Children of the Cosmos

Presenting a Toy Model of Science with a Supporting Cast of Infinitesimals

Sylvia Wenmackers

> *[A]ll our science, measured against reality, is primitive and childlike – and yet it is the most precious thing we have.*
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> Albert Einstein [1, p. 404]
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> *[…] I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.*
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> Isaac Newton [2, p. 54]

Abstract Mathematics may seem unreasonably effective in the natural sciences, in particular in physics. In this essay, I argue that this judgment can be attributed, at least in part, to selection effects. In support of this central claim, I offer four elements. The first element is that we are creatures that evolved within this Universe, and that our pattern finding abilities are selected by this very environment. The second element is that our mathematics—although not fully constrained by the natural world—is strongly inspired by our perception of it. Related to this, the third element finds fault with the usual assessment of the efficiency of mathematics: our focus on the rare successes leaves us blind to the ubiquitous failures (selection bias). The fourth element is that the act of applying mathematics provides many more degrees of freedom than those internal to mathematics. This final element will be illustrated by the usage of 'infinitesimals' in the context of mathematics and that of physics. In 1960, Wigner wrote an article on this topic [4] and many (but not all) later authors have echoed his assessment that the success of mathematics in physics is a mystery.

The above quote is attributed to Isaac Newton shortly before his death (so in 1727 our shortly before), from an anecdote in turn attributed to [Andrew Michael] Ramsey by J. Spence [2]. See also footnote 31 in [3].

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At the end of this essay, I will revisit Wigner and three earlier replies that harmonize with my own view. I will also explore some of Einstein's ideas that are connected to this. But first, I briefly expose my views of science and mathematics, since these form the canvass of my central claim.

Toy Model of Science

Science can be viewed as a long-lasting and collective attempt at assembling an enormous jigsaw puzzle. The pieces of the puzzle consist of our experiences (in particular those that are intersubjectively verifiable) and our argumentations about them (often in the form of mathematical models and theories). The search for additional pieces is part of the game. Any piece that we add to the puzzle at one time may be removed later on. Nobody knows how many pieces there are, what the shape of the border looks like, or whether the pieces belong to the same puzzle at all. We assume optimistically that this is the case, indeed, and we attempt to connect all the pieces of the puzzle that have been placed on the table so far.[1]

So, like Einstein and (allegedly) Newton in the quotes appearing on the title page, I view science as a playful and limited activity, which is at the same time a highly valuable and unprecedented one. Scientific knowledge is fallible, but there is no better way to obtain knowledge. Hence, it seems wise to base our other epistemic endeavors (such as philosophy) on science—a position known as 'naturalism'. In addition, there is no more secure foundation for scientific knowledge beyond science itself. The epistemic position of 'coherentism' lends support to the positive and optimistic project of science. It has been phrased most evocatively by Otto Neurath [7, p. 206]:

> Like sailors we are, who must rebuild their ship upon the open sea, without ever being able to put it in a dockyard to dismantle it and to reconstruct it from the best materials.[2]

We are in the middle of something and we are not granted the luxury of a fresh start. Hence, we cannot analyze the apparent unreasonable effectiveness of mathematics in science from any better starting point either. Condemned we are to deciphering the issue from the incomplete picture that emerges from the scientific puzzle itself, while its pieces keep moving. A dizzying experience.

Since I mentioned "mathematical models and theories", I should also express my view on those.[3] To me, mathematics is a long-lasting and collective attempt at

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[1] Making these connections involves developing narratives. Ultimately, science is about storytelling. "The anthropologists got it wrong when they named our species *Homo sapiens* ('wise man'). In any case it's an arrogant and bigheaded thing to say, wisdom being one of our least evident features. In reality, we are *Pan narrans*, the storytelling chimpanzee."—Ian Stewart, Jack Cohen, and Terry Pratchett (2002) [6, p. 32].

[2] This is my translation of the German quote [7, p. 206]: "*Wie Schiffer sind wir, die ihr Schiff auf offener See umbauen müssen, ohne es jemals in einem Dock zerlegen und aus besten Bestandteilen neu errichten zu können*."

[3] In the current context, I differentiate little between 'models' and 'theories'. For a more detailed account of scientific models, see [5].

thinking systematically about hypothetical structures—or imaginary puzzles, if you like. (More on this in section "Mathematics as Constrained Imagination" below.)

Selection Effects Behind Perceived Effectiveness of Mathematics in Physics

The four elements brought to the fore in this section collectively support my deflationary conclusion, that the effectiveness of mathematics is neither very surprising nor unreasonable.

A Natural History of Mathematicians

This section addresses the two following questions. What enables us to do mathematics at all? And how is it that we cannot simply describe real-world phenomena with mathematics, but even predict later observations with it? I think that we throw dust in our own eyes if we do not take into account to which high degree we—as a biological species, including our cognitive abilities that allow us to develop mathematics—have been selected by this reality.

To address the matter of whether mathematical success in physics is trick or truth (or something else), and in the spirit of naturalism and coherentism (section "Toy Model of Science"), we need to connect different pieces of the scientific puzzle. In the ancient Greek era, the number of available pieces was substantially smaller than it is now. Plato was amongst the first to postulate parallel worlds: alongside our concrete world, populated by imperfect particulars, he postulated a world of universal Ideas or ideal Forms, amongst which the mathematical Ideas sat on their thrones of abstract existence.[4] In this view, our material world is merely an imperfect shadow of the word of perfect Forms. Our knowledge of mathematics is then attributed to our soul's memories from a happier time, at which it had not yet been incarcerated in a body and its vista had not yet been limited by our unreliable senses.

This grand vision of an abstract world beyond our own has crippled natural philosophy ever since. The time has come to lay this view to rest and to search for better answers, guided by science. Although large parts of the scientific puzzle remain missing in our time, I do think that we are in a better position than the ancient Athenian scholars to descry the contours of an answer to the questions posed at the beginning of this section.

Let us first take stock of what is on the table concerning the origin of our mathematical knowledge. Is mathematical knowledge innate, as Plato's view implied? According to current science, the matter is a bit more subtle: mathematical knowledge

[4] I will have more to say on the ancient Greek view on mathematics and science in section "A Speculative Question Concerning the Unthinkable".

is not innate (unfortunately, since otherwise we would not have such a hard time learning or teaching mathematics), but there are robust findings that very young children (as well as newborns of non-human animals, for that matter) possess numerical abilities [8, 9]. So, we have innate cognitive abilities, that allow us to learn how to count and—with further effort—to study and to develop more abstract forms of mathematics.

This raises the further question as to the origin of these abilities. To answer it, we rely on the coherent picture of science, which tells us this: if our senses and reasoning did not work at all, at least to an approximation sufficient for survival, our ancestors would not have survived long enough to raise offspring and we would not have come into being. Among the traits that have been selected, our ancestors passed on to us certain cognitive abilities (as well as associated vices: more on this below). On this view, we owe our innate numerical abilities to the biological evolution of our species and its predecessors.

Let me give a number of examples to illustrate how our proto-mathematical capacities might have been useful in earlier evolutionary stages of our species. Being able to estimate and to compare the number of fruits hanging from different trees contributes to efficient foraging patterns. So does the recognition of regional and seasonal[5] patterns in the fruition of plants and the migration of animals. And the ability to plan future actions (rather than only being able to react to immediate incentives) requires a crude form of extrapolation of past observations. These traits, which turned out to be advantageous during evolution, lie at the basis of our current power to think abstractly and to act with foresight.

Our current abilities are advanced, yet limited. Let us first assess our extrapolative capacities: we are far from perfect predictors of the future. Sometimes, we fail to take into account factors that are relevant, or we are faced with deterministic, yet intrinsically chaotic systems. Consider, for example, a solar eclipse. An impending occultation is predicted many years ahead. However, whether the weather will be such that we can view the phenomenon from a particular position on the Earth's surface, that is something we cannot predict reliably a week ahead. Let us then turn to the more basic cognitive faculty of recognizing patterns. We are prone to patternicity, which is a bias that makes us see patterns in accidental correlations [10]. This patternicity also explains why we like to play 'connect the dots' while looking at the night sky: our brains are wired to see patterns in the stars, even though the objects we thus group into constellations are typically not in each other's vicinity; the patterns are merely apparent from our earthbound position.

In our evolutionary past, appropriately identifying many patterns yielded a larger advantage than the disadvantage due to false positives. In the case of a tiger, it is clear that one false negative can be lethal. But increasing appropriate positives invariable comes at the cost of increasing false positives as well.[6]

As a species, we must make do without venom or an exoskeleton, alas, but we have higher cognitive abilities that allow us to plan our actions and to devise mathematics.

[5] Or 'spatiotemporal', if you like to talk like a physicist.

[6] The same trade-off occurs, for instance, in medical testing and law cases.

These are our key traits for survival (although past success does not guarantee our future-proofness). In sum, mathematics is a form of human reasoning—the most sophisticated of its kind. When this reasoning is combined with empirical facts, we should not be perplexed that—on occasions—this allows us to effectively describe and even predict features of the natural world. The fact that our reasoning can be applied successfully to this aim is precisely why the traits that enable us to achieve this were selected in our biological evolution.

Mathematics as Constrained Imagination

In my view, mathematics is about exploring hypothetical structures; some call it the science of patterns. Where do these structures or patterns come from? Well, they may be direct abstractions of objects or processes in reality, but they may also be inspired by reality in a more indirect fashion. For instance, we could start from an abstraction of an actual object or process, only to negate one or more of its properties—just think of mathematics' ongoing obsession with the infinite (literally the non-finite). Examples involving such an explicit negation clearly demonstrate that the goal of mathematics is not representation of the real world or advancing natural science. Nevertheless, this playful and free exercise in pure mathematics may—initially unintended and finally unexpected—turn out to be applicable to abstractions of objects and processes in reality (completely different from the one we started from). Stated in this way, the effectiveness of mathematics surely seems unreasonable. However, I argue that there are additional factors at play that can explain this success—making these unintentional applications of mathematics more likely after all.

Let us return to the toy metaphor, assuming, for definiteness, that the puzzle of natural science appears to be a planar one. Of course, this is no reason for mathematicians not to think up higher dimensional puzzles, since their activity is merely imaginative play, unhindered by any of the empirical jigsaw pieces. However, it is plausible that the initial inspiration for considering, say, toroidal or hypercubic puzzles has been prompted by difficulties with fitting the empirical pieces into a planar configuration.[7] In addition, and irrespective of its source, this merely mathematical construct may subsequently prompt speculations about the status of the scientific puzzle. Due to feedback processes like these, the imaginative play is not as unconstrained as we might have assumed at the outset. The hypothetical structures of mathematics are not concocted in a physical or conceptual vacuum. Even in pure mathematics, this physical selection bias acts very closely to the source of innovation and creativity.

In the previous section, I highlighted that humans, including mathematicians, have evolved in this Universe. Mathematics itself also evolves by considering variations on earlier ideas and selection: this is a form of cultural evolution which allows changes on a much shorter time scale than biological evolution does. Just like in biology, this

[7]In this example, considering the negation of the planar assumption—rather than any of the other background assumptions—is prompted by troubles in physics.

variation produces many unviable results. Evolution is squandermanious—quite the opposite of efficient. The selection process is mainly driven by cultural factors, which are internal to mathematics (favoring theories that exhibit epistemic virtues such as beauty and simplicity). But, as we saw in the previous paragraph, empirical factors come into play as well, mediated by external interactions with science. Although mathematics is often described as an a priori activity, unstained by any empirical input, this description itself involves an idealization. In reality, there is no a priori.

Mathematics Fails Science More Often Than Not

For each abstraction, many variations are possible, the majority of which are not applicable to our world in any way. The effectiveness perceived by Wigner [4] may be due to yet another form of selection bias: one that makes us prone to focus on the winners, not the bad shots. Moreover, even scientific applications of mathematics that are widely considered to be highly successful have a limited range of applicability and even within that range they have a limited accuracy.

Among the mathematics books in university libraries, many are filled with theories for which not a single real world application has been found.[8] We could measure the efficiency of mathematics for the natural science as follows: divide the number of pages that contain scientifically applicable results by the total number of pages produced in pure mathematics. My conjecture is that, on this definition, the efficiency is very low. In the previous section we saw that research, even in pure mathematics, is biased towards the themes of the natural sciences. If we take this into account, the effectiveness of mathematics in the natural sciences does come out as unreasonable—unreasonably low, that is.[9]

Maybe it was unfair to focus on pure mathematics in the proposed definition for efficiency? A large part of the current mathematical corpus deals with applied mathematics, from differential equations to bio-statistics. If we measure the efficiency by dividing the number of 'applicable pages' by the total number of pages produced in all branches of mathematics, we certainly get a much higher percentage. But, now, the effectiveness of mathematics in the natural sciences appears reasonable enough, since research and publications in applied mathematics are (rightfully) biased towards real world applicability.

At this point, you may object that Wigner made a categorical point that there is some part of mathematics at all that works well, even if this does not constitute all or most of mathematics. I am sympathetic to this objection (and the current point is the least important one in my argument), but then what is the contrasting case: that

[8]This is fine, of course, since this is not the goal of mathematics.

[9]Here, I recommend humming a Shania Twain song: "So, you're a rocket scientist. That don't impress me much." If you are too young to know this song, consult your inner teenager for the appropriate dose of underwhelmedness.

no mathematics would describe anything in the Universe? I offer some speculations about this in section "A Speculative Question Concerning the Unthinkable".

Abundant Degrees of Freedom in Applying Mathematics: The Case of Infinitesimals

I once attended a lecture in which the speaker claimed that "There is a matter of fact about how many people are in this room". Unbeknownst to anyone else in that room, I was pregnant at the time, and I was unsure whether an unborn child should be included in the number of people or not. To me, examples like this show that we can apply mathematically crisp concepts (such as the counting numbers) to the world, but only because other concepts (like person or atom) are sufficiently vague.

The natural sciences aim to formulate their theories in a mathematically precise way, so it seems fitting to call them the 'exact sciences'. However, the natural sciences also allow—and often require—deviations from full mathematical rigor. Many practices that are acceptable to physicists—such as order of magnitude calculations, estimations of errors, and loose talk involving infinitesimals—are frowned upon by mathematicians. Moreover, all our empirical methods have a limited range and sensitivity, so all experiments give rise to measurement errors. Viewed as such, one may deny that any empirical science can be fully exact. In particular, systematic discrepancies between our models and the actual world can remain hidden for a long time, provided that the effects are sufficiently small, compared to our current background theories and empirical techniques.

Einstein put it like this: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" [11, p. 28]. To illustrate this point, I will concentrate on the calculus—the mathematics of differential and integral equations—and consider the role of infinitesimals in mathematics as well as in physics.

In mathematics, infinitesimals played an important role during the development of the calculus, especially in the work of Leibniz [12], but also in that of Newton (where they figure as 'evanescent increments') [13]. The development of the infinitesimal calculus was motivated by physics: geometric problems in the context of optics, as well as dynamical problems involving rates of change. Berkeley [14] ridiculed these analysts as employing "ghosts of departed quantities". It has taken a long time to find a consistent definition of infinitesimals that holds up to the current standards of mathematical rigour, but meanwhile this has been achieved [15]. The contemporary definition of infinitesimals considers them in the context of an incomplete, ordered field of 'hyperreal' numbers, which is non-Archimedean: unlike the field of real numbers, it does contain non-zero, yet infinitely small numbers (infinitesimals).[10] The alternative calculus based on hyperreal numbers, called 'non-standard analysis'

[10] Here, I mean by infinitesimals numbers larger than zero, yet smaller than $1/n$ for any natural number n.

(NSA), is conceptually closer to Leibniz's original work (as compared to standard analysis).

While infinitesimals have long been banned from mathematics, they remained in fashion within the sciences, in particular in physics: not only in informal discourse, but also in didactics, explanations, and qualitative reasoning. It has been suggested that NSA can provide a post hoc justification for how infinitesimals are used in physics [16]. Indeed, NSA seems a very appealing framework for theoretical physics: it respects how physicists are already thinking of derivatives, differential equations, series expansions, and the like, and it is fully rigorous.[11]

Rephrasing old results in the language of NSA may yield new insights. For instance, NSA can be employed to make sense of classical limits in physics: classical mechanics can be modelled as quantum mechanics with an infinitesimal Planck constant [22]. Likewise, Newtonian mechanics can be modelled as a relativity theory with an infinite maximal speed, c (or infinitesimal $1/c$).

Infinitesimal numbers are indistinguishable from zero (within the real numbers), yet distinct from zero (as can be made explicit in the hyperreal numbers). This is suggestive of a physical interpretation of infinitesimals as 'currently unobservable quantities'. The ontological status of unobservables is an important issue in the realism–anti-realism debate [23]. Whereas constructive empiricists interpret 'observability' as 'detectability by the human, unaided senses' [24], realists regard 'observability' as a vague, context-dependent notion [25]. When an apparatus with better resolving power is developed, some quantities that used to be unobservably small become observable [26, 27]. This shift in the observable-unobservable distinction can be modelled by a form of NSA, called relative analysis, as a move to a finer context level [28]. Doing so requires the existing static theory to be extended by new principles that constrain the allowable dynamics [29].

The interpretation of (relative) infinitesimals as (currently) unobservable quantities is suggestive of why the calculus is so applicable to the natural sciences: it appears that infinitesimals provide scientists with the flexibility they need to fit mathematical theories to the empirically accessible world. To return to the jigsaw puzzle analogy of section "Toy Model of Science": we need some tolerance at the edges of the pieces. If the fit is too tight, it becomes impossible to connect them at all.

A Speculative Question Concerning the Unthinkable

Could our cosmos have been different—so different that a mathematical description of it would have been fundamentally impossible (irrespective of whether life could

[11] It has been shown that physical problems can be rephrased in terms of NSA [17], both in the context of classical physics (Lagrangian mechanics [18]) and of quantum mechanics (quantum field theory [19], spin models [20], relativistic quantum mechanics [21], and scattering [18]). Apart from formal aspects (mathematical rigour), such a translation also offers more substantial advantages, such as easier (shorter) proofs.

emerge in it)? Some readers may have the impression that I have merely explored issues in the vicinity of this mystery, without addressing it directly.

Before I indulge in this speculation, it may be worthwhile to remember that the very notion of a 'cosmos' emerged in ancient Greek philosophy, with the school of Pythagoras, where it referred to the order of the Universe (not the Universe itself). It is closely related to the search for *archai* or fundamental ordering principles. It is well known that the Pythagorians took the whole numbers and—by extension—mathematics as the ordering principle of the Universe. Their speculations about a mathematically harmonious music of the spheres resonated with Plato and Johannes Kepler (the great astronomer, but also the last great neoplatonist). Since these *archai* had to be understandable to humans, without divine intervention or mystical revelation, they had to be limited in number and sufficiently simple. So, the idea that the laws of nature have to be such that they can be printed on the front of a T-shirt, goes back to long before the invention of the T-shirt.[12] In this sense, the answer to the speculative question at the start of this section is 'no' and trivially so, for otherwise it would not be a cosmos. Yet, even if we understand 'our cosmos' as 'the Universe', there is a strong cultural bias to answer the speculative question in the negative.

In section "A Natural History of Mathematicians", I considered our proto-mathematical abilities as well as their limits. At least in some areas, our predictions do better than mere guesses. This strongly suggests that there are patterns in the world itself—maybe not the patterns that we ascribe to it, since these may fail, but patterns all the same. It is then often taken to be self-evident that these patterns must be mathematical, but to me this is a substantial additional assumption. On my view of mathematics, the further step amounts to claiming that nature itself is—at least in principle—understandable by humans. I think that all we understand about nature are our mathematical representations of it.[13] Ultimately, reality is not something to be understood, merely to be. (And, for us, to be part of.)

When we try to imagine a world that would defy our mathematical prowess, it is tempting to think of a world that is totally random. However, this attempt is futile. Pure randomness is a human idealization of maximally unpredictable outcomes (like a perfectly fair lottery [30]). Yet, random processes are very well-behaved: they consist of events that may be maximally unpredictable in isolation, but collectively they produce strong regularities. It is no longer a mystery to us how order emerges from chaos. In fact, we have entire fields of mathematics for that, called probability theory and statistics, which are closely related to branches of physics, such as statistical mechanics.

As a second attempt, we could propose a Daliesque world, in which elements combine in unprecedented ways and the logic seems to change midgame: rigid clocks become fluid, elephants get stilts, and tigers emerge from the mouths of fish shooting

[12] In case this remark made you wonder: the T-shirt was invented about a century ago.

[13] My view of mathematics might raise the question: "Why, then, should we expect that anything as human and abstract as mathematics applies to concrete reality?" I think this question is based on a false assumption, due to prolonged exposure to Platonism—remnants of which are abundant in our culture.

from a pomegranate. Yet, even such surrealistic tableaus have meta-regularities of their own. Many people are able to recognize a Dalí painting instantly as his work, even if they have not seen this particular painting before. Since we started from human works of art, unsurprisingly, the strategy fails to outpace our own constrained imagination.

At best, I can imagine a world in which processes cannot be summarized or approximated in a meaningful way. Our form of intelligence is aimed at finding the gist in information streams, so it would not help us in this world (in which it would not arise spontaneously by biological evolution either). In any case, what I can imagine about such a world remains very vague—insufficient for any mathematical description. Maybe there are better proposals out there?

Max Tegmark has put forward an evocative picture of the ultimate multiverse as consisting of all the orderings that are mathematically possible [31]. (See also Marc Séguin's contribution [32].) Surely, this constitutes a luscious multiplicity. From my view of mathematics as constrained imagination, however, the idea of a mathematical multiverse is still restricted by what is thinkable by us, humans. Aristotle described us as thinking animals, but for the current purpose 'mathematizing mammals' may fit even better. My diagnosis of the situation is that the speculative question asks us to boldly go even beyond Tegmark's multiverse and thus to exceed the limits of our cognitive kung fu: even with mathematics, we cannot think the unthinkable.

Reflections on Wigner and Einstein

In this section, I return to Wigner [4] and compare my own reflections to earlier replies by Hamming [33], Grattan-Guinness [34], and Abbott [35]. I end with a short reflection on some ideas of Einstein [11] that predate Wigner's article by four decades.

Wigner's Two Miracles

Wigner wrote about "two miracles": "the existence of the laws of nature" and "the human mind's capacity to divine them" [4, p. 7]. First and foremost, I hope that my essay helps to see that we need not presuppose the former to understand the latter: it is by assuming that the Universe forms a cosmos that we have started reading laws into it. Galileo Galilei later told us that those laws are mathematical ones. The very term "laws of nature" may be misleading and for this reason, I avoided it so far (except for the remark of fitting them on a T-shirt). The fact that our so-called laws can be expressed with the help of mathematics should be telling, since that is *our* science of patterns. When we open Galileo's proverbial book of nature, we find it filled with our own handwriting.

To illustrate the unreasonable effectiveness of mathematics, Wigner offered the following analogy for it: consider a man with many keys in front of many doors, who "always hits on the right key on the first or second try" [4, p. 2]. Lucky streaks like this may seem to require further explanation. However, if there are many people, each with many keys, it becomes likely that at least one of them will have an experience like Wigner's man—and no further explanation is needed (see also Hand [36]). There are indeed many people active in mathematics and science, and few of them succeed "on the first or second try"—or at all. In the essay, I argued that the perceived effectiveness of mathematics in physics can be diagnosed in terms of selection bias. The same applies to the metaphor Wigner presented.

Wigner found it hard to believe that the perfection of our reasoning power was brought about by Darwin's process of natural selection [4, p. 3]. Ironically, the selection bias that he may have fallen prey to is a good illustration of the *lack* of perfection of our reasoning powers. To be clear, I do not claim that Darwin's theory of biological evolution suffices to explain the success of (certain parts of) mathematics in physics. However, a similar combination of variation and selection is at work in the evolution of mathematics and science. See also Pólya, as cited by Grattan-Guinness, who has given an iterative or evolutionary description of the development of science [37, Vol. 2, p. 158].

Previous Replies to Wigner

In his description of mathematics, Wigner wrote about "defining concepts beyond those contained in the axioms" [4, p. 3]. Wigner did not, however, reflect on where those axioms come from in the first place. This has been criticized by Hamming [33], a mathematician who worked on the Manhattan Project and for Bell Labs. Axioms or postulates are not specified upfront; instead, mathematicians may try various postulates until theorems follow that harmonize with their initial vague ideas. Hamming cited the Pythagorean theorem and Cauchy's theorem as examples: if mathematicians would have started out with a system in which those crucial results would not hold, then—according to Hamming—they would have changed their postulates until they did. And, of course, the initial vague ideas are thoughts produced by beings entrenched in the physical world. This brings us to Putnam, who pointed out that mathematical knowledge resembles empirical knowledge in many respects: "the criterion of success in mathematics is the success of its ideas in practice" [38, p. 529].

Wigner did concede that not any mathematical concept will do for the formulation of laws of nature in physics [4, p. 7], but he claimed that "in many if not most cases" these concepts were developed "*independently* by the physicist and *recognized* then as having been conceived before by the mathematician" [4, p. 7] (my emphasis). I think this part is misleading: there is a lot of interaction between mathematics and physics and what is being 'recognized' is actually the finding of a new analogy. We may think that we are merely discovering a similarity, when we are really forging new connections, which may subtly alter both sides. This is a creative element of great importance within mathematics as well as in finding applications to other fields, in which our patternicity may be a virtue rather than a vice.

Both aspects have been illuminated by Grattan-Guinness, a historian of mathematics, who argued that "much mathematics has been motivated by interpretations in the sciences" [34, p. 7]. He stressed the importance of analogies within mathematics and between mathematics and natural science and he gave historical examples in which mathematics and physics take turns in reshaping earlier concepts. Moreover, he remarked that there are many analogies that can be tried (somewhat similar to the ideas in section "Abundant Degrees of Freedom in Applying Mathematics: The Case of Infinitesimals"), but that only the successful ones are taken into account when assessing the effectiveness of mathematics (postselection as in section "Mathematics as Constrained Imagination").

My essay mainly focused on elementary mathematics and simple models. Of course, there are very complicated mathematical theories in use in advanced physics. In relation to this, Grattan-Guinness observed that "by around 1900 linearisation had become something of a fixation" [34, p. 11], but he also discussed the subsequent "desimplification" or "putting back in the theory effects and factors that had been deliberately left out" [34, p. 13].

In light of my discussion of infinitesimals in this essay, it is curious to observe that Grattan-Guinness and Hamming referred to them too. Grattan-Guinness spoke approvingly of the Leibniz-Euler approach to the calculus because it "often has a better analogy content to the scientific context" [34, p. 15]. Hamming even mentioned NSA, but only as an example of the observation that "logicians can make postulates that put still further entities on the real line" [33, p. 85].

Recently, the reply by Hamming has been developed further by Abbott, a professor in electrical engineering [35]. Whereas Hamming described his recurrent feeling that "God made the universe out of complex numbers" [33, p. 85], Abbott described the complex numbers as "simply a convenience for describing rotations", concluding that "the ubiquity of complex numbers is not magical at all" [35, p. 2148].

More specifically, Abbott adds two points to Hamming's earlier observations. Abbott's first addition is that "all physical laws and mathematical expressions of those laws are […] necessarily compressed due to the limitations of the human mind" [35, p. 2150]. He explains that the associated loss of information does not preclude usefulness "provided the effects we have neglected are small", which lends itself perfectly to a rephrasing in terms of 'my' relative infinitesimals. Abbott's second addition is that "the class of successful mathematical models is preselected", which he described as a "Darwinian selection process" [35, p. 2150]. Like I did here, Abbott warned his readers not to overstate the effectiveness of mathematics. Moreover, as an engineer, he is well aware that "when analytical methods become too complex, we simply resort to empirical models and simulations" [35, p. 2148].

The title of Abbot's piece, "The reasonable ineffectiveness of mathematics", and the general anti-Platonist stance agree with the views exposed in the current essay. In addition, Abbott tried to show that this debate is relevant, even for those who prefer to "shut up and calculate", because "there is greater freedom of thought, once we realize that mathematics is something we entirely invent as we go along" [35, p. 2152].

Einstein's Philosophy of Science

In 1921, so almost forty years before Wigner wrote his article, Einstein gave an address to the Prussian Academy of Sciences in Berlin titled "Geometrie und Erfahrung". The expanded and translated version contains the following passage (which includes the sentence already quoted in section "Abundant Degrees of Freedom in Applying Mathematics: The Case of Infinitesimals"):

> At this point an enigma presents itself which in all ages has agitated inquiring minds. How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things. In my opinion the answer to this question is, briefly, this: As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. [11, p. 28]

On the one hand, we read of "an enigma", which seems to set the stage for Wigner's later question. On the other hand, we are not dealing with a Platonist conception of mathematics here, since Einstein describes it as "a product of human thought". Yet, how are we to understand Einstein's addition that mathematics is "independent of experience"? This becomes clearer in the remainder of his text: geometry stems from empirical "earth-measuring", but modern axiomatic geometry, which allows us to consider multiple axiomatisations (including non-Euclidean ones), remains silent on whether any of these axiom schemes applies to reality [11, pp. 28–32]. Einstein refers approvingly to Schlick's view that axioms in the modern sense function as implicit definitions, which is in agreement with Hilbert's formalist position.[14]

Regarding his own answer to the question, Einstein further explains that axiomatic geometry can be supplemented with a proposition to relate mathematical concepts to objects of experience: "Geometry thus completed is evidently a natural science" [11, p. 32]. My essay agrees with Einstein's answer: the application of mathematics supplies additional degrees of freedom external to mathematics and we can never be sure that the match is perfect, since empirical precision is always limited. On other occasions, Einstein also pointed out that Kant's a priori would better be understood as 'conventional': a position close to that of Pierre Duhem (see for instance [39]).

In my late teens, I read a Dutch translation of essays by Einstein. The oldest essay in that collection stemmed from 1936 [40], so the text from which I quoted does not appear in that book. However, it is clear that similar ideas have influenced me in my formative years. It was a pleasure to reexamine some of them here. They contributed to my decision to become a physicist and a philosopher of physics, which in turn helped me to write this essay. Hence, the appearance of a text in this collection that is in agreement with some of Einstein's view on science may be a selection effect as well.

[14] For the influence of Moritz Schlick on Einstein's ideas, see [39].

Conclusion

In this essay, I have argued that:

- we are selected (A Natural History of Mathematicians);
- our mathematics is selected (Mathematics as Constrained Imagination);
- the application of mathematics has degrees of freedom beyond those internal to mathematics (Abundant Degrees of Freedom in Applying Mathematics: The Case of Infinitesimals);
- and, still, effective applications of mathematics remain the exception rather than the rule (Mathematics Fails Science More Often Than Not).

Too often, physicists have linearly approximated phenomena and considered themselves gods, only to discover much richer overtones of these phenomena later on. Let this be a lesson in modesty. While we are playing, things may appear to be very simple, but we should beware that (paraphrasing J.L. Austin[15]): it is not nature, it is scientists that are simple.

In the same spirit, I do not claim that my essay answers all questions concerning the relationship between mathematics and physics. Large gaps do remain between the pieces of this puzzle, but I found pleasure in arranging them in a way that suggests that further connections can be made. Feel free to pick up the pieces where I left them. Consider this essay your invitation to start playing.

Disclaimer
No parallel universes were postulated during the writing of this essay.

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