Chapter 2
Function Spaces

2.1 \( L^p, C^\alpha, \text{BMO}, L^{p,\lambda}, L^{p,\lambda}_w \)

Listed here are several classical function spaces defined with some of their properties useful in the sequel. Generally, \( \Omega \) will be a bounded domain in \( \mathbb{R}^n \), and when it matters, with smooth boundary \( \partial \Omega \) or at least a boundary of type A (as noted in the Introduction). \( L^p \) denotes the usual Lebesgue space of \( p \)th power integrable functions on \( \mathbb{R}^n \) or respectively on \( \Omega, L^p(\Omega) \): \( 1 \leq p < \infty \), and \( ||f||_{L^p} \) or \( ||f||_{L^p(\Omega)} \).

On the other hand, weak - \( L^p(L^p_w) \) consists of those \( f \) for which

\[
\sup_{t>0} \left[ t^p \mathcal{L}_n (\{ x \in \mathbb{R}^n : |f(x)| > t \}) \right]^{1/p} \equiv ||f||_{L^p_w} < \infty,
\]

and the corresponding \( L^p_w(\Omega).L^\infty \) is the essentially bounded functions on \( \mathbb{R}^n, L^\infty(\Omega) \) those on \( \Omega \). Whereas, \( C^\alpha \) is the usual space of Hölder continuous functions on \( \mathbb{R}^n \) with exponent \( \alpha \in (0, 1) \) and normed by

\[
||f||_{C^\alpha} = ||f||_{L^\infty} + [f]_\alpha < \infty,
\]

where,

\[
[f]_\alpha = \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}},
\]

and the corresponding \( C^\alpha(\Omega) \). We shall also have some need of the John-Nirenberg space of functions of bounded mean oscillation on a fixed cube \( Q_0 \subset \mathbb{R}^n \) or on \( \mathbb{R}^n \) itself, i.e., \( f \in \text{BMO}(Q_0) \) if

\[
\text{BMO}(Q_0) = \{ f \in L^1(Q_0) : \text{osc}_Q f < \infty \}
\]

where,

\[
\text{osc}_Q f = \sup_{x \in Q} f(x) - \inf_{y \in Q} f(y).
\]
\[ \sup_{Q \subset Q_0} \int_Q |f(x) - f_Q| dx = [f]_{*,Q_0} < \infty, \]

where \( Q \) is a cube in \( \mathbb{R}^n \) with sides parallel to the coordinate axes; \( f_Q \) is the integral average of \( f \) over \( Q \). For BMO on \( \mathbb{R}^n \), just replace \( Q_0 \) by \( \mathbb{R}^n \). And in particular, we have the celebrated J-N Lemma:

**Lemma 2.1.** If \( [f]_{*,Q_0} < \infty \), then there exist two constants \( c_1 \) and \( c_2 \) such that

\[ \mathcal{L}_n(\{x \in Q : |f(x) - f_Q| > \lambda\}) \leq c_1 e^{-c_2 \lambda/[f]_{*,Q_0} \cdot |Q|} \]

for all cubes \( Q \subset Q_0 \) and any \( \lambda > 0 \). Here \( |Q| = \mathcal{L}_n(Q) \).

(For this, see either [St2] or [To]). It thus follows that \( L^{\infty}(Q_0) \subset \text{BMO}(Q_0) \subset L^p(Q_0) \) for all \( p \geq 1 \).

### 2.2 The Campanato scale \( \mathcal{L}^{p,\lambda}, \mathcal{L}^{p,\lambda}(Q_0) \)

For \(-p < \lambda \leq n\), set

\[ \mathcal{L}^{p,\lambda}(Q_0) = \{f : ||f||_{L^p(Q_0)} + [f]_{p,\lambda;Q_0} < \infty\} \]

where

\[ [f]_{p,\lambda;Q_0} = \sup_{Q \subset Q_0} \left( |Q|^{-\lambda/n} \int_Q |f - f_Q|^p \, dx \right)^{1/p}. \]

As mentioned, this scale includes many of the classical function spaces of Harmonic Analysis, notably \( L^p(Q_0), L^{p,\lambda}(Q_0), \text{BMO}(Q_0), \) and \( C^\alpha(Q_0) \) - as well as versions over \( \mathbb{R}^n \). We will not give a complete proof of (1.6) even just over \( Q_0 \), but refer the reader to [Tr] for missing details. But because these notes are about Morrey Spaces, it seems appropriate to at least prove:

\[ \mathcal{L}^{p,\lambda}_0(Q_0) = L^{p,\lambda}(Q_0) \]  \hspace{1cm} (2.1)

with equivalence of norms: \( 0 < \lambda \leq n, \ 1 < p < \infty \). In fact this is a consequence of the following iteration lemma:

**Lemma 2.2.** Let \( \varphi(r) \) be a non-negative function on \((0, R]\) and suppose there are numbers \( \beta, \gamma > 0 \) and \( K > 1 \) such that

\[ \varphi(\rho) \leq K \left( \frac{\rho}{r} \right)^{\beta} \varphi(r) + K\rho^\gamma \]
whenever \( \frac{r}{s} \leq \rho < r \) and \( r \leq R \), for some \( s > 1 \) and \( R < \infty \). Then for any \( 0 < \epsilon < \beta - r \),

\[
\varphi(\rho) \leq K \left( \frac{\rho}{r} \right)^{\beta - \epsilon} \varphi(r) + KCp^\gamma
\]

(2.2)

for some constant \( C \) depending only on \( K, \beta, \gamma \). In particular, (2.2) holds for all \( 0 < \rho < r = R \). This Lemma is a special case of Lemma 1.18 of [Tr].

Applying this lemma to the estimate below gives the inclusion \( L^{p,\lambda}(Q_0) \subseteq L^{p,\lambda}(Q_0) \). The reverse is obvious. So let \( Q = Q_\rho = \) a cube with edge length \( \rho \), then

\[
\varphi(\rho) = \int_{Q_\rho} |u|^p \leq 2^p \int_{Q} |u - u_Q|^p + 2^p \int_{Q} |u_Q|^p
\]

\[
\leq 2^p \rho^{n-\lambda} [u]_{p,\lambda; Q_0} + 2^p \frac{\rho^n}{r^n} \int_{Q_r} |u|^p.
\]

Then with Lemma 2.2 and \( \gamma = n - \lambda < n - \epsilon \) for some \( \epsilon > 0 \), and \( \beta = n \), it follows that

\[
\varphi(\rho) \leq C \rho^{n-\lambda} \left( [u]_{p,\lambda; Q_0} + ||u||_{L^p(Q_0)}^p \right)
\]

for all \( 0 < \rho < R \).

For the case \( \lambda = 0 \), applying the J-N Lemma gives \( 1 < p < \infty \)

\[
\text{BMO}(Q_0) = \left\{ f : \sup_{Q \subset Q_0} \left( \int_{Q} |f - f_Q|^p \, dx \right)^{1/p} < \infty \right\}
\]

for any \( Q_0 \subseteq \mathbb{R}^n \).

Finally, when \( -p < \lambda < 0 \), we remark that

\[
\mathcal{L}^{p,\lambda}(Q_0) \equiv C^\alpha(Q_0), \alpha \equiv -\lambda/p > 0,
\]

(2.3)

was the consequence of independent investigations by S. Campanato [Ca] and N.G. Meyers [M2]. See [To], VIII.5. The proof in this case is a bit more delicate than that given above for \( 0 < \lambda \leq n \), but is repeated in several sources, namely in both [Tr] and [G].
2.3 Sobolev Spaces $W^{m,p}(\Omega), G_\alpha(L^p), I_\alpha(L^p)$

The space $W^{m,p}(\Omega)$ consists of those weakly differentiable functions $u(x)$ on $\Omega$ for which

$$\int_\Omega |D^m u|^p \, dx + \int_\Omega |u|^p \, dx \equiv \|u\|_{W^{m,p}(\Omega)}^p < \infty,$$

where $m = \text{positive integer}$, $D^m u$ is the set of all $m^{th}$ order derivatives of $u$ on $\Omega$. But for us, it will be essential to have a potential theoretic version of $W^{m,p}(\Omega)$. Here we refer to [AH] with $G_\alpha(x)$ the Bessel potential operator, $\alpha > 0$; $G_\alpha(x)$ is the Fourier transform of $G_\alpha = (1 + |\xi|^2)^{-\alpha/2}, \xi \in \mathbb{R}^n$. So when $\alpha = m$, a positive integer, then $W^{m,p}(\mathbb{R}^n) = G_\alpha(L^p(\mathbb{R}^n))$, and with $u(x) = G_m f$, $\|u\|_{W^{m,p}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}$, with equivalent norms; $1 < p < \infty$.

Now if $I_\alpha(x) = |x|^{\alpha-n}, 0 < \alpha < n$, and $f \in L^p(\mathbb{R}^n)$, $I_\alpha f(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha-n} f(y) \, dy$, the Riesz potential of $f$ of order $\alpha$ (which is finite a.e. when $\alpha p < n$), then

$$\|f\|_{L^p(\mathbb{R}^n)} + \|I_\alpha f\|_{L^p(\mathbb{R}^n)}$$

is an equivalent Sobolev norm, $\alpha = m < n, 1 < p < n/\alpha$. This is the result of the fact that the Calderon-Zymund singular integrals are bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$. See [St2] and [To].

2.4 Morrey-Sobolev Spaces $I_\alpha(L^{p,\lambda})$

From the above, we shall denote by $I_\alpha(L^{p,\lambda}(\mathbb{R}^n))$ a Morrey-Sobolev space, $0 < \lambda < n, 1 < p < n/\alpha, 0 < \alpha < n$. And often we will write $I_\alpha f$ when $f$ has compact support, and then from the results latter of Chapter 8, we set

$$\|u\|_{W^{m,p,\lambda}} = \|u\|_{L^{p,\lambda}(\mathbb{R}^n)} + \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

when $u(x) = I_m f$; i.e., $u$ and its $m$-th order derivatives belong to the Morrey Space $L^{p,\lambda}(\mathbb{R}^n)$.

2.5 Dense/non-dense subspaces, Zorko Spaces, $VL^{p,\lambda}$, VMO

It is well known that class $C_0^\infty(\Omega) = C^\infty$ functions on $\Omega$ with compact support in $\Omega$ is dense in $W^{m,p}$ and in $G_\alpha(L^p)$ when $\Omega = \mathbb{R}^n$. What we seek here is the density subspaces for $L^{p,\lambda}(\Omega)$. For this we turn to Zorko [Z].
Our first observation is: there are $f \in L^p$ that cannot be approximated even by continuous functions in the norm $\| \cdot \|_{L^p} + \| \cdot \|_{L^p}$. In fact Zorko shows that $f_{x_0}(x) = |x - x_0|^{-\lambda/p}$, $x_0 \in \Omega$, is one, for every $x_0$. In fact

$$\int_{|x-x_0|<\rho} |f_{x_0}(x) - g(x)|^p dx \geq 2^{-p} \int_{|x-x_0|<\rho} |f_{x_0}(x)|^p dx - \int_{|x-x_0|<\rho} |g(x)|^p dx$$

$$\geq \frac{2^{-p} W_{n-1}}{n - \lambda} \rho^{n-\lambda} - \|g\|_{L^{\infty}(B(x_0, \rho))} \cdot \frac{W_{n-1}}{n} \rho^n$$

$$= w_{n-1} \rho^{n-\lambda} \left( \frac{2^p}{n - \lambda} - \|g\|_{L^{\infty}(B(x_0, \rho))} \rho^\lambda \right).$$

So for $\rho \leq \rho_0 = $ sufficiently small,

$$\rho^{\lambda-n} \int_{|x-x_0|<\rho} |f_{x_0}(x) - g(x)|^p dx \geq c_0 > 0.$$

This motivates us to set

$$VL^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^{p,\lambda} : \|f(\cdot - y) - f(\cdot)\|_{L^{p,\lambda}} \to 0, \ |y| \to 0 \right\}.$$ 

Here the $V$ stands for “vanishing” honoring Sarason’s $VMO = \text{Vanishing mean oscillation function space}$; see [To]. We also at times will refer to this space as the Zorko subspace of $L^{p,\lambda}$, the Morrey Space on $\mathbb{R}^n$. We now note

**Theorem 2.3.** If $f \in VL^{p,\lambda}$, then $f$ can be approximated by $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\| \cdot \|_{L_p^{p,\lambda}} = \| \cdot \|_{L_p(\mathbb{R}^n)} + \| \cdot \|_{L_p^{p,\lambda}(\mathbb{R}^n)}.$$

**Proof.** Let $\varphi \in C_0^\infty(B(0,1)^+)$ with $\int \varphi(x) dx = 1$, then upon setting $\varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon)$, $\epsilon > 0$, and $\varphi_\epsilon * f$ the usual convolution, we get

$$\left( \int_{B(x_0, \rho)} |\varphi_\epsilon * f - f|^p dx \right)^{1/p} \leq \int \varphi_\epsilon(y) \left\{ \int_{B(x_0, \rho)} |f(x-y) - f(x)|^p dx \right\}^{1/p} dy$$

hence

$$\left( \rho^{\lambda-n} \int_{B(x_0, \rho)} |\varphi_\epsilon * f - f|^p dx \right)^{1/p} \leq \int \varphi_\epsilon(y) \|f(\cdot - y) - f(\cdot)\|_{L^{p,\lambda}} dy$$

or

$$\|\varphi_\epsilon * f - f\|_{L^{p,\lambda}} \leq \sup_{|y|<\epsilon} \|f(\cdot - y) - f(\cdot)\|_{L^{p,\lambda}}.$$
This may seem a bit disconcerting, but as it turns out, all is not lost, for we have:

**Theorem 2.4.** If \( f \in L^{p,\lambda} \), then \( f \in V^{p,\mu} \) for all \( \mu > \lambda, \ 0 < \lambda < n \).

**Proof.** Again by mollifying \( f \) as above, we can write

\[
 r^{\mu-n} \int_{B(x_0,r)} |f - f \ast \varphi_\epsilon|^p \leq \left( r^{\lambda-n} \int_{B(x_0,r)} |f - f \ast \varphi_\epsilon|^p \, dx \right)^{1/q} \left( \int_{B(x_0,r)} |f - f \ast \varphi_\epsilon|^p \, dx \right)^{1/q'}
\]

where \( \lambda < \mu < n, \ q = (n - \lambda)/(n - \mu) \), and \( q' = q/(q - 1) \). The first factor is bounded by \( C \|f\|_{L^{p,\lambda}} \) and the second factor tends to zero as \( \epsilon \to 0 \) due to the density of smooth functions in the \( L^p \) spaces. \( \square \)

### 2.6 Note

1. Extensions of the idea of a Morrey Space and the study of various operators (classical or not) on these spaces are numerous. We collect a few of these that have caught our attention in the Bibliography under “Generalized Morrey Spaces and some applications.”
Morrey Spaces
Adams, D.R.
2015, XVII, 124 p. 1 illus. in color., Softcover
ISBN: 978-3-319-26679-4
A product of Birkhäuser Basel