \tau_j) = (\partial_i \nu_{\Sigma} | \tau_j) = -(\tau_{ij} | \nu_{\Sigma}) = -l_{ij},$$

hence

$$\gamma_i^r = \gamma_i^k g_{kj} g^{jr} = -l_{ij} g^{jr} = -g^{jr} l_{ji} = -l_i^r,$$

where we used symmetry of $L$ and $G$. Therefore we have

$$\partial_i \nu_{\Sigma} = -l_i^r \tau_r = -L_{\Sigma} \tau_i, \quad i = 1, \ldots, n - 1,$$  (2.11)
the Weingarten relations. Furthermore,

$$0 = \partial_i(\nu_\Sigma|\partial_j \nu_\Sigma) = (\partial_i \nu_\Sigma|\partial_j \nu_\Sigma) + (\nu_\Sigma|\partial_i \partial_j \nu_\Sigma)$$

implies

$$-(\partial_i \nu_\Sigma|\partial_j \nu_\Sigma) = (\partial_i \nu_\Sigma|\partial_j \nu_\Sigma) = l'^r_i l'^s_j \tau_r \tau_s = l'^r_i g_{rs} l'^s_j = l'^r_i l_{rj}, \quad (2.12)$$

which are the entries of the matrix $LG^{-1}L$, i.e., the covariant components of $L^2_\Sigma$. This is the so-called third fundamental form of $\Sigma$. In particular this implies the relation

$$\text{tr} L^2_\Sigma = (L_\Sigma \tau^i|L_\Sigma \tau_i) = -g^{ij}(\partial_i \partial_j \nu_\Sigma|\nu_\Sigma), \quad (2.13)$$

which will be useful later on. Moreover, we deduce from (2.12)

$$\text{tr} L^2_\Sigma = (L_\Sigma \tau^i|L_\Sigma \tau_i) = g^{ij} l'^r_i l_{rj} = l'^r_i l'_i = \sum_{i=1}^{n-1} \kappa^2_i. \quad (2.14)$$

### 1.4 The Christoffel Symbols

The Christoffel symbols are defined according to

$$\Lambda_{ij|k} = (\tau_{ij}|\tau_k), \quad \Lambda^k_{ij} = g^{kr} \Lambda_{ij|r}. \quad (2.15)$$

Their importance stems from the representation of $\tau_{ij}$ in the basis $\{\tau_k, \nu_\Sigma\}$ of $\mathbb{R}^n$ via

$$\tau_{ij} = \Lambda^k_{ij} \tau_k + l_{ij} \nu_\Sigma. \quad (2.16)$$

Indeed, suppose $\tau_{ij} = a^k_{ij} \tau_k + b_{ij} \nu_\Sigma$. Then $l_{ij} = (\tau_{ij}|\nu_\Sigma) = b_{ij}$ and

$$\Lambda_{ij|k} = (\tau_{ij}|\tau_k) = (a^r_{ij} \tau_r|\tau_k) = g_{kr} a^r_{ij}.$$

Therefore, $a^k_{ij} = g^{sk} g_{kr} a^r_{ij} = g^{sk} \Lambda_{ij|k} = \Lambda^k_{ij}$. To express the Christoffel symbols in terms of the fundamental form $G$ we use the identities

$$\partial_k g_{ij} = \partial_k (\tau_i|\tau_j) = (\tau_{ik}|\tau_j) + (\tau_i|\tau_{jk}),$$

$$\partial_k g_{kj} = \partial_k (\tau_i|\tau_j) = (\tau_{ik}|\tau_j) + (\tau_k|\tau_{ij}),$$

$$\partial_j g_{ik} = \partial_j (\tau_i|\tau_k) = (\tau_{ij}|\tau_k) + (\tau_i|\tau_{jk}),$$

which yield

$$\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = 2(\tau_{ij}|\tau_k),$$

i.e.,

$$\Lambda_{ij|k} = \frac{1}{2}[\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}]. \quad (2.17)$$

Another important identity follows by differentiation of the relations $(\tau^j|\tau_k) = \delta^j_k$ and $(\tau^j|\nu_\Sigma) = 0$. We have

$$(\partial_i \tau^j|\tau_k) = -(\tau^j|\tau_{ik}) = -\Lambda^r_{ik} (\tau^j|\tau_r) = -\Lambda^j_{ik},$$
and
\[(\partial_i \tau_j | \nu_\Sigma) = -(\tau_j | \partial_i \nu_\Sigma) = (\tau_j | L_\Sigma \tau_i) = l^j_i,\]

hence
\[\partial_i \tau_j = -\Lambda^j_{ik} \tau^k + l^j_i \nu_\Sigma. \tag{2.18}\]

This gives another interpretation of the Christoffel symbols and of the second fundamental form.

1.5 The Surface Gradient
Let \(\rho\) be a scalar field on \(\Sigma\). The surface gradient \(\nabla_\Sigma \rho\) at \(p\) is a vector which belongs to the tangent space of \(\Sigma\) at \(p\). Thus it can be characterized by its
- **covariant components** \(a_i\), i.e., \(\nabla_\Sigma \rho = a_i \tau^i\), or by its
- **contravariant components** \(a^i\), i.e., \(\nabla_\Sigma \rho = a^i \tau_i\).

The chain rule
\[\partial_i (\rho \circ \phi) = (\nabla_\Sigma \rho | \tau_i)\]
yields \(a_i = \partial_i (\rho \circ \phi) = \partial_i \rho\). This implies
\[a_i = (\nabla_\Sigma \rho | \tau_i) = a^k (\tau_k | \tau_i) = a^k g_{ki},\]

hence
\[\nabla_\Sigma \rho = \tau^i \partial_i \rho = (g^{ij} \partial_j \rho) \tau_i. \tag{2.19}\]

Suppose \(\tilde{\rho}\) is a \(C^1\)-extension of \(\rho\) in a neighbourhood of \(\Sigma\). We then have
\[\nabla \tilde{\rho} = (\nabla \tilde{\rho} | \nu_\Sigma) \nu_\Sigma + (\nabla \tilde{\rho} | \tau_i) \tau^j = (\nabla \tilde{\rho} | \nu_\Sigma) \nu_\Sigma + (\nabla_\Sigma \rho | \tau_i) \tau^j,\]

and hence, the surface gradient \(\nabla_\Sigma \rho\) is the projection of \(\nabla \tilde{\rho}\) onto \(T_p \Sigma\), that is,
\[\nabla_\Sigma \rho = P_\Sigma \nabla \tilde{\rho}. \tag{2.20}\]

For a vector field \(f : \Sigma \to \mathbb{R}^m\) of class \(C^1\) we define similarly
\[\nabla_\Sigma f := g^{ij} \tau_i \otimes \partial_j f = \tau^j \otimes \partial_j f. \tag{2.21}\]

In particular, this yields for the identity map \(id_\Sigma\) on \(\Sigma\)
\[\nabla_\Sigma id_\Sigma = g^{ij} \tau_i \otimes \partial_j \phi = g^{ij} \tau_i \otimes \tau_j = P_\Sigma,\]

and by the Weingarten relations
\[\nabla_\Sigma \nu_\Sigma = g^{ij} \tau_i \otimes \partial_j \nu_\Sigma = -g^{ij} l^r_j \tau_i \otimes \tau_r = -l^{ij} \tau_i \otimes \tau_j = -L_\Sigma.\]

For the surface gradient of tangent vectors we have
\[\nabla_\Sigma \tau_k = g^{ij} \tau_i \otimes \partial_j \tau_k = g^{ij} \tau_i \otimes \tau_{jk} = g^{ij} \tau_i \otimes (\Lambda^r_{jk} \tau_r + l_{jk} \nu_\Sigma)\]
\[= g^{ij} \Lambda^r_{jk} \tau_r \otimes \tau_r + l^r_i \tau_i \otimes \nu_\Sigma = \Lambda^r_{kj} \tau^j \otimes \tau_r + (L_\Sigma \tau_k) \otimes \nu_\Sigma.\]
Finally we note that the surface gradient for tensors is defined according to
\[ \nabla_\Sigma K = \tau^j \otimes \partial_j K. \] (2.22)

1.6 The Surface Divergence

Let \( f \) be a tangential vector field on \( \Sigma \). As before, \( f^i = (f|\tau^i) \) denote the contravariant components of \( f \), and \( f_i = (f|\tau_i) \) the covariant components, respectively. The surface divergence of \( f \) is defined by
\[ \text{div}_\Sigma f = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} f^i) = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} g^{ij} f_j). \] (2.23)

As before, \( g := \det G \) denotes the determinant of \( G = [g_{ij}] \). This definition ensures that partial integration can be carried out as usual, i.e., that the surface divergence theorem holds for tangential \( C^1 \)-vector fields \( f \):
\[ \int_\Sigma (\nabla_\Sigma \rho | f)_\Sigma \, d\Sigma = -\int_\Sigma \rho \text{div}_\Sigma f \, d\Sigma. \] (2.24)

In fact, if e.g. \( \rho \) has support in a chart \( \phi(\Theta) \) at \( p \), then
\[
\int_\Sigma (\nabla_\Sigma \rho | f)_\Sigma \, d\Sigma = \int_\Theta \partial_i(\rho \circ \phi)[(f^i \circ \phi)\sqrt{g}]] \, d\theta
= -\int_\Theta (\rho \circ \phi) \frac{1}{\sqrt{g}} \partial_i[\sqrt{g}(f^i \circ \phi)]\sqrt{g} \, d\theta
= -\int_\Sigma \rho \text{div}_\Sigma f \, d\Sigma.
\]

Here we used that the surface measure in local coordinates is given by \( d\Sigma = \sqrt{gd\theta} \). The general case follows from this argument by using a partition of unity. There is another useful representation of surface divergence, given by
\[ \text{div}_\Sigma f = g^{ij}(\tau_j | \partial_i f) = (\tau_i | \partial_i f). \] (2.25)

It comes from
\[ \text{div}_\Sigma f = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} g^{ij} f_j) = \frac{1}{\sqrt{g}} \partial_i[\sqrt{g} g^{ij}(\tau_j | f)], \]

since
\[ (\partial_i(\sqrt{g} g^{ij} \tau_j | \tau_k) = 0, \quad k = 1, \ldots, n - 1. \] (2.26)

Here (2.26) follows from
\[
(\partial_i(\sqrt{g} g^{ij} \tau_j | \tau_k) = \partial_i(\sqrt{g} g^{ij} \tau_j | \tau_k) - \sqrt{g} g^{ij}(\tau_j | \tau_k) = \partial_k \sqrt{g} - \sqrt{g} g^{ij}(\tau_j | \tau_k)
= \partial_k \sqrt{g} - \frac{1}{2} \sqrt{g} g^{ij} \partial_k(\tau_j | \tau_i) = \frac{1}{2\sqrt{g}} (\partial_k g - g g^{ij} \partial_k g_{ij})
\]
and the relation 
\[ \partial_k g = g \text{ tr } [G^{-1} \partial_k G] = g g^{ij} \partial_k g_{ij}. \] (2.27)

The last assertion can be verified as follows:
\[
\partial_k g = \partial_k \det G = \sum_{j=1}^{n-1} \det \left( G^{-1} \left[ G_{1j} \cdots \partial_k G_{j} \cdots G_{n-1} \right] \right) = g \text{ tr } [G^{-1} \partial_k G],
\]
where \( G = [g_{ij}] = [G_1, \cdots, G_{n-1}] \), with \( G_j \) the \( j \)-th column of \( G \). From (2.25) follows
\[
\text{div}_\Sigma \tau_k = g^{ij} (\tau_j | \partial_i f) = g^{ij} \Lambda_{ki} = \Lambda_i^k.
\]
Equation (2.25) can be used as a definition of surface divergence for general, not necessarily tangential vector fields \( f \), i.e., we have
\[ \text{div}_\Sigma f := g^{ij} (\tau_j | \partial_i f) = (\tau^i | \partial_i f), \quad f \in C^1(\Sigma, \mathbb{R}^n). \] (2.28)

For example, consider \( f = \nu_\Sigma \). Then \( \partial_i \nu_\Sigma = -l_i^k \tau_k \) by the Weingarten relations and we obtain
\[ \text{div}_\Sigma \nu_\Sigma = g^{ij} (\tau_j | \partial_i \nu_\Sigma) = -g^{ij} l_{ij} = -\kappa_\Sigma. \]
This way we have derived the important relation
\[ \kappa_\Sigma = -\text{div}_\Sigma \nu_\Sigma. \] (2.29)

With this in hand, we can now deduce the relation
\[ \text{div}_\Sigma f = \text{div}_\Sigma P_\Sigma f - (f | \nu_\Sigma) \kappa_\Sigma. \] (2.30)

We remind that the surface divergence theorem (2.24) only holds for tangential vector fields. The surface divergence theorem for general vector fields reads as
\[ \int_\Sigma (\nabla_\Sigma \rho | f) \, d\Sigma = - \int_\Sigma \rho (\text{div}_\Sigma f + (f | \nu_\Sigma) \kappa_\Sigma) \, d\Sigma, \quad f \in C^1(\Sigma, \mathbb{R}^n). \] (2.31)

This follows from (2.24) and (2.30) by means of
\[
\int_\Sigma (\nabla_\Sigma \rho | f) \, d\Sigma = \int_\Sigma (\nabla_\Sigma \rho | P_\Sigma f) \, d\Sigma = - \int_\Sigma \rho \, \text{div}_\Sigma P_\Sigma f \, d\Sigma.
\]

Another representation of the surface divergence of a general vector field \( f \) is given by
\[ \text{div}_\Sigma f = (\tau^i | \partial_i f) = \text{tr} [\tau^i \otimes \partial_i f] = \text{tr} \nabla_\Sigma f. \] (2.32)
Suppose that $f \in C^1(\Sigma, \mathbb{R}^n)$ admits a $C^1$-extension $\tilde{f}$ in a neighbourhood of $\Sigma$. Then

$$\text{div}_\Sigma f = \text{div}_x \tilde{f} - (\nu_\Sigma |[\nabla_x \tilde{f}]^T \nu_\Sigma) - \left( \text{div}_x \tilde{f} - \left( \nu_\Sigma \left| \frac{\partial \tilde{f}}{\partial \nu_\Sigma} \right) \right),$$

as can be deduced from

$$\text{div}_\Sigma f = (\tau^i | \partial_i f) = (\tau^i |[\nabla_x \tilde{f}]^T \tau_i) = (\tau^i |[\nabla_x \tilde{f}]^T \tau_i) + (\nu_\Sigma |[\nabla_x \tilde{f}]^T \nu_\Sigma) - (\nu_\Sigma |[\nabla_x \tilde{f}]^T \nu_\Sigma).$$

Suppose now that $\nu_\Sigma$ admits a $C^1$-extension $\tilde{\nu}_\Sigma$ in a neighbourhood of $\Sigma$ such that $|\tilde{\nu}_\Sigma| = 1$ in this neighborhood. Then we have

$$2(\nu_\Sigma (p)|[\nabla_x \tilde{\nu}_\Sigma (p)]^T \nu_\Sigma (p)) = \frac{d}{dt} (\tilde{\nu}_\Sigma (p + t\nu_\Sigma (p)|\tilde{\nu} (p + t\nu_\Sigma (p))) \bigg|_{t=0} = 0,$$

and we obtain

$$\text{div}_x \tilde{\nu}_\Sigma = \text{div}_\Sigma \nu_\Sigma = -\kappa_\Sigma. \quad (2.33)$$

Consequently, if $\Sigma$ is given as the zero set of a $C^2$-level function $\varphi$ with $\nabla_x \varphi \neq 0$, with $\nabla_x \varphi$ pointing in the direction of $\nu_\Sigma$, we have the well-known formula

$$\kappa_\Sigma = -\text{div}_x \left( \frac{\nabla_x \varphi}{|\nabla_x \varphi|} \right).$$

Finally, the surface divergence for tensors is given by

$$\text{div}_\Sigma K = (\tau^j | \partial_j K) := (\partial_j K)^T \tau^j. \quad (2.34)$$

This immediately yields the important relation

$$\text{div}_\Sigma \mathcal{P}_\Sigma = \kappa_\Sigma \nu_\Sigma. \quad (2.35)$$

### 1.7 The Laplace-Beltrami Operator

The *Laplace-Beltrami* operator on $\Sigma$ is defined for scalar fields by means of

$$\Delta_\Sigma \rho = \text{div}_\Sigma \nabla_\Sigma \rho,$$

which in local coordinates reads

$$\Delta_\Sigma \rho = \frac{1}{\sqrt{g}} \partial_i \left[ \sqrt{g} g^{ij} \partial_j \rho \right].$$

Another representation of $\Delta_\Sigma$ is given by

$$\Delta_\Sigma \rho = g^{ij} \partial_i \partial_j \rho - g^{ij} \Lambda_{ij}^k \partial_k \rho. \quad (2.36)$$

This follows from (2.19), (2.25) and (2.18). Since at each point $p \in \Sigma$ we may choose a chart such that $g_{ij} = \delta_{ij}$ and $\Lambda_{ij}^k = 0$ at $p$, we see from this representation that
the Laplace-Beltrami operator is equivalent to the Laplacian at the point \( p \); see also Section 2.1.8 below.

To obtain another representation of \( \Delta_\Sigma \), for a scalar \( C^2 \)-function we compute
\[
\nabla^2_\Sigma \rho = \nabla_\Sigma (\tau^j \partial_j \rho) = \tau^i \otimes \partial_i (\tau^j \partial_j \rho).
\]
This yields with (2.18)
\[
\nabla^2_\Sigma \rho = (\partial_i \partial_j \rho) \tau^i \otimes \tau^j + (\partial_i \tau^j \partial_j \rho) \tau^i \otimes \tau^k + (L_\Sigma \nabla_\Sigma \rho) \otimes \nu_\Sigma.
\]
Taking traces gives
\[
\Delta_\Sigma \rho = \text{tr} \nabla^2_\Sigma \rho.
\]
Similarly, the Laplace-Beltrami operator applies to general vector fields \( f \) according to
\[
\Delta_\Sigma f = g^{ij} (\partial_i \partial_j f - \Lambda^r_{ij} \partial_r f).
\]
For example, this yields, for the identity map \( \text{id}_\Sigma \) on \( \Sigma \),
\[
\Delta_\Sigma \text{id}_\Sigma = g^{ij} (\partial_i \partial_j \phi - \Lambda^r_{ij} \partial_r \phi) = g^{ij} (\tau_{ij} - \Lambda^r_{ij} \tau_r),
\]
and hence by (2.16)
\[
\Delta_\Sigma \text{id}_\Sigma = g^{ij} l_{ij} \nu_\Sigma = \kappa_\Sigma \nu_\Sigma.
\]
Finally, we prove the important formula
\[
\Delta_\Sigma \nu_\Sigma = -\nabla_\Sigma \kappa_\Sigma - [\text{tr} L^2_\Sigma] \nu_\Sigma.
\]
(2.37)
In fact, we have from (2.12)
\[
(\Delta_\Sigma \nu_\Sigma | \nu_\Sigma) = g^{ij} (\partial_i \partial_j \nu_\Sigma - \Lambda^r_{ij} \partial_r \nu_\Sigma | \nu_\Sigma) = g^{ij} (\partial_i \nu_\Sigma | \nu_\Sigma) = -\text{tr} L^2_\Sigma.
\]
Next observe that
\[
(\partial_k \partial_j \nu_\Sigma | \tau_i) - (\partial_i \partial_j \nu_\Sigma | \tau_k) = \partial_k (\partial_j \nu_\Sigma | \tau_i) - \partial_i (\partial_j \nu_\Sigma | \tau_k)
\]
\[
= -\partial_k (\nu_\Sigma | \tau_{ij}) + \partial_i (\nu_\Sigma | \tau_{kj}) = \partial_k (\partial_i \nu_\Sigma | \tau_j) - \partial_i (\partial_k \nu_\Sigma | \tau_j)
\]
\[
= (\partial_i \nu_\Sigma | \tau_{kj}) - (\partial_k \nu_\Sigma | \tau_{ij}) = \Lambda^r_{kj} (\partial_i \nu_\Sigma | \tau_r) - \Lambda^r_{ij} (\partial_k \nu_\Sigma | \tau_r)
\]
\[
= \Lambda^r_{kj} (\partial_r \nu_\Sigma | \tau_i) - \Lambda^r_{ij} (\partial_r \nu_\Sigma | \tau_k),
\]
hence
\[
(\partial_k \partial_j \nu_\Sigma - \Lambda^r_{kj} \partial_r \nu_\Sigma | \tau_i) = (\partial_i \partial_j \nu_\Sigma - \Lambda^r_{ij} \partial_r \nu_\Sigma | \tau_k).
\]
This implies
\[
(\Delta_\Sigma \nu_\Sigma | \tau_i) = g^{ik} (\partial_k \partial_j \nu_\Sigma - \Lambda^r_{kj} \partial_r \nu_\Sigma | \tau_i) = (\partial_i \partial_j \nu_\Sigma - \Lambda^r_{ij} \partial_r \nu_\Sigma | \tau^j).
\]
On the other hand,

\[-(\nabla_\Sigma \kappa_\Sigma |\tau_i|) = -\partial_i \kappa_\Sigma = \partial_i (\partial_j \nu_\Sigma |\tau^j|)
\]

\[= (\partial_i \partial_j \nu_\Sigma |\tau^j|) + (\partial_i \nu_\Sigma |\partial_j \tau^j|)
\]

\[= (\partial_i \partial_j \nu_\Sigma - \Lambda^r_{ij} \partial_r \nu_\Sigma |\tau^j|).
\]

This proves formula (2.37).

1.8 The Case of a Graph over \(\mathbb{R}^{n-1}\)

Suppose that \(\Sigma\) is a graph over \(\mathbb{R}^{n-1}\), i.e., there is a function \(h \in C^2(\mathbb{R}^{n-1})\) such that the hypersurface \(\Sigma\) is given by the chart \(\phi(x) = [x^T, h(x)]^T, x \in \mathbb{R}^{n-1}\). Then the tangent vectors are given by \(\tau_i = [e_i^T, \partial_i h]^T\), where \(\{e_i\}\) denotes the standard basis in \(\mathbb{R}^{n-1}\). The (upward pointing) normal \(\nu_\Sigma\) is given by

\[\nu_\Sigma(x) = \beta(x) [-\nabla_x h(x)^T, 1]^T,\]

\[\beta(x) = 1/\sqrt{1 + |\nabla_x h(x)|^2}.
\]

The first fundamental form becomes \(g_{ij} = \delta_{ij} + \partial_i h \partial_j h\), hence

\[g^{ij} = \delta^{ij} - \beta^2 \partial_i h \partial_j h.
\]

This yields

\[\tau^i = [e_i - \beta^2 \partial_i h \nabla_x h]^T, \beta^2 \partial_i h]^T,
\]

and with \(\tau_{ij} = [0, \partial_i \partial_j h]^T\),

\[l_{ij} = (\tau_{ij} |\nu_\Sigma|) = \beta \partial_i \partial_j h,
\]

and therefore

\[\kappa_\Sigma = g^{ij} l_{ij} = \beta [\Delta_x h - \beta^2 (\nabla_x^2 h |\nabla_x h| \nabla_x h)] = \text{div}_x \left( \frac{\nabla_x h}{\sqrt{1 + |\nabla_x h|^2}} \right).
\]

The Christoffel symbols in this case are given by

\[\Lambda_{ij|k} = \partial_i \partial_j \partial_k h, \quad \Lambda^k_{ij} = \beta^2 \partial_i \partial_j h \partial_k h.
\]

Suppose that \(\mathbb{R}^{n-1} \times \{0\}\) is the tangent plane at \(\phi(0) = 0 \in \Sigma\). Then \(h(0) = 0\) and \(\nabla_x h(0) = 0\), hence at this point we have \(g_{ij} = \delta_{ij}, \tau_i = [e_i^T, 0]^T, \nu_\Sigma = [0, 1]^T\), \(\beta = 1\), and \(l_{ij} = \partial_i \partial_j h\). Thus the principal curvatures \(\kappa_i(0)\) are the eigenvalues of \(\nabla_x^2 h(0)\), the mean curvature is \(\kappa_\Sigma(0) = \Delta_x h(0)\), and \(\Lambda^k_{ij}(0) = 0\).

To obtain a representation of the surface gradient, let \(\rho : \Sigma \to \mathbb{R}\). Then

\[\nabla_\Sigma \rho = \tau^i \partial_i \rho = [[\nabla_x \rho - \beta^2 (\nabla_x \rho |\nabla_x h| \nabla_x h)^T, \beta^2 (\nabla_x \rho |\nabla_x h)]^T.
\]

Similarly, for \(f = (\tilde{f}, f^n) : \Sigma \to \mathbb{R}^{n-1} \times \mathbb{R}\) we obtain

\[\text{div}_\Sigma f = (\tau^i |\partial_i f) = \text{div}_x \tilde{f} + \beta^2 (\nabla_x h |\nabla_x h) f^n - (\nabla_x h \cdot \nabla_x f),
\]

and for the Laplace-Beltrami

\[\Delta_\Sigma \rho = \Delta_x \rho - \beta^2 (\nabla_x^2 \rho |\nabla_x h| \nabla_x h) - \beta^2 [\Delta_x h - \beta^2 (\nabla_x^2 h |\nabla_x h| \nabla_x h)] (\nabla_x h |\nabla_x \rho).
\]
2.2 Parameterized Hypersurfaces

We consider now a hypersurface $\Gamma = \Gamma_\rho$ which is parameterized over a fixed hypersurface $\Sigma$ according to

$$q = \psi_\rho(p) = p + \rho(p)\nu_\Sigma(p), \quad p \in \Sigma,$$

where as before $\nu_\Sigma = \nu_\Sigma(p)$ denotes the outer unit normal of $\Sigma$ at $p \in \Sigma$.

We want to derive the basic geometric quantities of $\Gamma$ in terms of $\rho$ and those of $\Sigma$. In the sequel we assume that $\rho$ is of class $C^1$ and small enough. A precise bound on $\rho$ will be given below.

2.1 The Fundamental Form

Differentiating (2.38) we obtain with the Weingarten relations (2.11)

$$\tau^\Gamma_i = \partial_i \psi_\rho = \tau_i + \rho \partial_i \nu_\Sigma + (\partial_i \rho)\nu_\Sigma = (I - \rho L_\Sigma)\tau_i + (\partial_i \rho)\nu_\Sigma. \quad (2.39)$$

We may then compute the fundamental form $G^\Gamma = [g^\Gamma_{ij}]$ of $\Gamma$ to the result

$$g^\Gamma_{ij} = (\tau^\Gamma_i | \tau^\Gamma_j) = ((I - \rho L_\Sigma)\tau_i + \partial_i \rho \nu_\Sigma | (I - \rho L_\Sigma)\tau_j + \partial_j \rho \nu_\Sigma)$$

$$= ((I - \rho L_\Sigma)\tau_i | (I - \rho L_\Sigma)\tau_j) + \partial_i \rho \partial_j \rho,$$

$$= (\tau_i | (I - \rho L_\Sigma)^2 \tau_j) + (\tau_i | \nabla_\Sigma \rho \otimes \nabla_\Sigma \rho) \tau_j$$

where we used that $((I - \rho L_\Sigma)\tau_k | \nu_\Sigma) = 0$. Hence

$$[g^\Gamma_{ij}] = [(I - \rho L_\Sigma)^2 + \nabla_\Sigma \rho \otimes \nabla_\Sigma \rho]_{ij}.$$
This yields the representation
\[
[g^{ij}_\Gamma] = [(I - \rho L \Sigma)^2[I + (I - \rho L \Sigma)^{-2}\nabla \Sigma \rho \otimes \nabla \Sigma \rho]]_{ij}
\]
\[
= [g_{ik}][(I - \rho L \Sigma)^2[I + (I - \rho L \Sigma)^{-2}\nabla \Sigma \rho \otimes \nabla \Sigma \rho]]_{k}^{j}.
\]
(2.40)

We then have
\[
g^{\Gamma} := \det G^{\Gamma} := \det [g^{ij}_\Gamma] = g \det [(I - \rho L \Sigma)^2[I + (I - \rho L \Sigma)^{-2}\nabla \Sigma \rho \otimes \nabla \Sigma \rho]].
\]

Since for any two vectors \( a, b \in \mathbb{R}^n \),
\[
\det(I + a \otimes b) = 1 + (a|b),
\]
we obtain
\[
g^{\Gamma} = g^{\alpha^2(\rho)\mu^2(\rho)},
\]
(2.41)

where
\[
\alpha(\rho) = \det(I - \rho L \Sigma) = \det(I - \rho K) = \prod_{i=1}^{n-1}(1 - \rho \kappa_i),
\]
and
\[
\mu(\rho) = (1 + ((I - \rho L \Sigma)^{-2}\nabla \Sigma \rho|\nabla \Sigma \rho))^{1/2}
\]
\[
= (1 + ((I - \rho L \Sigma)^{-1}\nabla \Sigma \rho|(I - \rho L \Sigma)^{-1}\nabla \Sigma \rho))^{1/2}.
\]

This yields for the surface measure \( d\Gamma \) on \( \Gamma_\rho \),
\[
d\Gamma = \sqrt{g^{\Gamma}} d\theta = \alpha(\rho)\mu(\rho)\sqrt{g} d\theta = \alpha(\rho)\mu(\rho) d\Sigma,
\]
(2.42)

hence
\[
|\Gamma_\rho| = \int_{\Gamma_\rho} d\Gamma = \int_{\Sigma} \alpha(\rho)\mu(\rho) d\Sigma,
\]
where \( |\Gamma_\rho| \) denotes the surface area of \( \Gamma_\rho \). Since
\[
(I + a \otimes b)^{-1} = I - \frac{a \otimes b}{1 + (a|b)},
\]
we obtain for \([G^{\Gamma}]^{-1}\) the identity
\[
[G^{\Gamma}]^{-1} = [g^{ij}_\Gamma] = [(I - \mu^{-2}(\rho)(I - \rho L \Sigma)^{-2}\nabla \Sigma \rho \otimes \nabla \Sigma \rho][I - \rho L \Sigma)^{-2}]_{k}^{i} [g^{kj}_\Sigma].
\]

All of this makes sense only for functions \( \rho \) such that \( I - \rho K \) is invertible, i.e., \( \alpha(\rho) \) should not vanish. Thus the precise bound for \( \rho \) is determined by the principle curvatures of \( \Sigma \), and we assume here and in the sequel that
\[
|\rho|_\infty < \frac{1}{\max\{|\kappa_i(p)|: i = 1, \ldots, n - 1, p \in \Sigma\}} =: \rho_0.
\]
(2.43)
2.2 The Normal at $\Gamma$
We next compute the outer unit normal at $\Gamma$. For this purpose we set
\[
\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho)),
\]
where $\beta(\rho)$ is a scalar and $a(\rho) \in T_p \Sigma$. Then $\beta(\rho) = (1 + |a(\rho)|^2)^{-1/2}$ and
\[
0 = (\nu_\Gamma|\tau^\Gamma_i) = \beta(\rho)(\nu_\Sigma - a(\rho))(I - \rho L_\Sigma)\tau_i + \nu_\Sigma \partial_i \rho),
\]
which yields
\[
0 = \partial_i \rho - (a(\rho)|(I - \rho L_\Sigma)\tau_i) = \partial_i \rho - ((I - \rho L_\Sigma)a(\rho)|\tau_i),
\]
by symmetry of $L_\Sigma$. But this implies $(I - \rho L_\Sigma)a(\rho) = \nabla_\Sigma \rho$, i.e., we have
\[
\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho))
\]
with
\[
a(\rho) = M_0(\rho)\nabla_\Sigma \rho, \quad M_0(\rho) = (I - \rho L_\Sigma)^{-1}, \quad \beta(\rho) = (1 + |a(\rho)|^2)^{-1/2} \quad (2.45)
\]
Note that $\mu(\rho) = \beta^{-1}(\rho)$, where $\mu(\rho)$ was introduced in the last section. By means of $a(\rho), \beta(\rho)$ and $M_0(\rho)$ this leads to another representation of $G^\Gamma$ and $G^{-1}_\Gamma$, namely
\[
[g_{ij}^\Gamma] = [(I - \rho L_\Sigma)(I + a(\rho) \otimes a(\rho))(I - \rho L_\Sigma)]_{ij},
\]
and
\[
[g^ij_\Gamma] = [M_0(\rho)(I - \beta^2(\rho)a(\rho) \otimes a(\rho))M_0(\rho)]^{ij}.
\]

2.3 The Surface Gradient and the Surface Divergence on $\Gamma$
It is of importance to have a representation for the surface gradient on $\Gamma$ in terms of $\Sigma$. For this purpose recall that
\[
\mathcal{P}_\Gamma = I - \nu_\Gamma \otimes \nu_\Gamma = g_{ij}^\Gamma \tau^\Gamma_i \otimes \tau^\Gamma_j,
\]
where $\nu_\Gamma = \beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho)$, and
\[
\tau^\Gamma_i = (I - \rho L_\Sigma)\tau^\Sigma_i + \partial_i \rho \nu_\Sigma.
\]
By virtue of $L_\Sigma \nu_\Sigma = 0$, the latter implies
\[
\tau^\Gamma_i = (I - \rho L_\Sigma)(\tau^\Sigma_i + \partial_i \rho \nu_\Sigma).
\]
As remarked before we do not distinguish between $L_\Sigma \in \mathcal{B}(\mathbb{R}^n)$ and its restriction to $T_p \Sigma$. With this identification, and by the fact that $(I - \rho L_\Sigma) = I$ on $\mathbb{R} \nu_\Sigma$, we have
\[
(I - \rho L_\Sigma)(\rho) \in \text{Isom}(\mathbb{R}^n, \mathbb{R}^n) \cap \text{Isom}(T_p \Sigma, T_p \Sigma),
\]
provided \( \rho \) satisfies (2.43). As before, \( \rho L_\Sigma \) is the short form for \( \rho(p)L_\Sigma(p) \). Hence, 
\[
M_0(\rho)(p) \in \text{Isom}(\mathbb{R}^n, \mathbb{R}^n) \cap \text{Isom}(T_p\Sigma, T_p\Sigma).
\]
We conclude that 
\[
M_0(\rho)\tau_i^\Gamma = \tau_i^\Sigma + (\partial_i \rho)\nu_\Sigma,
\]
and therefore 
\[
P_\Sigma M_0(\rho)\tau_i^\Gamma = \tau_i^\Sigma.
\]
(2.46)

On the other hand, we have 
\[
P_\Gamma M_0(\rho)\tau_i^\Gamma = g_{ij}^\Gamma \tau_j^\Gamma M_0(\rho)\tau_i^\Sigma = \tau_j^i(M_0(\rho)\tau_j^\Sigma),
\]
hence 
\[
P_\Gamma M_0(\rho)\tau_i^\Sigma = \tau_i^\Gamma.
\]
(2.47)

(2.46) and (2.47) allow for an easy change between the bases of \( T_p\Sigma \) and \( T_q\Gamma \), where \( q = \psi_\rho(p) = p + \rho(p)\nu_\Sigma(p) \). (2.47) implies for a scalar function \( \varphi \) on \( \Gamma \),
\[
\nabla_\Gamma \varphi = \tau_i^\Gamma \partial_i \varphi = P_\Gamma M_0(\rho)\tau_i^\Sigma \partial_i \varphi^* = P_\Gamma M_0(\rho)\nabla_\Sigma \varphi^*, \quad \varphi^* = \varphi \circ \psi_\rho
\]
which leads to the identity
\[
\nabla_\Gamma \varphi = P_\Gamma M_0(\rho)\nabla_\Sigma \varphi^*.
\]
Similarly, if \( f \) denotes a vector field on \( \Gamma \), then
\[
\nabla_\Gamma f = P_\Gamma M_0(\rho)\nabla_\Sigma f^*,
\]
and so 
\[
\text{div}_\Gamma f = (\tau_i^\Gamma | \partial_i f) = (P_\Gamma M_0(\rho)\tau_i^\Sigma | \partial_i f) = \text{tr} [P_\Gamma M_0(\rho)\nabla_\Sigma f^*].
\]
As a consequence, we obtain for the Laplace-Beltrami operator on \( \Gamma \),
\[
\Delta_\Gamma \varphi = \text{tr} [P_\Gamma M_0(\rho)\nabla_\Sigma (P_\Gamma M_0(\rho)\nabla_\Sigma \varphi^*)],
\]
which can be written as
\[
\Delta_\Gamma \varphi = M_0(\rho)P_\Gamma M_0(\rho) : \nabla_\Sigma^2 \varphi^* + (b(\rho, \nabla_\Sigma \rho, \nabla_\Sigma^2 \rho)|\nabla_\Sigma \varphi^*),
\]
with \( b = \partial_i(M_0(\rho)/P_\Gamma)P_\Gamma M(\rho)\tau_i^\Sigma \). One should note that the structure of the Laplace-Beltrami operator on \( \Gamma \) in local coordinates is
\[
\Delta_\Gamma \varphi = a^{ij}(\rho, \partial \rho)\partial_i \partial_j \varphi^* + b^k(\rho, \partial \rho, \partial^2 \rho) \partial_k \varphi^*
\]
with 
\[
a^{ij}(\rho, \partial \rho) = (P_\Gamma M_0(\rho)\tau_i^\Sigma | P_\Gamma M_0(\rho)\tau_j^\Sigma) = (\tau_i^\Gamma | \tau_j^\Gamma) = g_{ij}^\Gamma
\]
and 

\[ b^k(\rho, \partial \rho, \partial^2 \rho) = (\mathcal{P}_\Gamma M_0(\rho) \tau^k_\Sigma | \partial_i (M_0(\rho) \mathcal{P}_\Gamma \tau^k)) = (\tau^i_\Gamma | \partial_i \tau^k_\Gamma) = -g^i_\Gamma \Lambda^k_{ij}. \]

This shows that \(-\Delta_\Gamma\) is strongly elliptic on the reference manifold \(\Sigma\) as long as \(|\rho|_\infty < \rho_0\).

### 2.4 Normal Variations

For \(\rho, h \in C(\Sigma)\) sufficiently smooth and \(F(\rho) : \Sigma \to \mathbb{R}^k\) we define

\[ F'(\rho)h := \frac{d}{d\varepsilon} F(\rho + \varepsilon h) \bigg|_{\varepsilon=0}. \]

First we have

\[ M'_0(\rho) = M_0(\rho)L_\Sigma M_0(\rho), \quad M'_0(0) = L_\Sigma, \]

as \(M_0(0) = I\). Next

\[ \beta'(\rho)h = -\beta(\rho)^3(M_0(\rho)\nabla_\Sigma \rho | M'_0(\rho)h \nabla_\Sigma \rho + M_0(\rho)\nabla_\Sigma h), \]

which yields \(\beta'(0) = 0\), as \(\beta(0) = 1\). From this we get for the normal

\[ \nu(\rho) = \nu_\Gamma = \beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho) \]

the relation

\[ \nu'(\rho)h = \beta'(\rho)h(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho) - \beta(\rho)(M'_0(\rho)h \nabla_\Sigma \rho + M_0(\rho)\nabla_\Sigma h), \]

which yields

\[ \nu'(0)h = -\nabla_\Sigma h. \]

This in turn implies for the projection \(P(\rho) := \mathcal{P}_\Gamma\)

\[ P'(\rho)h = -\nu'(\rho)h \otimes \nu(\rho) - \nu(\rho) \otimes \nu'(\rho)h, \]

hence

\[ P'(0)h = \nabla_\Sigma h \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma h =: [\nabla_\Sigma \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma]h. \]

Applying these relations to \(\nabla(\rho) := \nabla_\Gamma = P(\rho)M_0(\rho)\nabla_\Sigma\) yields

\[ (\nabla'(0)h)\varphi = [P'(0)h + P(0)M'(0)h] \nabla_\Sigma \varphi \]

\[ = [\nabla_\Sigma h \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma h + hL_\Sigma] \nabla_\Sigma \varphi = [\nu_\Sigma \otimes \nabla_\Sigma h + hL_\Sigma] \nabla_\Sigma \varphi, \]

and for a not necessarily tangent vector field \(f\)

\[ (\nabla'(0)h)f = \nu_\Sigma \otimes (\nabla_\Sigma h|\nabla_\Sigma) f + hL_\Sigma \nabla_\Sigma f \]

where \((\nabla_\Sigma h|\nabla_\Sigma)f := (\nabla_\Sigma h|\tau^j) \partial_j f\). For the divergence of the vector field \(f\) this implies

\[ [\text{div}'(0)h]f = (\nu_\Sigma|[(\nabla_\Sigma h|\nabla_\Sigma)f] + h \text{tr}[L_\Sigma \nabla_\Sigma f]. \]
Finally, the variation of the Laplace-Beltrami operator $\Delta(\rho) := \Delta_\Gamma$ becomes

\[
(\Delta'(0)h)\varphi = h \text{tr}[L_\Sigma \nabla^2_\Sigma \varphi + \nabla_\Sigma (L_\Sigma \nabla_\Sigma \varphi)] + 2(L_\Sigma \nabla_\Sigma h|\nabla_\Sigma \varphi) - \kappa(\nabla_\Sigma h|\nabla_\Sigma \varphi).
\]

Note that in local coordinates we have

\[
\text{tr}[L_\Sigma \nabla^2_\Sigma \varphi] = I^i_j(\partial_i \partial_j \varphi - \Lambda^k_{ij} \partial_k \varphi),
\]

hence with

\[
\text{tr}[\nabla_\Sigma (L_\Sigma \nabla_\Sigma \varphi)] = \text{tr}[L_\Sigma \nabla^2_\Sigma \varphi] + (\text{div}_\Sigma L_\Sigma|\nabla_\Sigma \varphi),
\]

we may write alternatively

\[
(\Delta'(0)h)\varphi = 2h \text{tr}[L_\Sigma \nabla^2_\Sigma \varphi] + (h \text{div}_\Sigma L_\Sigma + [2L_\Sigma - \kappa_\Sigma]|\nabla_\Sigma h|\nabla_\Sigma \varphi).
\]

### 2.5 The Weingarten Tensor and the Mean Curvature of $\Gamma$

In invariant formulation we have

\[
L(\rho) := L_\Gamma = -\nabla_\Gamma \nu_T = -P(\rho)M_0(\rho)\nabla_\Sigma \{\beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho)\}.
\]

Thus for the variation of $L_\Gamma$ at $\rho = 0$ we obtain with $P(0) = \mathcal{P}_\Sigma$, $\beta(0) = 1$, $M_0(0) = I$, and $P'(0) = \nabla_\Sigma \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma; \beta'(0) = 0, M'_0(0) = L_\Sigma$,

\[
L'(0) = \nu_\Sigma \otimes L_\Sigma \nabla_\Sigma + L^2_\Sigma + \nabla^2_\Sigma.
\]

In particular, for $\kappa(\rho) := \kappa_\Gamma$ we have

\[
\kappa(\rho) = -\text{tr}[\nabla_\Gamma \nu_T] = \text{tr} L(\rho),
\]

hence

\[
\kappa'(0) = \text{tr} L^2_\Sigma + \Delta_\Sigma. \quad (2.48)
\]

Let us take another look at the mean curvature $\kappa(\rho) := \kappa_\Gamma$. By the relations $\tau^\rho_T = \mathcal{P}_\Gamma M_0(\rho)\tau^\Sigma_T$ and $\nu_T = \beta(\rho)(\nu_\Sigma - a(\rho))$ we obtain

\[
\kappa(\rho) = -\tau^\rho_T|\partial_j \nu_T = \mathcal{P}_\Gamma M_0(\rho)\tau^\Sigma_T|\partial_j (\beta(\rho)/\beta(\rho))\nu_T + \beta(\rho)(\partial_j \nu_\Sigma - \partial_j a(\rho))
\]

\[
= \beta(\rho)(\mathcal{P}_\Gamma M_0(\rho)\tau^\Sigma_T|L_\Sigma \tau^\Sigma_j + \partial_j a(\rho))
\]

\[
= \beta(\rho)(M_0(\rho)\tau^\Sigma_T|L_\Sigma \tau^\Sigma_j + \partial_j a(\rho)) - \beta(\rho)(\nu_T|M_0(\rho)\tau^\Sigma_T)(\nu_T|L_\Sigma \tau^\Sigma_j + \partial_j a(\rho)).
\]

Since $(M_0(\rho)\tau^\Sigma_T|L_\Sigma \tau^\Sigma_j) = \text{tr}[M_0(\rho)L_\Sigma]$ as well as

\[
(M_0(\rho)\tau^\Sigma_T|\partial_j a(\rho)) = \text{tr}[M_0(\rho)\nabla_\Sigma a(\rho)],
\]

and $(\nu_T|M_0(\rho)\tau^\Sigma_T) = -\beta(\rho)[M_0(\rho)a(\rho)]^j$, we obtain

\[
\kappa(\rho) = \beta(\rho)\{\text{tr}[M_0(\rho)(L_\Sigma + \nabla_\Sigma a(\rho))] + \beta^2(\rho)[M_0(\rho)a(\rho)]^j ([\nu_\Sigma|\partial_j a(\rho)) - (a(\rho)|\partial_j a(\rho)) - (a(\rho)|L_\Sigma \tau^\Sigma_j]\}
\]

\[
= \beta(\rho)\{\text{tr}[M_0(\rho)(L_\Sigma + \nabla_\Sigma a(\rho))] - \beta^2(\rho)(M_0(\rho)a(\rho)|\nabla_\Sigma a(\rho)a(\rho))\},
\]
as \((\nu_\Sigma | a(\rho)) = 0\) implies

\[
(\nu_\Sigma | \partial_j a(\rho)) = -(\partial_j \nu_\Sigma | a(\rho)) = (L_\Sigma \tau_j^{\Sigma} | a(\rho)).
\]

This yields the final form for the mean curvature of \(\Gamma\).

\[
\kappa(\rho) = \beta(\rho) \left\{ \text{tr} \left[ M_0(\rho) (L_\Sigma + \nabla_\Sigma a(\rho)) \right] - \beta^2(\rho) (M_0(\rho) a(\rho) | [\nabla_\Sigma a(\rho)] a(\rho)) \right\}. \tag{2.49}
\]

Recall that \(a(\rho) = M_0(\rho) \nabla_\Sigma \rho\).

We can write the curvature of \(\Gamma\) in local coordinates in the following form.

\[
\kappa(\rho) = c_{ij}(\rho, \partial \rho) \partial_i \partial_j \rho + g(\rho, \partial \rho),
\]

with

\[
c_{ij}(\rho, \partial \rho) = \beta(\rho) \left[ M_0^2(\rho) \right]_{ij} - \beta^3(\rho) [M_0^2(\rho) \nabla_\Sigma \rho]^i [M_0^2(\rho) \nabla_\Sigma \rho]^j.
\]

A simple computation yields for the symbol \(c(\rho, \xi) = c_{ij}(\rho, \partial \rho) \xi_i \xi_j\) of the principal part of \(-\kappa(\rho)\)

\[
c(\rho, \xi) = \beta(\rho) \{ |M_0(\rho) \xi|^2 - \beta^2(\rho) (a(\rho) | M_0(\rho) \xi)^2 \} \geq \beta^3(\rho) |M_0(\rho) \xi|^2 \geq \eta |\xi|^2,
\]

for \(\xi = \xi^k \tau_k^p \in T_p \Sigma\), as long as \(|\rho|_\infty < \rho_0\). Therefore, \(-\kappa(\rho)\) is a quasilinear strongly elliptic differential operator on \(\Sigma\), acting on the parameterization \(\rho\) of \(\Gamma\) over \(\Sigma\).

### 2.6 The Area Functional

As shown before, the area functional for the surface \(\Gamma_\rho = \{ p + \rho(\rho) \nu_\Sigma(p) : p \in \Sigma \}\) is given by

\[
\Phi(\rho) = \int_{\Gamma_\rho} d\Gamma = \int_\Sigma \alpha(\rho) \mu(\rho) d\Sigma.
\]

Here we use the notation

\[
\alpha(\rho) = \det(I - \rho K) = \prod_{i=1}^{n-1} (1 - \rho \kappa_i), \quad \mu(\rho) = (1 + |a(\rho)|^2)^{1/2},
\]

with \(a(\rho)\) defined in (2.45).

We compute its first variation to the result

\[
\langle \Phi'(\rho)|h \rangle = \int_\Sigma [(\mu(\rho) \alpha'(\rho) + \alpha(\rho) \mu'(\rho)) h] d\Sigma.
\]

For the derivatives of \(\alpha\) and \(\mu\) we get

\[
\alpha'(\rho) = \alpha(\rho) \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \rho \kappa_i}, \quad \mu'(\rho) h = \mu(\rho)^{-1} (a(\rho) | a'(\rho)| h).
\]

In particular, at \(\rho = 0\) we get with \(\alpha(0) = \mu(0) = 1\) and \(a(0) = 0\)

\[
\alpha'(0) = -\kappa_\Sigma, \quad \mu'(0) = 0.
\]
This implies for the first variation of $\Phi$ at $\rho = 0$,

$$\langle \Phi'(0)|h \rangle = -\int_{\Sigma} \kappa_{\Sigma} h d\Sigma. \quad (2.50)$$

This shows, in particular, that the critical points of the area functional $\Phi$ are hypersurfaces with mean curvature $\kappa_{\Sigma} = 0$. Such surfaces are called minimal surfaces.

Similarly, the second variation becomes

$$\langle \Phi''(\rho)|h|k \rangle = \int_{\Sigma} [\mu(\rho)\alpha''(\rho) + \alpha(\rho)\mu''(\rho)]hk d\Sigma$$

$$+ \int_{\Sigma} [\alpha'(\rho)\mu'(\rho)k + \alpha'(\rho)k\mu'(\rho)h] d\Sigma.$$

Since $\alpha(0) = \mu(0) = 1$ and $\mu'(0) = 0$ we get

$$\langle \Phi''(0)|h|k \rangle = \int_{\Sigma} [\alpha''(0) + \mu''(0)]hk d\Sigma.$$

We have

$$\alpha''(\rho) = \alpha(\rho) \left[ \left( \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \rho \kappa_i} \right)^2 - \sum_{i=1}^{n-1} \frac{\kappa_i^2}{(1 - \rho \kappa_i)^2} \right],$$

hence

$$\alpha''(0) = \left( \sum_{i=1}^{n-1} \kappa_i \right)^2 - \sum_{i=1}^{n-1} \kappa_i^2 = (\text{tr } K)^2 - \text{tr } K^2,$$

which is the second invariant of the shape operator $K$.

In particular, in case $\Sigma$ is a sphere of radius $R$ we have $\kappa_i = -1/R$, hence $\alpha''(0) = (n-1)(n-2)/R^2$.

For the second derivative of $\mu$ at $\rho = 0$ we obtain

$$\mu''(0)hk = (a'(0)h|a'(0)k) = (\nabla_{\Sigma} h|\nabla_{\Sigma} k).$$

This yields the following representation for the second variation of $\Phi$ at $\rho = 0$,

$$\langle \Phi''(0)|h|k \rangle = \int_{\Sigma} \{[(\text{tr } K)^2 - \text{tr } K^2]h + (\nabla_{\Sigma} h|\nabla_{\Sigma} k)\} d\Sigma. \quad (2.51)$$

By means of the surface divergence theorem (2.24), this representation can be rewritten as

$$\langle \Phi''(0)|h|k \rangle = \int_{\Sigma} \{[(\text{tr } K)^2 - \text{tr } K^2]h - \Delta_{\Sigma} h\} k d\Sigma,$$

and therefore

$$\Phi''(0)h = [(\text{tr } K)^2 - \text{tr } K^2]h - \Delta_{\Sigma} h, \quad (2.52)$$
2.2. Parameterized Hypersurfaces

i.e., $\Phi''(0)$ is the *Jacobi operator*, (sometimes also called the stability operator). Thus we see that

$$\Phi''(0) = -\kappa'(0) + \kappa^2_\Sigma.$$

In the next section we will come back to this relation.

2.7 The Volume Functional

Let $\Omega_\rho$ denote the domain bounded by the surface $\Gamma_\rho = \{ p + \rho(p)\nu_\Sigma(p) : p \in \Sigma \}$. We define the volume functional $\Psi$ by means of

$$\Psi(\rho) := |\Omega_\rho|.$$  \hspace{1cm} (2.53)

In order to obtain the variation of $\Psi(\rho)$ we rewrite the volume functional by means of the divergence theorem as

$$n \Psi(\rho) = \int_{\Omega_\rho} \text{div} \ x \ dx = \int_{\Gamma_\rho} (x|\nu_\Gamma) \ d\Gamma = \int_{\Sigma} (\text{id}_\Sigma + \rho \nu_\Sigma|\nu_\Gamma) \alpha(\rho) \mu(\rho) \ d\Sigma,$$

which yields, with $\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho))$,

$$n \Psi(\rho) = \int_{\Sigma} [\rho + (\text{id}_\Sigma|\nu_\Sigma - a(\rho))] \alpha(\rho) \ d\Sigma,$$

where as before $\alpha(\rho) = \det(I - \rho K) = \Pi_{i=1}^{n-1}(1 - \rho \kappa_i)$. The first variation of $\Psi$ then is

$$n \langle \Psi'(\rho)|h \rangle = \int_{\Sigma} \{ [\rho + (\text{id}_\Sigma|\nu_\Sigma - a(\rho))] \alpha'(\rho) h + [h - (\text{id}_\Sigma|a'(\rho) h)] \alpha(\rho) \} \ d\Sigma.$$

From $\alpha(0) = 1$, $\alpha'(0) = -\kappa_\Sigma$ and $a'(0)h = \nabla_\Sigma h$ follows

$$n \langle \Psi'(0)|h \rangle = \int_{\Sigma} [1 - (\text{id}_\Sigma|\nu_\Sigma) \kappa_\Sigma] h \ d\Sigma - \int_{\Sigma} (\text{id}_\Sigma|\nabla_\Sigma h) \ d\Sigma$$

$$= \int_{\Sigma} (1 + \text{div}_\Sigma \text{id}_\Sigma) h \ d\Sigma,$$

where we used the surface divergence theorem (2.31) in the last step. From

$$\text{div}_\Sigma \text{id}_\Sigma = (\tau^i | \partial_i \text{id}_\Sigma) = (\tau^i | \tau_i) = (n - 1)$$

follows the well-known formula for the *first variation* of the volume functional

$$\langle \Psi'(0)|h \rangle = \int_{\Sigma} h \ d\Sigma.$$  \hspace{1cm} (2.54)

Now we reconsider the area functional $\Phi$. We want to minimize surface area of $\Sigma$ under the constraint that the volume of the domain bounded by $\Sigma$ is a given
constant $\Psi_0$. The method of Lagrange multipliers yields a number $\lambda \in \mathbb{R}$ such that $\Phi' - \lambda \Psi' = 0$. According to (2.50) and (2.54), this means

$$0 = \langle \Phi' - \lambda \Psi' | h \rangle = - \int_{\Sigma} (\kappa_{\Sigma} + \lambda) h \, d\Sigma = 0,$$

for all functions $h$. This implies $\kappa_{\Sigma} \equiv -\lambda$, i.e., $\Sigma$ must be a sphere since $\Sigma$ is an embedded closed and compact hypersurface. But then the value $\Phi$ is given by the constraint, i.e.,

$$\Phi(S_R(x_0)) = \omega_n R^{n-1}, \quad \kappa_{\Sigma} = -(n-1)/R, \quad \lambda = (n-1)/R, \quad (\omega_n/n) R^n = \Psi_0.$$

The second variation of $\Psi$ can be computed as follows.

$$n \langle \Psi''(0) | h \rangle = \int_{\Sigma} (\text{id}_\Sigma | \nu_\Sigma) \alpha''(0) h k \, d\Sigma$$

$$+ \int_{\Sigma} \{(k - (\text{id}_\Sigma | \nabla_\Sigma k)) h + (h - (\text{id}_\Sigma | \nabla_\Sigma h)) k\} \alpha'(0) \, d\Sigma$$

$$- \int_{\Sigma} (\text{id}_\Sigma | a''(0) h k) \alpha(0) \, d\Sigma.$$

We observe that

$$(\text{id}_\Sigma | \nabla_\Sigma k) h + (\text{id}_\Sigma | \nabla_\Sigma h) k = (\text{id}_\Sigma | \nabla_\Sigma (hk))$$

and

$$a''(0) h k = M''_0(0) k \nabla_\Sigma h + M'_0(0) h \nabla_\Sigma k = L_\Sigma [k \nabla_\Sigma h + h \nabla_\Sigma k] = L_\Sigma \nabla_\Sigma (hk).$$

Collecting terms this yields

$$\langle \Psi''(0) | h \rangle = \frac{1}{n} \int_{\Sigma} [(\text{id}_\Sigma | \nu_\Sigma) \alpha''(0) + 2 \alpha'(0)] h k \, d\Sigma$$

$$- \frac{1}{n} \int_{\Sigma} (\text{id}_\Sigma | [\alpha'(0) I + L_\Sigma] \nabla_\Sigma (hk)) \, d\Sigma. \quad (2.55)$$

Here we recall that $\alpha'(0) = -\kappa_{\Sigma}$ and $\alpha''(0) = (\text{tr} L_\Sigma)^2 - \text{tr} L_\Sigma^2$.

In particular, for a sphere of radius $R$ centered at the origin we get $\text{id}_\Sigma = R \nu_\Sigma$, and hence

$$\langle \Psi''(0) | h \rangle = \frac{1}{n} \int_{\Sigma} \left[ \frac{(R-1)(n-2)}{R^2} + \frac{2(n-1)}{R} \right] h k \, d\Sigma = \frac{n-1}{R} \int_{\Sigma} h k \, d\Sigma.$$

This implies at a stationary point of the surface functional $\Phi$ with constraint $\Psi(\rho) = c$ with $\Phi' + \lambda \Psi' = 0$ and $\lambda = \kappa_{\Sigma}$,

$$\Phi'' + \lambda \Psi'' = -\Delta_{\Sigma} - (n-1)/R^2 = -\kappa'(0).$$
2.3 Approximation of Hypersurfaces

3.1 The Tubular Neighbourhood of a Hypersurface

Let \( \Sigma \) be a compact connected \( C^2 \)-hypersurface bounding a domain \( \Omega \subset \mathbb{R}^n \), and let \( \nu_\Sigma \) be the outer unit normal field on \( \Sigma \) with respect to \( \Omega \). Then \( \Sigma \) satisfies the uniform interior and exterior ball condition, i.e., there is a number \( a > 0 \) such that for each point \( p \in \Sigma \) there are balls \( B(x_1, a) \subset \Omega \) and \( B(x_2, a) \subset \Omega^c \), such that \( \Sigma \cap B(x_i, a) = \{p\} \). Choosing the radius \( a_0 \) maximal, we set \( a = a_0/2 \) in the sequel. Consider the mapping

\[
\Lambda: \Sigma \times (-a, a) \to \mathbb{R}^n, \quad \Lambda(p, r) := p + r\nu_\Sigma(p).
\]

We claim that \( \Lambda \) is a \( C^1 \)-diffeomorphism onto its image

\[
U_a := \text{im}(\Lambda) = \{x \in \mathbb{R}^n : \text{dist}(x, \Sigma) < a\}.
\]

Note that the centers of the balls \( B(x_i, a) \) necessarily are equal to \( x_1 = p - a\nu_\Sigma(p) \) and \( x_2 = p + a\nu_\Sigma(p) \). To prove injectivity of \( \Lambda \), suppose

\[
p_1 + r_1\nu_\Sigma(p_1) = p_2 + r_2\nu_\Sigma(p_2),
\]

where we may assume w.l.o.g. that \( r_2 \leq r_1 < a \). But then

\[
p_2 - (p_1 + r_1\nu_\Sigma(p_1)) = -r_2\nu_\Sigma(p_2),
\]

hence \( p_2 \in B(p_1 + r_1\nu_\Sigma(p_1), r_1) \cap \Sigma = \{p_1\} \), which shows \( p_1 = p_2 \) and then also \( r_1 = r_2 \). The set \( U_a \) will be called the tubular neighbourhood of \( \Sigma \) of of width \( a \).

To prove that \( \Lambda \) is a diffeomorphism, fix a point \( (p_0, r_0) \in \Sigma \times (-a, a) \) and a chart \( \phi \) for \( p_0 \). Then the function \( f(\theta, r) = \Lambda(\phi(\theta), r) \) has derivative

\[
Df(0, r_0) = [[I - r_0L_\Sigma(p_0)]\phi'(0), \nu_\Sigma(p_0)].
\]

It follows from (2.58) that \( [I - r_0L_\Sigma(p_0)] \in \mathcal{B}(T_{p_0} \Sigma) \) is invertible, and consequently, \( Df(0, r_0) \in \mathcal{B}(\mathbb{R}^n) \) is invertible as well. The inverse function theorem implies that \( \Lambda \) is locally invertible with inverse of class \( C^1 \).

It will be convenient to decompose the inverse of \( \Lambda \) into \( \Lambda^{-1} = (\Pi_\Sigma, d_\Sigma) \) such that

\[
\Pi_\Sigma \in C^1(U_a, \Sigma), \quad d_\Sigma \in C^1(U_a, (-a, a)).
\]

\( \Pi_\Sigma(x) \) is the nearest point on \( \Sigma \) to \( x \), \( d_\Sigma(x) \) is the signed distance from \( x \) to \( \Sigma \).

From the uniform interior and exterior ball condition follows that the number \( 1/a_0 \) bounds the principal curvatures of \( \Sigma \), i.e.,

\[
\max\{\kappa_i(p) : p \in \Sigma, \ i = 1, \ldots, n - 1\} \leq 1/a_0.
\]

(2.58)

A remarkable fact is that the signed distance \( d_\Sigma \) is even of class \( C^2 \). To see this, we use the identities

\[
x - \Pi_\Sigma(x) = d_\Sigma(x)\nu_\Sigma(\Pi_\Sigma(x)), \quad d_\Sigma(x) = (x - \Pi_\Sigma(x)|\nu_\Sigma(\Pi_\Sigma(x))).
\]
Differentiating w.r.t. \( x_k \) this yields
\[
\partial_{x_k} d_\Sigma(x) = (e_k - \partial_{x_k} \Pi_\Sigma(x) | \nu_\Sigma(\Pi_\Sigma(x))) + (x - \Pi_\Sigma(x)) | \partial_{x_k} (\nu_\Sigma \circ \Pi_\Sigma)(x))
\]
\[
= \nu_\Sigma^k(\Pi_\Sigma(x)) + d_\Sigma(x)(\nu_\Sigma(\Pi_\Sigma(x))) | \partial_{x_k} (\nu_\Sigma \circ \Pi_\Sigma(x))
\]
\[
= \nu_\Sigma^k(\Pi_\Sigma(x)),
\]

since \( \partial_{x_k} \Pi_\Sigma(x) \) belongs to the tangent space \( T_{\Pi_\Sigma(x)} \Sigma \), as does \( \partial_{x_k} (\nu_\Sigma \circ \Pi_\Sigma(x)) \), since \( |\nu_\Sigma \circ \Pi_\Sigma(x)| \equiv 1 \). Thus we have the formula
\[
\nabla_x d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad x \in U_a.
\]

This shows, in particular, that \( d_\Sigma \) is of class \( C^2 \).

It is useful to also have a representation for the derivative \( \partial \Pi_\Sigma(x) \) of \( \Pi_\Sigma(x) \). With
\[
I - \partial \Pi_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)) \otimes \nabla_x d_\Sigma(x) + d_\Sigma(x) \nu_\Sigma(\Pi_\Sigma(x)) \partial \Pi_\Sigma(x),
\]
and (2.59), we obtain
\[
\partial \Pi_\Sigma(x) = M_0(d_\Sigma(x))(\Pi_\Sigma(x)) \mathcal{P}_\Sigma(\Pi_\Sigma(x)),
\]
(2.60)

where \( M_0(p) := (I - rL_\Sigma(p))^{-1} \). This shows that \( \partial \Pi_\Sigma(p) = \nabla_x \Pi_\Sigma(p) = \mathcal{P}_\Sigma(p) \), the orthogonal projection onto the tangent space \( T_p \Sigma \).

### 3.2 The Level Function

Let \( \Sigma \) be a compact connected hypersurface of class \( C^2 \) bounding the domain \( \Omega \) in \( \mathbb{R}^n \). According to the previous section, \( \Sigma \) admits a tubular neighbourhood \( U_a \) of width \( a > 0 \). We may assume w.l.o.g. \( a \leq 1 \). The signed distance function \( d_\Sigma(x) \) in this tubular neighbourhood is of class \( C^2 \) as well, and since
\[
\nabla_x d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad x \in U_a,
\]
we can view \( \nabla_x d_\Sigma(x) \) as a \( C^1 \)-extension of the normal field \( \nu_\Sigma(x) \) from \( \Sigma \) to the tubular neighbourhood \( U_a \) of \( \Sigma \). Computing the second derivatives \( \nabla_x^2 d_\Sigma \) we obtain
\[
\nabla_x^2 d_\Sigma(x) = \nabla_x \nu_\Sigma(\Pi_\Sigma(x)) = -L_\Sigma(\Pi_\Sigma(x)) \mathcal{P}_\Sigma(\Pi_\Sigma(x))(I - d_\Sigma(x)L_\Sigma(\Pi_\Sigma(x)))^{-1}
\]
\[
= -L_\Sigma(\Pi_\Sigma(x))(I - d_\Sigma(x)L_\Sigma(\Pi_\Sigma(x)))^{-1},
\]
for \( x \in U_a \), as \( L_\Sigma(p) = L_\Sigma(p) \mathcal{P}_\Sigma(p) \). Taking traces then yields
\[
\Delta d_\Sigma(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i(\Pi_\Sigma(x))}{1 - d_\Sigma(x)\kappa_i(\Pi_\Sigma(x))}, \quad x \in U_a.
\]

(2.61)

In particular, this implies
\[
\nabla_x^2 d_\Sigma(p) = -L_\Sigma(p), \quad \Delta_x d_\Sigma(p) = -\kappa_\Sigma(p), \quad p \in \Sigma.
\]

(2.62)
2.3. Approximation of Hypersurfaces

Therefore the norm of $\nabla^2 d_\Sigma$ is equivalent to the maximum of the moduli of the curvatures of $\Sigma$ at a fixed point. Hence we find a constant $c$, depending only on $n$, such that

$$c|\nabla^2 d_\Sigma|_\infty \leq \max\{|\kappa_i(p)| : i = 1, \ldots, n-1, p \in \Sigma\} \leq c^{-1}|\nabla^2 d_\Sigma|_\infty.$$

Next we extend $d_\Sigma$ as a function $\varphi$ to all of $\mathbb{R}^n$. For this purpose we choose a $C^\infty$-function $\chi(s)$ such that $\chi(s)=1$ for $|s| \leq 1$, $\chi(s)=0$ for $|s| \geq 2$, $0 \leq \chi(s) \leq 1$. Then we set

$$\varphi(x) := \begin{cases} d_\Sigma(x)\chi(3d_\Sigma(x)/a) + \text{sign}(d_\Sigma(x))(1-\chi(3d_\Sigma(x)/a)), & x \in U_a, \\ \chi_{\Omega_{\text{ex}}}(x) - \chi_{\Omega_{\text{in}}}(x), & x \notin U_a, \end{cases}$$

where $\Omega_{\text{ex}}$ and $\Omega_{\text{in}}$ denote the exterior and interior component of $\mathbb{R}^n \setminus U_a$, respectively. This function $\varphi$ is then of class $C^2$, $\varphi(x) = d_\Sigma(x)$ for $x \in U_{a/3}$, and

$$\varphi(x) = 0 \iff x \in \Sigma.$$

Thus $\Sigma$ is given as zero-level set of $\varphi$, i.e., $\Sigma = \varphi^{-1}(0)$. $\varphi$ is called a canonical level function for $\Sigma$. It is a special level function for $\Sigma$, as

$$\nabla_x \varphi(x) = \nu_\Sigma(\Pi_\Sigma(x)) \quad \text{for } x \in U_{a/3}.$$ 

3.3 Existence of Parameterizations

Recall the Hausdorff metric on the set $K$ of compact subsets of $\mathbb{R}^n$ defined by

$$d_H(K_1, K_2) = \max\{ \sup_{x \in K_1} \text{dist}(x, K_2), \sup_{y \in K_2} \text{dist}(y, K_1) \}.$$ 

Suppose $\Sigma$ is a compact connected closed hypersurface of class $C^2$ bounding a bounded domain in $\mathbb{R}^n$. As before, let $U_a$ be its tubular neighbourhood, $\Pi_\Sigma : U_a \to \Sigma$ the projection and $d_\Sigma : U_a \to \mathbb{R}$ the signed distance. We want to parameterize hypersurfaces $\Gamma$ which are close to $\Sigma$ as

$$\Gamma = \{ p + \rho(p)\nu_\Sigma(p) : p \in \Sigma \},$$

where $\rho : \Sigma \to \mathbb{R}$ is then called the normal parameterization of $\Gamma$ over $\Sigma$. For this to make sense, $\Gamma$ must belong to the tubular neighbourhood $U_a$ of $\Sigma$. Therefore, a natural requirement would be $d_H(\Gamma, \Sigma) < a$. We then say that $\Gamma$ and $\Sigma$ are $C^0$-close (of order $\varepsilon$) if $d_H(\Gamma, \Sigma) < \varepsilon$.

However, this condition is not enough to allow for existence of a normal parameterization, since it is not clear that the map $\Pi_\Sigma$ is injective on $\Gamma$: small Hausdorff distance does not prevent $\Gamma$ from folding within the tubular neighbourhood. We need a stronger assumption to prevent this. If $\Gamma$ is a hypersurface of class $C^1$ we may introduce the normal bundle $\mathcal{N}\Gamma$ defined by

$$\mathcal{N}\Gamma := \{ (q, \nu_\Gamma(q)) : q \in \Gamma \} \subset \mathbb{R}^{2n}.$$
Suppose $\Gamma$ is a compact, connected $C^1$-hypersurface in $\mathbb{R}^n$. We say that $\Gamma$ and $\Sigma$ are $C^1$-close (of order $\varepsilon$) if $d_H(\mathcal{N}T, \mathcal{N}\Sigma) < \varepsilon$. We are going to show that $C^1$-hypersurfaces $\Gamma$ which are $C^1$-close to $\Sigma$ can in fact be parametrized over $\Sigma$.

For this purpose observe that, in case $\Gamma$ and $\Sigma$ are $C^1$-close of order $\varepsilon$, whenever $q \in \Gamma$, there is $p \in \Sigma$ such that $|q - p| + |\nu_T(q) - \nu_\Sigma(p)| < \varepsilon$. Hence $|q - \Pi_\Sigma q| < \varepsilon$, with $\Pi_\Sigma q := \Pi_\Sigma(q)$, and

$$|\nu_T(q) - \nu_\Sigma(\Pi_\Sigma q)| \leq |\nu_T(q) - \nu_\Sigma(p)| + |\nu_\Sigma(\Pi_\Sigma q) - \nu_\Sigma(p)| \leq \varepsilon + L|\Pi_\Sigma q - p|,$$

which yields with $|\Pi_\Sigma q - p| \leq |\Pi_\Sigma q - q| + |p - q| < 2\varepsilon$,

$$|q - \Pi_\Sigma q| + |\nu_T(q) - \nu_\Sigma(\Pi_\Sigma q)| \leq 2(1 + L)\varepsilon,$$

where $L$ denotes the Lipschitz constant of the normal of $\Sigma$. In particular, the tangent space $T_q\Gamma$ is transversal to $\nu_\Sigma(\Pi_\Sigma q)$, for each $q \in \Gamma$, that is,

$$\langle \nu_\Sigma(\Pi_\Sigma q), \nu_T(q) \rangle \neq 0, \quad q \in \Gamma.$$

Now fix a point $q_0 \in \Gamma$ and set $p_0 = \Pi_\Sigma q_0$. Since the tangent space $T_{q_0}\Gamma$ is transversal to $\nu_\Sigma(p_0)$, we infer that $\Pi_\Sigma'(q_0) : T_{q_0}\Gamma \to T_{p_0}\Sigma$ is invertible. The inverse function theorem yields an open neighbourhood $V(p_0) \subset \Sigma$ and a $C^1$-map $g : V(p_0) \to \Gamma$ such that $g(p_0) = q_0$, $g(V(p_0)) \subset \Gamma$, and $\Pi_\Sigma g(p) = p$ in $V(p_0)$. Therefore we obtain

$$q = g(p) = \Lambda \circ (\Pi_\Sigma, d_S)g(p) = \Pi_\Sigma g(p) + d_S(g(p))\nu_\Sigma(\Pi_\Sigma g(p)) = p + \rho(p)\nu_\Sigma(p),$$

with

$$\rho(p) := d_S(g(p)).$$

Thus we have a local normal parameterization of $\Gamma$ over $\Sigma$. We may extend $g$ to a maximal domain $V \subset \Sigma$, e.g. by means of Zorn’s lemma. Clearly $V$ is open in $\Sigma$ and we claim that $V = \Sigma$. If not, then the boundary of $V$ in $\Sigma$ is nonempty and hence we find a sequence $p_n \in V$ such that $p_n \to p_\infty \in \partial V$. Since $\rho_n = \rho(p_n)$ is bounded, we may assume w.l.o.g. that $\rho_n \to \rho_\infty$. But then $q_\infty = p_\infty + \rho_\infty\nu_\Sigma(p_\infty)$ belongs to $\Gamma$ as $\Gamma$ is closed. Now we may apply the inverse function theorem again to see that $V$ cannot be maximal. Since the map $\Phi(p) = p + \rho(p)\nu_\Sigma(p)$ is a local $C^1$-diffeomorphism, it is also open. Hence $\Phi(\Sigma) \subset \Gamma$ is open and compact, i.e., $\Phi(\Sigma) = \Gamma$ by connectedness of $\Gamma$. The map $\Phi$ is therefore a $C^1$-diffeomorphism from $\Sigma$ to $\Gamma$. In case $\Sigma$ is of class $C^{k+1}$ and $\Gamma$ is of class $C^k$ for $k \geq 1$ the proof above immediately implies that $\Phi \in \text{Diff}^k(\Sigma, \Gamma)$.

Observe that because of $x = \Pi_\Sigma x + d_S(x)\nu_\Sigma(\Pi_\Sigma x)$ in $U_a$ we have $x \in \Gamma$ if and only if $d_S(x) = \rho(\Pi_\Sigma x)$. This property can be used to construct a $C^1$-function $\psi$ on $\mathbb{R}^n$ such that $\Gamma = \psi^{-1}(0)$, i.e., a level function for $\Gamma$. For example we may take

$$\psi(x) = \varphi(x) - \rho(\Pi_\Sigma x)\chi(3d_S(x)/a), \quad x \in \mathbb{R}^n,$$

provided $\varepsilon < a/3$, where $\varphi$ and $\chi$ are defined in (2.63).
3.4 Approximation of Hypersurfaces
Suppose as before that $\Sigma$ is a compact connected hypersurface of class $C^2$ bounding a bounded domain $\Omega$ in $\mathbb{R}^n$. We may use the level function $\varphi : \mathbb{R}^n \to \mathbb{R}$ introduced in (2.63) to construct a real analytic hypersurface $\Sigma_\epsilon$ such that $\Sigma$ appears as a $C^2$-graph over $\Sigma_\epsilon$. In fact, we show that there is $\varepsilon_0 \in (0, a/3)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is an analytic manifold $\Sigma_\epsilon$ and a function $\rho_\epsilon \in C^2(\Sigma_\epsilon)$ with the property that

$$\Sigma = \{ p + \rho_\epsilon(p) \nu_{\Sigma_\epsilon}(p) : p \in \Sigma_\epsilon \}$$

and

$$|\rho_\epsilon|_\infty + |\nabla_{\Sigma_\epsilon} \rho_\epsilon|_\infty + |\nabla^2_{\Sigma_\epsilon} \rho_\epsilon|_\infty \leq \varepsilon.$$  

For this purpose, choose $R > 0$ such that $\varphi(x) = 1$ for $|x| > R/2$. Then define

$$\psi_k(x) = c_k \left(1 - \frac{|x|^2}{R^2}\right)^k, \quad x \in \mathbb{R}^n,$$

where $c_k > 0$ is chosen such that $\int_{\mathbb{R}^n} \psi_k(x) dx = 1$. Then $c_k \sim \alpha k^{-n/2}$ as $k \to \infty$, with some number $\alpha = \alpha(n, R)$. Indeed, we have

$$\int_{B(0,R)} \left(1 - \frac{|x|^2}{R^2}\right)^k dx = \omega_n R^n \int_0^1 (1 - r^2)^k r^{n-1} dr = \omega_n R^n \frac{\Gamma(n/2)}{\Gamma(k + n/2)} \int_0^1 (1 - t)^k t^{n/2 - 1} dt,$$

where $\omega_n = |\partial B(0,1)|$. Using the well-known relations

$$\int_0^1 (1 - t)^k t^{n/2 - 1} dt = B\left(\frac{n}{2}, k + 1\right) = \frac{\Gamma(n/2) \Gamma(k + 1)}{\Gamma(k + 1 + n/2)} \sim \Gamma(n/2) k^{-n/2}$$

with $B$ the Beta function and $\Gamma$ the Gamma function, the claim follows, with $\alpha = ((\omega_n R^n/2) \Gamma(n/2))^{-1} = (\pi R^2)^{-n/2}$.

Then as $k \to \infty$, we have $\psi_k(x) \to 0$, uniformly for $|x| \geq \eta > 0$, since $k^{n/2} q^k \to 0$ for any fixed $q \in (0, 1)$. Consequently, $\psi_k \ast f \to f$ in $C^m_{ub}(\mathbb{R}^n)$, whenever $f \in C^m_{ub}(\mathbb{R}^n)$. Let $\varphi_k = 1 + \psi_k \ast (\varphi - 1)$. Then

$$\varphi_k \to \varphi \quad \text{in} \quad C^2_{ub}(\mathbb{R}^n). \quad (2.65)$$

Moreover,

$$\psi_k \ast (\varphi - 1)(x) = \int_{\mathbb{R}^n} (\varphi(y) - 1) \psi_k(x - y) dy = \int_{|y| \leq R/2} (\varphi(y) - 1) \psi_k(x - y) dy.$$  

For $|x|, |y| < R/2$ follows $|x - y| < R$, and hence $\psi_k(x - y) = c_k (1 - |x - y|^2/R^2)^k$ is polynomial in $x, y$. But then $\varphi_k(x)$ is a polynomial for such values of $x$; in particular, $\varphi_k$ is real analytic in $U_a$. Choosing $k$ large enough, we have $|\varphi - \varphi_k|_{C^2(\mathbb{R}^n)} < \varepsilon$.

Now suppose $\varphi_k(x) = 0$. Then $|\varphi(x)| < \varepsilon$, hence $x \in U_a$ and then $|d\Sigma(x)| < \varepsilon$. This shows that the set $\Sigma_k := \varphi_k^{-1}(0)$ is in the $\varepsilon$-tubular neighbourhood around
Moreover, $|\nabla \varphi_k - \nabla \varphi|_\infty < \varepsilon$ yields $\nabla \varphi_k(x) \neq 0$ in $U_a$, and therefore $\Sigma_k$ is a manifold, which is real analytic.

Next we show that $\Sigma$ and $\Sigma_k$ are $C^1$-diffeomorphic. For this purpose, fix a point $q_0 \in \Sigma_k$. Then $q_0 = p_0 + r_0\nu_\Sigma(p_0)$, where $p_0 = \Pi\Sigma q_0 \in \Sigma$ and $r_0 = d_\Sigma(q_0)$. Consider the equation $g_k(p, r) := \varphi_k(p + r\nu_\Sigma(p)) = 0$ near $(p_0, r_0)$. Since
\[
\partial_r g_k(p_0, r_0) = (\nabla_x \varphi_k(p + r\nu_\Sigma(p))| \nu_\Sigma(p))
\]
we have
\[
\partial_r g_k(p_0, r_0) = (\nabla_x \varphi_k(q_0)| \nabla_x \varphi(p_0))
\geq 1 - |\nabla_x \varphi_k(q_0) - \nabla_x \varphi(q_0)| - |\nabla_x \varphi(q_0) - \nabla_x \varphi(p_0)|
\geq 1 - |\varphi_k - \varphi|_{C^1_b(\mathbb{R}^n)} - \varepsilon|\nabla^2 \varphi|_{C^b(\mathbb{R}^n)} > 0.
\]
Therefore, we may apply the implicit function theorem to obtain an open neighbourhood $V(p_0) \subset \Sigma$ and a $C^1$-function $r_k : V(p_0) \to \mathbb{R}$ such that $r_k(p_0) = r_0$ and $p + r_k(p)\nu_\Sigma(p) \in \Sigma_k$ for all $p \in V(p_0)$. We can now proceed as in the previous subsection to extend $r_k(\cdot)$ to a maximal domain $V \subset \Sigma$, which coincides with $\Sigma$ by compactness and connectedness of $\Sigma$.

Thus we have a well-defined $C^1$-map $f_k : \Sigma \to \Sigma_k$, $f_k(p) = p + r_k(p)\nu_\Sigma(p)$, which is injective and a diffeomorphism from $\Sigma$ to its range. We claim that $f_k$ is also surjective. If not, there is some point $q \in \Sigma_k$, $q \notin f_k(\Sigma)$. Set $p = \Pi\Sigma q$. Then $q = p + d_\Sigma(p)\nu_\Sigma(p)$ with $d_\Sigma(p) \neq r_k(p)$. Thus, there are at least two numbers $\beta_1, \beta_2 \in (-a, a)$ with $p + \beta_i\nu_\Sigma(p) \in \Sigma_k$. This implies with $\nu_\Sigma = \nu_\Sigma(p)$
\[
0 = \varphi_k(p + \beta_2\nu_\Sigma) - \varphi_k(p + \beta_1\nu_\Sigma) = (\beta_2 - \beta_1) \int_0^1 (\nabla_x \varphi_k(p + (\beta_1 + t(\beta_2 - \beta_1))\nu_\Sigma)| \nu_\Sigma) dt,
\]
which yields $\beta_2 - \beta_1 = 0$ since
\[
\int_0^1 (\nabla_x \varphi_k(p + (\beta_1 + t(\beta_2 - \beta_1))\nu_\Sigma)| \nu_\Sigma) dt \geq 1 - \varepsilon - \varepsilon|\nabla^2 \varphi|_{C^b(\mathbb{R}^n)} > 0,
\]
as above. Therefore, the map $f_k$ is also surjective, and hence $f_k \in \text{Diff}^1(\Sigma, \Sigma_k)$.

This implies, in particular, that $\Sigma_k = f_k(\Sigma)$ is connected. For later use we note that
\[
|r_k|_\infty + |\nabla_\Sigma r_k|_\infty \to 0 \text{ as } k \to \infty,
\]
as can be inferred from $\partial_3 r_k(p) = (\tau_1^\Sigma_k(p + r_k(p)\nu_\Sigma(p))| \nu_\Sigma(p))$ for $p \in \Sigma$, see (2.39).

Next we show that the mapping
\[
\Lambda_k : \Sigma_k \times (-a/2, a/2) \to U(\Sigma_k, a/2), \quad \Lambda_k(q, s) := q + s \nu_k(q)
\]
is a $C^1$-diffeomorphism for $k \geq k_0$, with $k_0 \in \mathbb{N}$ sufficiently large. In order to see this, we use the diffeomorphism $f_k$ constructed above to rewrite $\Lambda_k$ as
\[
\Lambda_k(q, s) = \Lambda_k(f_k(p), s) = p + s \nu_\Sigma(p) + r_k(p)\nu_\Sigma(p) + s[\nu_k(p + r_k(p)\nu_\Sigma(p)) - \nu_\Sigma(p)] =: \Lambda(p, s) + G_k(p, s) =: H_k(p, s).
\]
2.3. Approximation of Hypersurfaces

Clearly $H_k \in C^1(\Sigma \times (-a/2, a/2), \mathbb{R}^n)$ and $\Lambda \in \text{Diff}^1(\Sigma \times (-a, a), U(\Sigma, a))$. It is not difficult to see that

$$|G_k(p, s)| + |DG_k(p, s)| \to 0 \text{ as } k \to \infty, \text{ uniformly in } (p, s) \in \Sigma \times [-a/2, a/2].$$

Consequently, $DH_k(p, s) : T_p \Sigma \times (-a/2, a/2) \to \mathbb{R}^n$ is invertible for $k \geq k_0$, and by the inverse function theorem, $H_k$ is a local $C^1$-diffeomorphism. We claim that $H_k$ is injective for all $k$ sufficiently large. For this purpose, note that due to compactness of $\Sigma \times [-a/2, a/2]$ and injectivity of $\Lambda$ there exists a constant $c > 0$ such that

$$|\Lambda(p, s) - \Lambda(\bar{p}, \bar{s})| \geq c(|p - \bar{p}| + |s - \bar{s}|), \quad (p, s), \ (\bar{p}, \bar{s}) \in \Sigma \times [-a/2, a/2].$$

The properties of $G_k$ and compactness of $\Sigma \times [-a/2, a/2]$ imply, in turn, that the estimate above remains true for $\Lambda$ replaced by $H_k$, and $c$ replaced by $c/2$, provided $k \geq k_0$ with $k_0$ sufficiently large. Hence $H_k$ is a $C^1$-diffeomorphism onto its image for $k$ sufficiently large, as claimed. This shows that $\Sigma_k$ has a uniform tubular neighbourhood of width $a/2$ for any $k \geq k_0$, and it follows that $\Sigma \subset U(\Gamma_k, a/2)$. $\Sigma$ and $\Sigma_k$ are compact connected closed $C^1$ hypersurfaces, and we may now apply the results of the previous subsection, showing that $\Sigma$ can be parameterized over $\Sigma_k$ by means of

$$p \mapsto p + \rho_k(p)\nu_k(p) \text{ with } \rho_k \in C^2(\Sigma_k, \mathbb{R}),$$

with $\nu_k := \nu_{\Sigma_k}$.

Finally we show that $|\rho_k|_\infty + |\nabla_{\Sigma_k} \rho_k|_\infty + |\nabla^2_{\Sigma_k} \rho_k|_\infty \leq \varepsilon$ for $k$ sufficiently large. We already know from the construction that $|\rho_k|_\infty \to 0$ as $k \to \infty$. However, we need the following estimate on the rate of convergence: there exists $k_0 \in \mathbb{N}$ and a constant $C = C(k_0)$ such that

$$|\rho_k|_\infty \leq Ck^{-1/2}, \quad k \geq k_0. \quad (2.66)$$

In order to see this, we first observe that, for $|x| \leq R/2$,

$$|\varphi(x) - \varphi_k(x)| = \left| \int_{\mathbb{R}^n} [\varphi(x) - \varphi(x - y)]\psi_k(y)\, dy \right| \leq |\nabla \varphi|_\infty \int_{|y| \leq R} |y|\psi_k(y)\, dy$$

$$= |\nabla \varphi|_\infty C(n, R)c_k B\left(\frac{n+1}{2}, k+1\right).$$

Using similar arguments as above for the asymptotics of $c_k$ and $B((n+1)/2, k+1)$ this yields constants $k_0 \in \mathbb{N}$ and $C = C(k_0)$ such that $|\varphi(x) - \varphi_k(x)| \leq Ck^{-1/2}$, whenever $|x| \leq R/2$ and $k \geq k_0$. Let $p \in \Sigma_k$ be given, and let $q = p + \rho_k(p)\nu_k(p)$. Then $|\varphi_k(q)| = |\varphi_k(q) - \varphi(q)| \leq Ck^{-1/2}$ for $k \geq k_0$. On the other hand,

$$|\varphi_k(q)| = |\varphi_k(q) - \varphi_k(p)| = \rho_k(p)\left| \int_0^1 (\nabla \varphi_k(p + t\rho_k(p)\nu_k(p))\nu_k(p))\, dt \right| \geq \frac{1}{2}\rho_k(p),$$

provided $k$ is sufficiently large, and this implies (2.66).
Next we show that there exists $k_0 \in \mathbb{N}$ and $C = C(k_0)$ such that $|\partial^\alpha \varphi_k(x)| \leq Ck^{1/2}$ whenever $k \geq k_0$, $|x| \leq R/2$, and $|\alpha| = 3$. Indeed this follows from

$$|\partial_\ell \partial_\ell \partial_j \varphi_k(x)| = (2/R^2)ck_k \left| \int_{|y| \leq R} \partial_\ell \partial_\ell \partial_j \varphi_k(x-y)y_\ell(1-|y|^2/R^2)^{k-1}dy \right|$$

$$\leq Ckck_k \mathbb{E}\left( \frac{n+1}{2}, k \right) \sim ck^{1/2},$$

where $c$ is an appropriate constant. Combining with (2.66) we have shown that there are constants $k_0 \in \mathbb{N}$ and $C = C(k_0)$ such that

$$\rho_k(p)|\partial^\alpha \varphi_k(x)| \leq C,$$  \hspace{1cm} (2.67)

for $k \geq k_0$, $|\alpha| = 3$, $p \in \Sigma_k$, and $|x| \leq R/2$.

In order to show smallness of $|\nabla_{\Sigma_k} \rho_k|_\infty + |\nabla_{\Sigma_k}^2 \rho_k|_\infty$, we consider the relation

$$\varphi(\psi_k(\theta)) + (\rho_k \nu_k)(\psi_k(\theta)) = 0, \quad \theta \in \Theta_k,$$  \hspace{1cm} (2.68)

where $\psi_k : \Theta_k \rightarrow \Sigma_k$ is a $C^2$-parameterization of $\Sigma_k$ around a point $p_k = f_k(q)$ for some $q \in \Sigma$. Since $\Sigma_k = \varphi_k^{-1}(0)$ and $\varphi_k \rightarrow \varphi$ in $C^2_{\text{ub}}(\mathbb{R}^n)$ one shows that $|\partial_j \psi_k(0)|$ is uniformly bounded in $k$ for $k$ sufficiently large.

Let $\tilde{\nu}_k(x) := \nabla_x \varphi_k(x)/|\nabla_x \varphi_k(x)|$ for $x \in \mathbb{R}^n$. Clearly, $\nu_k(\psi_k(\theta)) = \tilde{\nu}_k(\psi_k(\theta))$. Taking partial derivatives in (2.68) and using the orthogonality relation $(\nabla_x \varphi_k(\psi_k(\theta))) = (\partial_j \psi_k(\theta)) = 0$ yields

$$\partial_j(\rho_k \circ \psi_k)(\theta)(\nabla_x \varphi(q_k(\theta))) = (\nu_k \circ \psi_k)(\theta))$$

$$= (\nabla_x \varphi(q_k(\theta)) - \nabla_x \varphi(\psi_k(\theta))) | \partial_j \psi_k(\theta))$$

$$- (\rho_k \circ \psi_k)(\theta)(\nabla_x \varphi(q_k(\theta))) | \partial_j (\tilde{\nu}_k \circ \psi_k)(\theta)) \hspace{1cm} (2.69)$$

where, for brevity, we set $q_k(\theta) = \psi_k(\theta) + (\rho_k \nu_k)(\psi_k(\theta))$. It follows from (2.65) and uniform continuity that

$$|\nabla_x \varphi(q_k(\theta)))| |(\nu_k \circ \psi_k)(\theta))| \geq 1/2, \hspace{1cm} (2.70)$$

provided $k \geq k_0$ with $k_0$ sufficiently large. The fact that $\partial_j \psi_k(\theta)$ is uniformly bounded for $k \geq k_0$ and (2.65) implies that the right-hand side in (2.69) converges to zero as $k \rightarrow \infty$. We have shown that $|\partial_j \rho_k(p_k)| \leq \varepsilon$, provided that $k \geq k_0$ with $k_0$ sufficiently large.

Next, we take an additional derivative $\partial_i = \partial_{\psi_i}$ in (2.69). This will produce the terms

$$\partial_i \partial_j (\rho_k \circ \psi_k)(\theta) (\nabla_x \varphi(q_k(\theta))) = (\nu_k \circ \psi_k)(\theta))$$

$$+ \partial_j (\rho_k \circ \psi_k)(\theta) \partial_i (\nabla_x \varphi(q_k(\theta))) (\nu_k \circ \psi_k)(\theta))$$

on the left-hand side. From the previous step for $\partial_j (\rho_k \circ \psi_k)$ we conclude that the second term converges to 0 as $k \rightarrow \infty$. Thus it follows from (2.70) that
\[\partial_t \partial_j (\rho_k \circ \psi_k)\] converges to 0 as \(k \to \infty\), provided we can show that the derivatives of the right-hand side in (2.69) converge to zero as \(k \to \infty\). A moment of reflection shows that this is indeed the case, with the possible exception of the term \(\rho_k (\psi_k (\theta)) (\nabla x \varphi (q_k (\theta))) (\partial_t \partial_j (\tilde{\nu}_k \circ \psi_k (\theta)))\) which is problematic as \(\partial_t \partial_j (\tilde{\nu}_k \circ \psi_k)\) involves third-order derivatives of \(\varphi_k\). Since \((\tilde{\nu}_k \circ \psi_k (\theta) \mid \partial_j (\tilde{\nu}_k \circ \psi_k (\theta)) = 1\) we get
\[(\tilde{\nu}_k \circ \psi_k (\theta) \mid \partial_t \partial_j (\tilde{\nu}_k \circ \psi_k (\theta))) = - (\partial_t (\tilde{\nu}_k \circ \psi_k (\theta) \mid \partial_j (\tilde{\nu}_k \circ \psi_k (\theta))).\]

With \(\nabla_x \varphi (q_k (\theta)) = \nu_\Sigma (q_k (\theta))\) this yields
\[\rho_k (\psi_k (\theta)) (\nabla_x \varphi (q_k (\theta))) (\partial_t \partial_j (\tilde{\nu} \circ \psi_k (\theta))) = (\nu_\Sigma (q_k (\theta)) - \tilde{\nu}_k (\psi_k (\theta)) \mid \rho_k (\psi_k (\theta)) \partial_t \partial_j (\tilde{\nu}_k \circ \psi_k (\theta))
+ \rho_k (\psi_k (\theta)) (\partial_t (\tilde{\nu}_k \circ \psi_k (\theta) \mid \partial_j (\tilde{\nu}_k \circ \psi_k (\theta))).\]

Convergence to 0 of the first term on the right-hand side follows from (2.67) and (2.65), while the second term converges to 0 since \(r_k\) has this property.

Since \(f_k : \Sigma \to \Sigma_k\) is a bijection, the assertion holds true for any point \(p_k \in \Sigma_k, k \geq k_0\), and hence the claim follows.

## 2.4 The Manifold of Hypersurfaces in \(\mathbb{R}^n\)

### 4.1 Compact Connected Hypersurfaces of Class \(C^2\)

Consider the set \(\mathcal{MH}^2\) of all compact connected \(C^2\)-hypersurfaces \(\Sigma\) in \(\mathbb{R}^n\). Let \(\mathcal{N}_\Sigma\) denote their associated normal bundles. The second normal bundle of \(\Sigma\) is defined by
\[\mathcal{N}^2 \Sigma = \{(p, \nu_\Sigma (p), \nabla_\Sigma \nu_\Sigma (p)) : p \in \Sigma\}.

We introduce a metric \(d_{\mathcal{MH}^2}\) on \(\mathcal{MH}^2\) by means of \(d_{\mathcal{MH}^2} (\Sigma_1, \Sigma_2) = d_\mu (\mathcal{N}_1, \mathcal{N}_2)\). This way \(\mathcal{MH}^2\) becomes a metric space. We want to show that \(\mathcal{MH}^2\) is a Banach manifold.

Fix a hypersurface \(\Sigma \in \mathcal{MH}^2\) of class \(C^3\). Then we define a chart over the Banach space \(X_\Sigma := C^2(\Sigma, \mathbb{R})\) as follows. \(\Sigma\) has a tubular neighbourhood \(U_a\) of width \(a\). For a given function \(\rho \in B_{X_\Sigma} (0, a/3)\) we obtain a hypersurface \(\Gamma_\rho^\Sigma\) by means of the map
\[\Phi_\Sigma (\rho) (p) := p + \rho (p) \nu_\Sigma (p), \quad p \in \Sigma.\]

According to Section 2.3, this yields a hypersurface \(\Gamma_\rho^\Sigma\) of class \(C^2\), diffeomorphic to \(\Sigma\). Moreover, with some constant \(C_a^\Sigma\), we have
\[d_{\mathcal{MH}^2} (\Gamma_\rho^\Sigma, \Sigma) \leq C_a^\Sigma |\rho|_{C^2_\Sigma},\]
which shows that the map \(\Phi_\Sigma : B_{X_\Sigma} (0, a/3) \to \mathcal{MH}^2\) is continuous. Conversely, given \(\Gamma \in \mathcal{MH}^2\) which is \(C^2\)-close to \(\Sigma\), the results in Section 2.3.3 show that \(\Gamma\) can be parameterized by a function \(\rho \in C^2 (\Sigma, \mathbb{R})\), such that \(|\rho|_{C^2_\Sigma} < a/3\).
We now determine the tangent space $T_\Sigma \mathcal{MH}^2$ at some fixed $C^3$-hypersurface $\Sigma \in \mathcal{MH}^2$. For this purpose we take a differentiable curve $\Gamma : (-\delta_0, \delta_0) \to \mathcal{MH}^2$ with $\Gamma(0) = \Sigma$. According to Section 2.3.3, there is $\delta \in (0, \delta_0)$ such that for each $t \in (-\delta, \delta)$ we find a normal parameterization $\rho(t) \in C^2(\Sigma, \mathbb{R})$ of $\Gamma(t)$. Then in these coordinates we have

$$v := \frac{d}{dt} \Gamma(0) = \frac{d}{dt} \rho(0) \nu_\Sigma \in C^2(\Sigma, T^\perp \Sigma \mathcal{MH}^2).$$

On the other hand, if $v = \rho \nu_\Sigma$ is a normal field on $\Sigma$ with $\rho \in \mathcal{X}_\Sigma$ we obtain a curve $\Gamma : (-\delta, \delta) \to \mathcal{MH}^2$ by means of $\Gamma(t)(p) = p + t \rho(p) \nu_\Sigma(p)$. Clearly, $\Gamma(0) = \Sigma$ and $\rho \nu_\Sigma = \frac{d}{dt} \Gamma(0) \in T_\Sigma \mathcal{MH}^2$. In other words, the tangent space $T_\Sigma \mathcal{MH}^2$ consists of all normal fields $v$ on $\Sigma$ which are of class $C^2$.

There is one shortcoming with this approach, namely the need to require that $\Sigma \in C^3$. This is due to the fact that we are losing one derivative when forming the normal $\nu_\Sigma$. However, since we may approximate a given hypersurface of class $C^2$ by a real analytic one in the second normal bundle, this defect can be avoided by only parameterizing over real analytic hypersurfaces, which will be sufficient below.

### 4.2 Compact Hypersurfaces with Uniform Ball Condition

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider a closed compact connected $C^2$-hypersurface $\Gamma \subset \Omega$. This hypersurface separates $\Omega$ into two disjoint open connected sets $\Omega_1$ and $\Omega_2$, the interior and the exterior of $\Gamma$ w.r.t. $\Omega$. By means of the level function $\varphi_\Gamma$ of $\Gamma$ we have $\Omega_1 = \varphi_\Gamma^{-1}(-\infty, 0)$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. Then $\partial \Omega_1 = \Gamma$ and $\partial \Omega_2 = \partial \Omega \cup \Gamma$.

The hypersurface $\Gamma$ satisfies the ball condition, i.e., there is a radius $r > 0$ such that for each point $p \in \Gamma$ there are balls $B(x_i, r) \subset \Omega_i$ such that $\Gamma \cap \overline{B}(x_i, r) = \{p\}$. The set of hypersurfaces of class $C^2$ contained in $\Omega$ satisfying the ball condition with radius $r > 0$ will be denoted by $\mathcal{MH}^2(\Omega, r)$. Note that hypersurfaces in this class have uniformly bounded principal curvatures.

The elements of $\mathcal{MH}^2(\Omega, r)$ have a tubular neighbourhood of width $a$ larger than $r$. Therefore the construction of the level function $\varphi_\Gamma$ of $\Gamma$ from Section 2.3.2 can be carried out with the same $a$ and the same cut-off function $\chi$ for each $\Gamma \in \mathcal{MH}^2(\Omega, r)$. More precisely, we have

$$\varphi_\Gamma(x) = g(d_\Gamma(x)), \quad x \in \Omega,$$

with

$$g(s) = s\chi(3s/a) + \text{sgn}(s)(1 - \chi(3s/a)), \quad s \in \mathbb{R};$$

note that $g$ is strictly increasing and equals $\pm 1$ for $\pm s > 2a/3$. This induces an injective map

$$\Phi : \mathcal{MH}^2(\Omega, r) \to C^2(\overline{\Omega}), \quad \Phi(\Gamma) := \varphi_\Gamma. \tag{2.71}$$

$\Phi$ is in fact a homeomorphism of $\mathcal{MH}^2(\Omega, r)$ onto $\Phi(\mathcal{MH}^2(\Omega, r)) \subset C^2(\overline{\Omega})$. 

This can be seen as follows. Let $\varepsilon > 0$ be small enough. If $|\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{2,\infty} \leq \varepsilon$, then $d_{\Gamma_1}(x) \leq \varepsilon$ on $\Gamma_2$ and $d_{\Gamma_2}(x) \leq \varepsilon$ on $\Gamma_1$, which implies $d_H(\Gamma_1, \Gamma_2) \leq \varepsilon$. Moreover, we also have $|\nabla_x \varphi_{\Gamma_1}(x) - \nu_{\Gamma_2}(x)| \leq \varepsilon$ on $\Gamma_2$ and $|\nabla_x \varphi_{\Gamma_2}(x) - \nu_{\Gamma_1}(x)| \leq \varepsilon$ on $\Gamma_1$ which yields $d_H(\mathcal{N}\Gamma_1, \mathcal{N}\Gamma_2) \leq C_0\varepsilon$. Then the hypersurfaces $\Gamma_j$ can both be parameterized over a $C^3$-hypersurface $\Sigma$, and therefore $d_H(\mathcal{N}^2\Gamma_1, \mathcal{N}^2\Gamma_2) \leq \varepsilon$ if and only if

$$
|\rho_1 - \rho_2|_{\infty} + |\nabla \Sigma(\rho_1 - \rho_2)|_{\infty} + |\nabla^2 \Sigma(\rho_1 - \rho_2)|_{\infty} \leq C_1\varepsilon.
$$

This in turn is equivalent to $|\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{2,\infty} \leq C_2\varepsilon$.

Let $s - (n - 1)/p > 2$. For $\Gamma \in \mathcal{M}H^2(\Omega, r)$ we then define

$$
\Gamma \in W^s_p(\Omega) \text{ if } \varphi_{\Gamma} \in W^s_p(\Omega),
$$

and

$$
\text{dist}_{W^s_p(\Omega)}(\Gamma_1, \Gamma_2) := |\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{W^s_p(\Omega)}.
$$

In this case the local charts for $\Gamma$ can be chosen of class $W^s_p$ as well. A subset $A \subset W^s_p(\Omega, r)$ is said to be (relatively) compact, if $\Phi(A) \subset W^s_p(\Omega)$ is (relatively) compact. In particular, it follows from Rellich’s theorem that $W^s_p(\Omega, r)$ is a compact subset of $W^\sigma_q(\Omega, r)$, whenever $s - n/p > \sigma - n/q$, and $s > \sigma$.

## 2.5 Moving Hypersurfaces and Domains

In this section we consider the situation of moving hypersurfaces, that is, hypersurfaces that are time dependent. We first introduce the notion of normal velocity, and we then prove a transport theorem for moving surfaces. A special case is the well-known formula for the change of surface area. In addition, we prove a transport theorem for moving domains, and derive the change of volume formula.

### 5.1 Moving Hypersurfaces

Let $\{\Gamma(t) : t \in I\}$ be a family of compact connected closed $C^2$-hypersurfaces in
$\mathbb{R}^n$ bounding domains $\Omega(t) \subset \mathbb{R}^n$, with $I \subset \mathbb{R}$ an open interval. In the following, we write $\nu_\Gamma(t, \cdot)$, $\kappa_\Gamma(t, \cdot)$, and $L_\Gamma(t, \cdot)$ for the unit normal, the mean curvature and the Weingarten tensor of $\Gamma(t)$, respectively. Let

$$\mathcal{M} = \bigcup_{t \in I} \{ \{t\} \times \Gamma(t) \}. \quad (2.74)$$

By definition, $\mathcal{M}$ is of class $C^{1,2}$ if it is a $C^1$-hypersurface in $\mathbb{R}^{n+1}$ and, moreover, $\nu_\Gamma \in C^1(\mathcal{M}, \mathbb{R}^n)$.

We now show that for every $t_0 \in I$ there is a closed, compact, analytic hypersurface $\Sigma$, an interval $I_0 := (t_0 - \delta, t_0 + \delta) \subset I$ and a function $\rho : I_0 \times \Sigma \to \mathbb{R}$ with $\rho \in C^1(I_0 \times \Sigma)$, $\nabla_\Sigma \rho \in C^1(I_0 \times \Sigma, \mathbb{R}^n)$

$$\Gamma(t) = \{ \xi + \rho(t, \xi)\nu_\Sigma(\xi) : t \in I_0, \xi \in \Sigma \}. \quad (2.75)$$

This is obtained as follows. Let $t_0 \in I$ be fixed. The assumption that $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{n+1}$ implies that for every $\varepsilon > 0$ there exits $\delta > 0$ such that $d_H(\Gamma(t_0), \Gamma(t)) \leq \varepsilon$ whenever $|t - t_0| \leq \delta$. In order to prove the assertion, it suffices to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{dist}(p, \Gamma(t_0)) \leq \varepsilon \quad \text{for all } p \in \Gamma(t) \text{ and all } |t - t_0| \leq \delta.$$

Suppose the latter assertion is not true. Then there exists $\alpha > 0$, a sequence $(p_n)_{n \in \mathbb{N}}$ in $\Gamma(t)$, and a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \to t_0$ such that $\text{dist}(p_n, \Gamma(t_n)) \geq \alpha$ for all $n \in \mathbb{N}$. As $\Gamma(t_0)$ is compact, we find $p \in \Gamma(t_0)$ and a subsequence of $(p_n)_{n \in \mathbb{N}}$, again denoted by $(p_n)_{n \in \mathbb{N}}$, such that $p_n \to p$. Therefore, $\text{dist}(p, \Gamma(t_n)) \geq \alpha$ for $n \geq N$, with $N$ sufficiently large. This shows that $\{ \{t_n\} \times \Gamma(t_n) \} \cap (\mathbb{R} \times \mathbb{B}_{\mathbb{R}^n}(p, \alpha)) = \emptyset$ for $n \geq N$, contradicting the assumption that $\mathcal{M}$ is a manifold. As $\nu_\Gamma$ is continuous on $\mathcal{M}$ we conclude that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_H(\mathcal{M}(t), \mathcal{M}(t_0)) \leq \varepsilon, \quad \text{whenever } |t - t_0| \leq \delta.$$

According to the approximation result in Section 2.3.4 we can find an analytic hypersurface $\Sigma$ which approximates $\Gamma(t_0)$. We can assume that $\Gamma(t) \subset U_{\varepsilon/2}(\Sigma)$ for $t \in I_0$, that is, $\Gamma(t)$ is contained in the tubular neighbourhood $U_{\varepsilon/2}(\Sigma)$ of $\Sigma$ of width $\varepsilon/2$. By Section 2.3.3, for every $t \in I_0$ there exists a function $\rho(t, \cdot) \in C^2(\Sigma)$ such that (2.76) holds. It remains to show that $\rho$ satisfies the regularity assumptions claimed in (2.75). In order to see this, let us consider the mapping

$$\tilde{\Pi}_\Sigma : \mathcal{M}(I_0) \to I_0 \times \Sigma, \quad \tilde{\Pi}_\Sigma(t, p) = (t, \Pi_\Sigma(p)), \quad \text{where } \mathcal{M}(I_0) := \bigcup_{t \in I_0} \{ \{t\} \times \Gamma(t) \}.$$

We note that $\tilde{\Pi}_\Sigma$ is well-defined, as $\Gamma(t) \subset U_{\varepsilon/2}(\Sigma)$ for each $t \in I_0$. Moreover, we have

$$\tilde{\Pi}_\Sigma \in C^1(\mathcal{M}(I_0), I_0 \times \Sigma), \quad \tilde{\Pi}_\Sigma(t, \cdot) = \Pi_\Sigma|_{\Gamma(t)}.$$
An inspection of the proof in Section 2.3.3 shows that
\[ \hat{\Pi}_I \in \text{Diff}^1(\mathcal{M}(I_0), I_0 \times \Sigma), \quad (\hat{\Pi}_I)^{-1}(t, \xi) = \Phi(t, \xi) := (t, \xi + \rho(t, \xi)\nu_\Sigma(\xi)). \]
This yields, in particular, \( \rho \in C^1(I_0 \times \Sigma) \) and it remains to show the additional regularity claimed in (2.75). We recall from (2.44) that
\[ \nu_T(\Phi(t, \xi)) = (\beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho))(t, \xi), \quad (t, \xi) \in I_0 \times \Sigma. \tag{2.77} \]
This representation, in conjunction with the regularity \( \rho \in C^1(I_0 \times \Sigma) \) already established, implies that
\[ \nu_T \in C^1(\mathcal{M}(I_0), \mathbb{R}^n) \iff \nabla_\Sigma \rho \in C^1(I_0 \times \Sigma, \mathbb{R}^n), \]
as we will see next. Clearly, \( \nu_T \in C^1(\mathcal{M}(I_0), \mathbb{R}^n) \) iff \( \nu_T \circ \Phi \in C^1(I_0 \times \Sigma, \mathbb{R}^n) \).
Suppose that \( \nu_T \in C^1(\mathcal{M}(I_0), \mathbb{R}^n) \). Thanks to \( \beta(\rho)(t, \xi) = (\nu_T(\Phi(t, \xi)) | \nu_\Sigma(\xi)) \) we have \( \beta(\rho) \in C^1(I_0 \times \Sigma) \) and this, in turn, implies
\[ \nabla_\Sigma \rho = (I - \rho L_\Sigma)(\nu_\Sigma - (1/\beta(\rho)(\nu_T \circ \Phi)) \in C^1(I_0 \times \Sigma). \]
On the other hand, if \( \rho \) satisfies the regularity assumptions in (2.75) and the family \( \{\Gamma(t) : t \in I_0\} \) is given by (2.76), then it is not difficult to verify that \( \mathcal{M}(I_0) \) is a hypersurface of class \( C^{1,2} \).

We now state a useful variant of (2.76). The result reads as follows: for every fixed \( t \in I \) there exists a number \( \delta > 0 \) and a function \( \rho \in C^1((-\delta, \delta) \times \Sigma) \), where \( \Sigma = \Gamma(t) \), such that
\[ \Gamma(t + s) = \{p + \rho(s, p)\nu_\Sigma(p) : s \in (-\delta, \delta), \ p \in \Sigma\}, \quad \Sigma := \Gamma(t). \tag{2.78} \]
This follows by an obvious modification of the arguments given above. In fact, the proof is less involved, as there is no need to generate a smooth approximation for \( \Gamma(t) \).

### 5.2 The Normal Velocity

Let \( \mathcal{M} \) be as above. Suppose \( I_0 \) is a subinterval of \( I \) and \( \gamma : I_0 \to \mathbb{R}^n \) is a \( C^1 \)-curve. Then \( \gamma \) is called a \( C^1 \)-curve on \( \mathcal{M} \) if \( \gamma(t) \in \Gamma(t) \) for each \( t \in I_0 \). Hence, \( \gamma \) is a \( C^1 \)-curve on \( \mathcal{M} \) iff \( (t, \gamma(t)) \in \mathcal{M} \) for \( t \in I_0 \). If \( \gamma \) is \( C^1 \)-curve on \( \mathcal{M} \), then
\[ V_T(t, p) := (\gamma'(t)|\nu_T(t, p)), \quad p = \gamma(t), \tag{2.79} \]
is called the normal velocity of \( \{\Gamma(t) : t \in I\} \) at the point \( (t, p) \). The normal velocity \( V_T \) is well-defined, that is, \( V_T(t, p) \) does not depend on the choice of a \( C^1 \)-curve on \( \mathcal{M} \) through \( p \in \Gamma(t) \). Indeed, let \( \gamma : I_0 \to \mathbb{R}^n \) be an arbitrary \( C^1 \)-curve on \( \mathcal{M} \) and let \( p = \gamma(t) \). We can assume, by possibly shrinking \( I_0 \), that the representation (2.76) holds. Therefore, the curve \( \gamma \) can be expressed by
\[ \gamma(t) = \xi(t) + \rho(t, \xi(t))\nu_\Sigma(\xi(t)), \quad t \in I_0, \ \xi(t) \in \Sigma, \]
and hence,
\[
\gamma'(t) = (I - \rho(t, \xi(t))L_{\Sigma}(\xi(t)))\xi'(t) + (\partial_t \rho(t, \xi(t)) + (\nabla_\Sigma \rho(t, \xi(t)))[\xi'(t)]\nu_{\Sigma}(\xi(t)).
\]

Using (2.77), and suppressing the variables, we obtain
\[
V_\Gamma = (\gamma'|\nu_\Gamma) = \beta(\rho)\{\partial_t \rho + (\nabla_\Sigma \rho)\xi'\} - ((I - \rho L_{\Sigma})\xi' M_0(\rho) \nabla_\Sigma \rho\} = \beta(\rho) \partial_t \rho,
\]
or in more precise notation, \( V_\Gamma(t, p) = (V_\Gamma \circ \Phi)(t, \xi) = \beta(\rho(t))(\xi) \partial_t \rho(t, \xi) \). This expression does not refer to the curve \( \gamma \), and this shows that the definition (2.79) is independent of a particular curve. Moreover, this also shows that we can, alternatively, define the normal velocity by
\[
V_\Gamma = \beta(\rho) \partial_t \rho, \tag{2.80}
\]
provided \( \{\Gamma(t) : t \in I_0\} \) is represented by (2.76), which can always be assumed.

For later use we note that
\[
[1, V_\Gamma \nu_\Gamma]^T \in T_{(t,p)} \mathcal{M}, \tag{2.81}
\]
i.e., \([1, (V_\Gamma \nu_\Gamma)(t,p)]^T\) is a tangent vector for \( \mathcal{M} \) at the point \((t,p)\). This can be seen as follows. Suppose \( \gamma : I_0 \to \mathbb{R}^n \) is a \( C^1 \)-curve on \( \mathcal{M} \). Then \((t, \gamma(t)) \in \mathcal{M} \) for \( t \in I_0 \) and consequently, \([1, \gamma'(t)]^T \in T_{(t,p)} \mathcal{M} \) with \( p = \gamma(t) \). Hence, by (2.79),
\[
[1, (V_\Gamma \nu_\Gamma)(t,p)]^T = [1, (\gamma'(t) \nu_\Gamma(t,p))\nu_\Gamma(t,p)]^T
= [1, \gamma'(t)]^T - [0, \mathcal{P}_{\Gamma(t)}(p) \gamma'(t)]^T \in T_{(t,p)} \mathcal{M},
\]
as \([0, v]^T \in T_{(t,p)} \mathcal{M} \) for any vector \( v \in T_p \Gamma(t) \).

5.3 The Lagrange Derivative for Moving Surfaces
Suppose that \( u_\Gamma(t, \cdot) := u_\Gamma(t)(\cdot) : \Gamma(t) \to \mathbb{R}^n \) is a vector field for each \( t \in I \). Hence \( u_\Gamma \) is defined on \( \mathcal{M} \) and we assume that \( u_\Gamma \in C^1(\mathcal{M}, \mathbb{R}^n) \). Then \( u_\Gamma \) is called a \( C^1 \)-velocity field for the family \( \{\Gamma(t) : t \in I\} \) if
\[
u_\Gamma = u_\Gamma|\nu_\Gamma, \tag{2.82}
\]
or more precisely, if \( V_\Gamma(t, p) = (u_\Gamma(t, p)|\nu_\Gamma(t, p)) \) for \((t,p) \in \mathcal{M} \).

A velocity field \( u_\Gamma \) is called a normal velocity field for \( \{\Gamma(t) : t \in I\} \) if \( u_\Gamma(t, \cdot) \in T_\perp \Gamma(t) \), i.e., \( u_\Gamma(t, \cdot) \) lies in the normal bundle of \( \Gamma(t) \) for each \( t \in I \). Hence,
\[
 u_\Gamma \text{ is a normal velocity field } \iff u_\Gamma = V_\Gamma \nu_\Gamma. \tag{2.83}
\]
Although only normal velocity fields matter from a geometric point of view, we nevertheless need to consider general velocity fields in order to treat the motion of fluid particles in fluid flows subject to phase transitions.
We note that if $u_\Gamma$ is a velocity field for $\{\Gamma(t) : t \in I\}$ then

$$[1, u_\Gamma]^T \in T\mathcal{M}.$$ \hspace{1cm} (2.84)

This can be deduced from (2.81), (2.82), and the decomposition

$$[1, u_\Gamma] = [1, (u_\Gamma|\nu_\Gamma)\nu_\Gamma] + [0, \mathcal{P}_\Gamma u_\Gamma] = [1, V_\Gamma \nu_\Gamma] + [0, \mathcal{P}_\Gamma u_\Gamma],$$

where, as before, we use the fact that $[0, v]^T \in T\mathcal{M}$ for any vector $v \in T\Gamma(t)$.

Next we show that for every $C^1$-velocity field $u_\Gamma$ and every $p \in \Gamma(t)$, with $t$ fixed, there exists $\delta > 0$ and a unique $C^1$-curve $[s \mapsto x(t+s)] : (-\delta, \delta) \to \mathbb{R}^n$ such that

$$\frac{d}{ds} x(t+s) = u_\Gamma(t+s, x(t+s)), \quad x(t+s) \in \Gamma(t+s), \quad s \in (-\delta, \delta),$$ \hspace{1cm} (2.85)

The solution to (2.85), in the sequel denoted by $x(t+s, t, p)$, is then called a trajectory or a flow line on $\mathcal{M}$ through $p \in \Gamma(t)$, generated by the velocity field $u_\Gamma$. The existence of such a trajectory can be seen by the following argument. Setting

$$z(s) := [t+s, x(t+s)]^T$$

we see that $x(t+s) \in \Gamma(t+s)$ is equivalent to $z(s) \in \mathcal{M}$ for $s \in (-\delta, \delta)$. Therefore, (2.85) has a (unique) solution if and only if the differential equation

$$\dot{z}(s) = [1, u_\Gamma(z(s))]^T, \quad s \in (-\delta, \delta), \quad z(0) = (t, p),$$ \hspace{1cm} (2.86)

has a (unique) solution. Existence and uniqueness of a solution $z(s) = z(s, (t, p))$ to (2.86) follows from the fact that the vector field $[1, u_\Gamma]^T$ is tangential to $\mathcal{M}$, see (2.84), and well-known results from the theory of differential equations. Moreover, we conclude that

$$[(s, (t, p)) \mapsto z(s, (t, p))] \in C^1((-\delta, \delta) \times \mathcal{M}, \mathcal{M}),$$

and this implies

$$[(s, p) \mapsto x(t+s, t, p)] \in C^1((-\delta, \delta) \times \Gamma(t), \Gamma(t)).$$

We note that

$$u_\Gamma \text{ is a } C^1\text{-velocity field : } \iff V_\Gamma = (u_\Gamma|\nu_\Gamma) \iff [1, u_\Gamma]^T \in T\mathcal{M}.$$ \hspace{1cm} (2.87)

The first equivalence follows by definition, while the second implication “ $\Rightarrow$ ” has been shown above. Suppose that $[1, u_\Gamma]^T \in T\mathcal{M}$. Then (2.85) admits a $C^1$-solution $[s \mapsto x(t+s, t, p)]$, and the definition of $V_\Gamma$ in (2.79) implies

$$V_\Gamma(t, p) = \left( \frac{d}{ds} x(t+s, t, p) \big|_{s=0} \big| \nu_\Gamma(t, p) \right) = (u_\Gamma(t, p)|\nu_\Gamma(t, p)).$$
Chapter 2. Tools from Differential Geometry

It is illustrative to point out an alternative way to establish existence of solutions to (2.85). By (2.76) we can assume that \( \{ \Gamma(t+s) : s \in (-\delta, \delta) \} \) is given by

\[
\Gamma(t+s) = \{ \xi + \rho(s, \xi) \nu_\Sigma(\xi) : s \in (-\delta, \delta), \; \xi \in \Sigma \},
\]

where \( \Sigma \) is a smooth hypersurface. Then the curve \( x(s) = \xi(s) + \rho(s, \xi(s)) \nu_\Sigma(\xi(s)) \), with \( \xi(s) \in \Sigma \), satisfies (2.76) if and only if

\[
\begin{align*}
\xi'(s) &= (I - \rho L_\Sigma)^{-1} P_\Sigma(\xi(s)) u_\Gamma(s, \xi(s) + \rho(s, \xi(s)) \nu_\Sigma(\xi(s))) \\
\xi(t) &= \xi_0,
\end{align*}
\]

(2.88)

where \((I - \rho L_\Sigma)\) is the short form for \((I - \rho(s, \xi(s)) L_\Sigma(\xi(s)))\). Indeed, applying the projection \( P_\Sigma \) to the equation

\[
(I - \rho L_\Sigma) \xi'(s) + [\partial_s \rho(s, \xi(s)) + (\nabla_\Sigma \rho(s, \xi(s)) \xi'(s)) \nu_\Sigma(\xi(s)) = u_\Gamma(s, x(s))
\]

yields (2.88), while the projection onto \( T^\perp \Sigma \) trivializes, i.e., we automatically have

\[
\partial_s \rho(s, \xi(s)) + (\nabla_\Sigma \rho(s, \xi(s)) \xi'(s)) = (u_\Gamma(s, x(s)) \nu_\Sigma(\xi(s))).
\]

The last assertion follows from

\[
\begin{align*}
\beta(\rho(s))(u_\Gamma(s, x(s)) \nu_\Sigma(\xi(s))) \\
&= (u_\Gamma(s, x(s)) u_\Gamma(s, x(s))) + \beta(\rho(s))(u_\Gamma(s, x(s)) M_0(\rho(s)) \nabla_\Sigma \rho(s, \xi(s))) \\
&= \beta(\rho(s)) [\partial_s \rho(s, \xi(s)) + (M_0(\rho(s)) P_\Sigma(\xi(s)) u_\Gamma(s, x(s)) \nabla_\Sigma \rho(s, \xi(s))] \\
&= \beta(\rho(s)) [\partial_s \rho(s, \xi(s)) + (\xi'(s)) \nabla_\Sigma \rho(s, \xi(s))],
\end{align*}
\]

where we employed (2.77), (2.80) and (2.88). It remains to observe that the ordinary differential equation (2.88), defined on \( \Sigma \), admits a unique solution as \((I - \rho L_\Sigma)^{-1} P_\Sigma u_\Gamma \in T \Sigma \).

Suppose that \( u_\Gamma \) is a \( C^1 \)-velocity field for \( \{ \Gamma(t) : t \in I \} \) and \( f_\Gamma \in C^1(M, \mathbb{R}) \). Then we define the Lagrange derivative of \( f_\Gamma \) (sometimes also called the material derivative of \( f_\Gamma \)) with respect to the velocity field \( u_\Gamma \) at the point \((t, p) \in M \) by

\[
\frac{D}{Dt} f_\Gamma(t, p) := \frac{Du_\Gamma}{Dt} f_\Gamma(t, p) := \frac{d}{ds} f_\Gamma(t + s, x(t + s, t, p)) \bigg|_{s=0}
\]

where \([s \mapsto x(s + t, t, p)]\) denotes the solution of (2.85). In case \( u_\Gamma \) is a normal \( C^1 \)-velocity field, in which case \( u_\Gamma = V_\Gamma \nu_\Gamma \), the Lagrange derivative is called the normal derivative, and we set

\[
\frac{D_n}{Dt} := \frac{D_{V_\Gamma \nu_\Gamma}}{Dt}.
\]

Then the following relation holds.

\[
\frac{Du_\Gamma}{Dt} f_\Gamma(t, p) = \frac{D_n}{Dt} f_\Gamma(t, p) + (u_\Gamma(t, p) | \nabla_\Gamma(t)) f_\Gamma(t, p).
\]

(2.89)
2.5. Moving Hypersurfaces and Domains

In order to see this, let us consider an extension \( \tilde{f}_\Gamma \) of \( f_\Gamma \) in an open neighbourhood of \( \mathcal{M} \) in \( \mathbb{R}^{n+1} \). Such an extension can, for instance, be obtained on the neighbourhood

\[
\mathcal{U}_a(\mathcal{M}) := \bigcup_{t \in I} (\{t\} \times U_a(\Gamma(t))),
\]

where \( U_a(\Gamma(t)) \) is a tubular neighbourhood of \( \Gamma(t) \) of with \( a \), by setting

\[
\tilde{f}_\Gamma(t, x) := f_\Gamma(t, p), \quad (t, x) \in \mathcal{U}_a(\mathcal{M}), \quad p = \Pi_{\Gamma(t)}(x).
\]

Then one obtains

\[
\frac{D_{ur}}{Dt} f_\Gamma(t, p) = \partial_t \tilde{f}_\Gamma(t, p) + (u_\Gamma(t, p)|\nabla_x) \tilde{f}_\Gamma(t, p).
\]

By the same argument one has

\[
\frac{D_n}{Dt} f_\Gamma(t, p) = \frac{d}{ds} f_\Gamma(t + s, y(t + s, t, p)) \bigg|_{s=0} = \frac{d}{ds} \tilde{f}_\Gamma(t + s, y(t + s, t, p)) \bigg|_{s=0} = \partial_t \tilde{f}_\Gamma(t, p) + V_\Gamma(t, p)(\nu_\Gamma(t, p)|\nabla_x) \tilde{f}_\Gamma(t, p),
\]

where \( y(\cdot) \) is the solution of (2.85) with respect to the normal velocity field \( V_\Gamma \nu_\Gamma \).

Using the relation

\[
\nabla_x \tilde{f}_\Gamma = (\nabla_x \tilde{f}_\Gamma|\nu_\Gamma) \nu_\Gamma + \mathcal{P}_\Gamma \nabla_x \tilde{f}_\Gamma = (\nabla_x \tilde{f}_\Gamma|\nu_\Gamma) \nu_\Gamma + \nabla \Gamma \tilde{f}_\Gamma,
\]

see (2.20), we conclude with (2.82)

\[
\frac{D_{ur}}{Dt} f_\Gamma(t, p) = \partial_t \tilde{f}_\Gamma(t, p) + V_\Gamma(t, p)(\nu_\Gamma(t, p)|\nabla_x) \tilde{f}_\Gamma(t, p) + (u_\Gamma(t, p)|\nabla \Gamma(t)) \tilde{f}_\Gamma(t, p) = \frac{D_n}{Dt} f_\Gamma(t, p) + (u_\Gamma(t, p)|\nabla \Gamma(t)) \tilde{f}_\Gamma(t, p).
\]

5.4 The Transport Theorem for Moving Hypersurfaces

Suppose \( u_\Gamma \) is a \( C^1 \)-velocity field for \( \{\Gamma(t) : t \in I\} \) and \( f_\Gamma \in C^1(\mathcal{M}, \mathbb{R}^n) \). The transport theorem for moving surfaces states that

\[
\frac{d}{dt} \int_{\Gamma(t)} f_\Gamma(t, x) d\Gamma = \int_{\Gamma(t)} \left[ \frac{D_{ur}}{Dt} f_\Gamma(t, x) + f_\Gamma(t, x) \text{div}_\Gamma u_\Gamma(t, x) \right] d\Gamma \]

\[
= \int_{\Gamma(t)} \left[ \frac{D_n}{Dt} f_\Gamma(t, x) - f_\Gamma(t, x) \kappa_\Gamma(t, x) V_\Gamma(t, x) \right] d\Gamma.
\]

Proof. Let \( (t, p) \in \mathcal{M} \) be fixed let \( \phi(t, \cdot) : \Theta \subset \mathbb{R}^{n-1} \to \Gamma(t) \) be a sufficiently smooth parameterization of an open neighbourhood of \( p \) in \( \Gamma(t) \). Then

\[
\phi(t + s, \cdot) := x(t + s, t, \phi(t, \cdot)) : \Theta \to \Gamma(t + s), \quad s \in (-\delta, \delta),
\]
defines a $C^1$-parameterization of a neighbourhood of $x(t + s, t, p)$ in $\Gamma(t + s)$. We first suppose that supp $f_\Gamma \subset \subset U := \{\phi(t + s, \theta) : (s, \theta) \in (-\delta, \delta) \times \Theta\}$. Let

$$g_{ij}(t + s, \theta) := (\partial_i \phi(t + s, \theta) | \partial_j \phi(t + s, \theta)), \quad G(t + s, \theta) := [g_{ij}(t + s, \theta)].$$

Hence, $G(t + s, \theta)$ is the fundamental matrix of $\Gamma(t + s)$ with respect to the parameterization $\phi(t + s, \cdot)$. With $g(t + s, \cdot) := \det G(t + s, \cdot)$ we obtain

$$\int_{\Gamma(t + s)} f_\Gamma(t + s, \tau) \, d\Gamma = \int_{\Omega} f_\Gamma(t + s, \phi(t + s, \theta)) \sqrt{g(t + s, \theta)} \, d\theta,$$

and hence

$$\frac{d}{ds} \int_{\Gamma(t + s)} f_\Gamma(t + s, \tau) \, d\Gamma \bigg|_{s=0} = \int_{\Omega} \left( \frac{D}{Dt} f_\Gamma(t, \phi(t, \theta)) \right) \sqrt{g(t, \theta)} + f_\Gamma(t, \phi(t, \theta)) \frac{\partial}{\partial s} \sqrt{g(t + s, \theta)} \bigg|_{s=0} d\theta.$$

As in (2.27) we obtain

$$\frac{\partial}{\partial s} \sqrt{g(t + s, \theta)} = \frac{1}{2\sqrt{g(t + s, \theta)}} \frac{\partial}{\partial s} g(t + s, \theta)$$

$$= \frac{1}{2} \sqrt{g(t + s, \theta)} g^{ij}(t + s, \theta) \frac{\partial}{\partial s} g_{ij}(t + s, \theta).$$

From

$$\partial_x \partial_s x(t + s, t, \phi(t, \theta)) = \partial_t \partial_s x(t + s, t, \phi(t, \theta)) = \partial_t u_\Gamma(t + s, x(t + s, t, \phi(t, \theta)))$$

follows

$$\frac{1}{2} g^{ij}(t + s, \theta) \frac{\partial}{\partial s} g_{ij}(t + s, \theta) \bigg|_{s=0}$$

$$= \frac{1}{2} g^{ij}(t, \theta) \left[ (\partial_i u_\Gamma(t, \phi(t, \theta)) | \partial_j \phi(t, \theta)) + (\partial_i \phi(t, \theta) | \partial_j u_\Gamma(t, \phi(t, \theta))) \right]$$

$$= \frac{1}{2} g^{ij}(t, \theta) \left[ (\partial_i u_\Gamma(t, \phi(t, \theta)) | \Gamma^{(t)}_j(\phi(t, \theta))) + (\Gamma^{(t)}_i(\phi(t, \theta)) | \partial_j u_\Gamma(t, \phi(t, \theta))) \right]$$

$$= \frac{1}{2} \left[ (\partial_i u_\Gamma(t, \phi(t, \theta)) | \tau^{(t)}_j(\phi(t, \theta))) + (\tau^{(t)}_i(\phi(t, \theta)) | \partial_j u_\Gamma(t, \phi(t, \theta))) \right]$$

$$= \div_\Gamma(t) u_\Gamma(t, \phi(t, \theta)).$$

Combining all steps yields

$$\frac{d}{ds} \int_{\Gamma(t + s)} f_\Gamma(t + s, \tau) \, d\Gamma \bigg|_{s=0} = \int_{\Omega} \left[ \frac{D}{Dt} f_\Gamma(t, \phi(t, \theta)) + f_\Gamma(t, \phi(t, \theta)) \div_\Gamma(t) u_\Gamma(t, \phi(t, \theta)) \right] \sqrt{g(t, \theta)} \, d\theta$$

$$= \int_{\Gamma(t)} \left[ \frac{D}{Dt} f_\Gamma(t, y) + f_\Gamma(t, y) \div_\Gamma(t) u_\Gamma(t, y) \right] \, d\Gamma.$$
For a more general function $f_\Gamma$ we can apply the result above in conjunction with a partition of unity for $\mathcal{M}$. Hence, we have shown that
\[
\frac{d}{dt} \int_{\Gamma(t)} f_\Gamma(t, x) \, d\Gamma = \int_{\Gamma(t)} \left[ \frac{D u_\Gamma}{Dt} f_\Gamma(t, x) + f_\Gamma(t, x) \text{div}_\Gamma u_\Gamma(t, x) \right] \, d\Gamma.
\]
The second assertion in (2.91) follows from the surface divergence theorem (2.31) and (2.89).

We note that (2.91) implies, in particular, the well-known change of area formula
\[
\frac{d}{dt} |\Gamma(t)| = - \int_{\Gamma(t)} \kappa_\Gamma V_\Gamma \, d\Gamma.
\] (2.92)

It is worthwhile to point out that (2.92) can also be derived from (2.50). This can be obtained as follows. Using the representation (2.78) we have $|\Gamma(t+s)| = \Phi(\rho(s))$ for $t$ fixed, and the change of area formula (2.50) in conjunction with the relation $\rho(0) = 0$ immediately yields
\[
\frac{d}{ds} |\Gamma(t+s)| \bigg|_{s=0} = \langle \Phi'(0), \partial_s \rho(0) \rangle = - \int_\Sigma \kappa_\Sigma \partial_s \rho(0) \, d\Sigma = - \int_{\Gamma(t)} \kappa_\Gamma V_\Gamma \, d\Gamma.
\]

### 5.5 The Transport Theorem for Moving Domains

Suppose $\{\Gamma(t) : t \in I\}$ is a family of compact connected closed $C^2$-hypersurfaces in $\mathbb{R}^n$, bounding domains $\Omega(t) \subset \mathbb{R}^n$. We assume again that
\[
\mathcal{M} = \bigcup_{t \in I} (\{t\} \times \Gamma(t))
\]
is a $C^{1,2}$-hypersurface in $\mathbb{R}^{n+1}$, and we set
\[
\mathcal{Q} = \bigcup_{t \in I} (\{t\} \times \Omega(t)).
\]
Let $f \in C^1(\overline{\mathcal{Q}})$. Then we have the transport theorem for moving domains:
\[
\frac{d}{dt} \int_{\Omega(t)} f(t, x) \, dx = \int_{\Omega(t)} \partial_t f(t, x) \, dx + \int_{\Gamma(t)} f(t, x) V_\Gamma(t, x) \, d\Gamma.
\] (2.93)

**Proof.** We first show that for each fixed $t \in I$ there exists a family of mappings
\[
\Phi(t + s, \cdot) : \overline{\Omega}(t) \rightarrow \overline{\Omega}(t+s), \quad s \in (-\delta, \delta),
\]
such that
\[
\Phi(t + s, \cdot) \in \text{Diff}^1(\Omega(t), \Omega(t+s)) \cap \text{Diff}^1(\Gamma(t), \Gamma(t+s)), \quad s \in (-\delta, \delta),
\] (2.94)
where $\text{Diff}^1(U,V)$ denotes the set of all $C^1$-diffeomorphisms from $U$ into $V$. The mappings $\Phi(t + s, \cdot)$ can, for instance, be constructed as follows. According to (2.78) we know that

$$\phi(t + s, p) := p + \rho(s, p)\nu_{\Sigma}(p), \quad p \in \Sigma := \Gamma(t), \quad s \in (-\delta, \delta),$$

(2.95)

satisfies $\phi(t + s, \cdot) \in \text{Diff}^1(\Gamma(t), \Gamma(t + s))$. By means of a Hanzawa transform we can extend $\phi(t + s, \cdot)$ to $\overline{\Omega}(t)$ such that (2.94) holds. In more detail, let

$$\Phi(t + s, x) = x + \chi(d_{\Sigma}(x)/a)\rho(s, \Pi_{\Sigma}(x))\nu_{\Sigma}(\Pi_{\Sigma}(x)), \quad x \in \overline{\Omega}(t).$$

Here $d_{\Sigma}$ and $\Pi_{\Sigma}$ have the same meaning as in (2.57), and $\chi$ is a suitable cut-off function, say $\chi \in D(\mathbb{R}), \ 0 \leq \chi \leq 1$, $\chi(r) = 1$ for $|r| < 1/3$, and $\chi(r) = 0$ for $|r| > 2/3$.

Clearly, $\Phi(t + s, p) = \phi(t + s, p)$ for $p \in \Gamma(t)$. Since $\Phi(t, \cdot) = \text{id}_{\overline{\Omega}(t)}$ we can assume that $\det \partial_x \Phi(t + s, x) > 0$ for $(s, x) \in (-\delta, \delta) \times \Omega(t)$ by choosing $\delta$ small enough. Next we observe that by (2.27)

$$\frac{d}{ds} \det \partial_x \Phi(t + s, x) = \det \partial_x \Phi(t + s, x) \text{tr}([\partial_y \Phi(t + s, x)]^{-1}[\partial_x \partial_s \Phi(t + s, x)]),$$

and therefore,

$$\frac{d}{ds} \det \partial_x \Phi(t + s, x) \bigg|_{s=0} = \text{tr}[\partial_x \partial_s \Phi(t, x)] = \text{div}_x \partial_x \Phi(t, x).$$

Employing the transformation rule for integrals yields

$$\int_{\Omega(t+s)} f(t + s, y) dy = \int_{\Omega(t)} f(t + s, \Phi(t + s, x)) \det \partial_x \Phi(t + s, x) dx,$$

and hence,

$$\frac{d}{ds} \int_{\Omega(t+s)} f(t + s, y) dy \bigg|_{s=0}
= \int_{\Omega(t)} \left[ \partial_t f(t, x) + (\nabla_x f(t, x)|_{\partial_x \Phi(t, x)}) + f(t, x)\text{div}_x \partial_x \Phi(t, x) \right] dx$$

$$= \int_{\Omega(t)} \left[ \partial_t f(t, x) + \text{div}_x (f(t, x)\partial_x \Phi(t, x)) \right] dx$$

(2.96)

$$= \int_{\Omega(t)} \partial_t f(t, x) dx + \int_{\Gamma(t)} f(t, x)(\partial_x \Phi(t, x)|_{\nu_{\Gamma}(t, x)}) d\Gamma$$

$$= \int_{\Omega(t)} \partial_t f(t, x) dx + \int_{\Gamma(t)} f(t, x)V_{\Gamma}(t, x) d\Gamma,$$

where we used (2.79) in the last step. This completes the proof. \qed
The relation (2.93) immediately yields the well-known change of volume formula

\[
\frac{d}{dt} |\Omega(t)| = \int_{\Gamma(t)} V_\Gamma \, d\Gamma.
\]

(2.97)

We point out that (2.97) can also be derived from (2.54). Indeed, using once more the representation

\[
\Gamma(t + s) = \{ p + \rho(s, p) \nu_\Sigma(p) : s \in (-\delta, \delta), \ p \in \Sigma \}, \quad \Sigma := \Gamma(t),
\]

we have \(|\Omega(t + s)| = \Psi(\rho(s))\), with \(\Psi\) the volume functional introduced in Section 2.2.7. Then the first variation formula (2.54) and the relation \(\rho(0) = 0\) imply

\[
\left. \frac{d}{ds} |\Omega(t + s)| \right|_{s=0} = \langle \Psi'(0), \partial_s \rho(0) \rangle = \int_\Sigma \partial_s \rho(0) \, d\Sigma = \int_{\Gamma(t)} V_\Gamma \, d\Gamma.
\]

We now consider the more special case where the moving domain \(\Omega(t)\) is transported by a velocity field \(u\). Suppose then that \(J \subset \mathbb{R}\) is an open interval, \(G \subset \mathbb{R}^n\) is an open set, and \(u \in C^1(J \times \Omega, \mathbb{R}^n)\). We assume that solutions to the ordinary differential equation

\[
y'(t) = u(t, y(t)), \quad y(\tau) = \xi,
\]

exist on \(I\) for all \((\tau, \xi) \in J \times G\), and we denote the unique solution with initial value \(y(\tau) = \xi\) by \(y(t, \tau, \xi)\). Let \(\Omega_0 \subset \subset G\) be a \(C^2\)-domain, \(t_0 \in I\) a fixed number, and suppose that the family \(\{\Omega(t) : t \in I\}\) of moving domains is given by

\[
\Omega(t) = y(t, t_0, \cdot)|_{\Omega_0} = \{ y(t, t_0, x_0) : x_0 \in \Omega_0 \}, \quad t \in I.
\]

Suppose that \(f \in C^1(J \times G)\). Then the Reynolds transport theorem states that

\[
\frac{d}{dt} \int_{\Omega(t)} f(t, x) \, dx = \int_{\Omega(t)} \left[ \partial_t f(t, x) + \text{div}_x (f(t, x)u(t, x)) \right] \, dx.
\]

(2.98)

Proof. Let \(t \in I\) be fixed and let \(\Phi(s, t) := y(s, t, x)\) for \((s, x) \in J \times G\). From the theory of ordinary differential equations follows that

\[
\Phi(t + s, \cdot) \in \text{Diff}^1(\Omega(t), \Omega(t + s)), \quad s \in (-\delta, \delta),
\]

(2.99)

with \(\phi^{-1}(t + s, \cdot) = y(t, t + s, \cdot)\). We can now follow the computations in (2.93) to the result

\[
\frac{d}{dt} \int_{\Omega(t)} f(x, t) \, dx = \int_{\Omega(t)} \left[ \partial_t f(t, x) + \text{div}_x (f(t, x)\partial_s \Phi(t, x)) \right] \, dx
\]

\[
= \int_{\Omega(t)} \left[ \partial_t f(t, x) + \text{div}_x (f(t, x)u(t, x)) \right] \, dx
\]

and this completes the proof. \(\square\)
5.6 The Transport Theorem for Two-Phase Moving Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain in $\mathbb{R}^n$ with $C^2$-boundary $\partial \Omega$. Suppose that $\{\Gamma(t) : t \in I\}$ is a family of closed compact $C^2$-hypersurfaces with $\Gamma(t) \subset \Omega$, such that $\Gamma(t)$ encloses a region $\Omega_1(t) \subset \Omega$, and such that $\partial \Omega_1(t) = \Gamma(t)$ for each $t \in I$. Let $\Omega_2(t) := \Omega \setminus \Omega_1(t)$. Then

$$\overline{\Omega} = \overline{\Omega}_1(t) \cup \overline{\Omega}_2(t), \quad \overline{\Omega}_1(t) \cap \overline{\Omega}_2(t) = \Gamma(t), \quad \partial \Omega_2(t) = \Gamma(t) \cup \partial \Omega, \quad t \in I.$$  

Hence, $\Gamma(t)$ separates $\Omega$ into an ‘inner’ region $\Omega_1(t)$ and an ‘outer’ region $\Omega_2(t)$, with $\Omega_2(t)$ being in contact with the boundary $\partial \Omega$. Then $\nu_{\Gamma(t)}$ denotes the outward pointing unit normal field for $\Omega_1(t)$ on $\Gamma(t)$. Let

$$\mathcal{Q}_j = \bigcup_{t \in I} \{\{t\} \times \Omega_j(t)\}, \quad j = 1, 2.$$  

As above, we assume that $\mathcal{M}$ is a $C^{1,2}$-hypersurface. Let $f_j : \mathcal{Q}_j \to \mathbb{R}$ be given. Then we set

$$f(t, x) := \begin{cases} f_1(t, x), & x \in \Omega_1(t), \\ f_2(t, x), & x \in \Omega_2(t), \end{cases}$$  

so that $f : \mathcal{Q}_1 \cup \mathcal{Q}_2 \to \mathbb{R}$. In case $f_j$ admits a continuous extension $\tilde{f}_j \in C(\overline{\mathcal{Q}}_j)$ we define the jump of $f$ across $\Gamma(t)$ by means of

$$\jump{f(t, p)} := \tilde{f}_2(t, p) - \tilde{f}_1(t, p), \quad p \in \Gamma(t). \quad (2.100)$$

Suppose that the functions $f_j$ admit extensions $\tilde{f}_j \in C^1(\overline{\mathcal{Q}}_j)$, $j = 1, 2$. Then the transport theorem for two-phase moving domains states that

$$\frac{d}{dt} \int_{\Omega \setminus \Gamma(t)} f(t, x) \, dx = \int_{\Omega \setminus \Gamma(t)} \partial_t f(t, x) \, dx - \int_{\Gamma(t)} \jump{f(t, x)} V_\Gamma(t, x) \, d\Gamma. \quad (2.101)$$

**Proof.** Let $t \in I$ be fixed. As in the proof of (2.93) we extend the family of diffeomorphisms $\phi(t + s, \cdot) : \Gamma(t) \to \Gamma(t + s)$ given in (2.95) by means of

$$\Phi(t + s, x) = x + \chi(d\Sigma(x)/a)\rho(s, \Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \overline{\Omega},$$

to a family of diffeomorphisms $\Phi(t + s, \cdot) : \Omega \to \Omega$ such that

$$\Phi_j(t + s, \cdot) := \Phi(t + s, \cdot)|_{\Omega_j(t+s)} \in \text{Diff}^1(\Omega_j(t), \Omega_j(t + s)), \quad j = 1, 2, \quad s \in (-\delta, \delta).$$

By choosing $a$ small enough we can assume that a tubular neighbourhood of $\Gamma(t)$ of width $a$ is contained in $\Omega$, and hence that $\Phi(t + s, \cdot) = \text{id}_{\mathbb{R}^n}$ in a neighbourhood of $\partial \Omega$. We can now proceed as in the proof of (2.93) to obtain

$$\frac{d}{dt} \int_{\Omega_j(t)} f(t, x) \, dx = \int_{\Omega_j(t)} \partial_t f(t, x) \, dx - \int_{\Gamma(t)} (-1)^j \jump{\tilde{f}_j(t, x)} V_\Gamma(t, x) \, d\Gamma,$$

and (2.101) then follows from (2.100). \qed
Moving Interfaces and Quasilinear Parabolic Evolution Equations
Prüss, J.; Simonett, G.
2016, XIX, 609 p. 7 illus., Hardcover
ISBN: 978-3-319-27697-7
A product of Birkhäuser Basel