2.1 \( L^p, C^\alpha, \text{BMO}, L^{p,\lambda}, L^{p,\lambda}_w \)

Listed here are several classical function spaces defined with some of their properties useful in the sequel. Generally, \( \Omega \) will be a bounded domain in \( \mathbb{R}^n \), and when it matters, with smooth boundary \( \partial \Omega \) or at least a boundary of type A (as noted in the Introduction). \( L^p \) denotes the usual Lebesgue space of \( p \)-th power integrable functions on \( \mathbb{R}^n \) or respectively on \( \Omega \), \( L^p(\Omega) \); \( 1 \leq p < \infty \), and \( ||f||_{L^p} \) or \( ||f||_{L^p(\Omega)} \). On the other hand, weak - \( L^p(\mathbb{R}^n) \) consists of those \( f \) for which

\[
\sup_{t>0} \left[ t^p \mathcal{L}_n \left( \{ x \in \mathbb{R}^n : |f(x)| > t \} \right) \right]^{1/p} \equiv ||f||_{L^p_w} < \infty,
\]

and the corresponding \( L^p_w(\mathbb{R}^n) \). \( L^\infty \) is the essentially bounded functions on \( \mathbb{R}^n \), \( L^\infty(\Omega) \) those on \( \Omega \). Whereas, \( C^\alpha \) is the usual space of Hölder continuous functions on \( \mathbb{R}^n \) with exponent \( \alpha \in (0, 1) \) and normed by

\[
||f||_{C^\alpha} = ||f||_{L^\infty} + [f]_\alpha < \infty,
\]

where,

\[
[f]_\alpha = \sup_{x, y \in \mathbb{R}^n \atop x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},
\]

and the corresponding \( C^\alpha(\Omega) \). We shall also have some need of the John-Nirenberg space of functions of bounded mean oscillation on a fixed cube \( Q_0 \subset \mathbb{R}^n \) or on \( \mathbb{R}^n \) itself, i.e., \( f \in \text{BMO}(Q_0) \) if

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where $Q$ is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes; $f_Q = \text{the integral average of } f \text{ over } Q$. For BMO on $\mathbb{R}^n$, just replace $Q_0$ by $\mathbb{R}^n$. And in particular, we have the celebrated J-N Lemma:

**Lemma 2.1.** If $[f]_{*, Q_0} < \infty$, then there exist two constants $c_1$ and $c_2$ such that

$$\mathcal{L}_n(\{|x| \in Q : |f(x) - f_Q| > \lambda\}) \leq c_1 e^{-c_2 \lambda/[f]_{*, Q_0} \cdot |Q|}$$

for all cubes $Q \subset Q_0$ and any $\lambda > 0$. Here $|Q| = \mathcal{L}_n(Q)$.

(For this, see either [St2] or [To]). It thus follows that $L^\infty(Q_0) \subset \text{BMO}(Q_0) \subset L^p(Q_0)$ for all $p \geq 1$.

### 2.2 The Campanato scale $\mathcal{L}^{p, \lambda}, \mathcal{L}^{p, \lambda}(Q_0)$

For $-p < \lambda \leq n$, set

$$\mathcal{L}^{p, \lambda}(Q_0) = \{f : ||f||_{L^p(Q_0)} + [f]_{p, \lambda; Q_0} < \infty\}$$

where

$$[f]_{p, \lambda; Q_0} = \sup_{Q \subset Q_0} \left(\frac{1}{|Q|} \int_Q |f - f_Q|^p \, dx\right)^{1/p}.$$

As mentioned, this scale includes many of the classical function spaces of Harmonic Analysis, notably $L^p(Q_0), L^{p, \lambda}(Q_0), \text{BMO}(Q_0)$, and $C^\alpha(Q_0)$ - as well as versions over $\mathbb{R}^n$. We will not give a complete proof of (1.6) even just over $Q_0$, but refer the reader to [Tr] for missing details. But because these notes are about Morrey Spaces, it seems appropriate to at least prove:

$$\mathcal{L}^{p, \lambda}_0(Q_0) = L^{p, \lambda}(Q_0)$$

(2.1)

with equivalence of norms: $0 < \lambda \leq n$, $1 < p < \infty$. In fact this is a consequence of the following iteration lemma:

**Lemma 2.2.** Let $\varphi(r)$ be a non-negative function on $(0, R]$ and suppose there are numbers $\beta, \gamma > 0$ and $K > 1$ such that

$$\varphi(\rho) \leq K \left(\frac{\rho}{r}\right)^\beta \varphi(r) + K \rho^\gamma$$
whenever \( \frac{r}{s} \leq \rho < r \) and \( r \leq R \), for some \( s > 1 \) and \( R < \infty \). Then for any \( 0 < \epsilon < \beta - r \),

\[
\varphi(\rho) \leq K \left( \frac{\rho}{r} \right)^{\beta - \epsilon} \varphi(r) + KCP^\gamma
\]  

(2.2)

for some constant \( C \) depending only on \( K, \beta, \gamma \). In particular, (2.2) holds for all \( 0 < \rho < r = R \). This Lemma is a special case of Lemma 1.18 of [Tr].

Applying this lemma to the estimate below gives the inclusion \( \mathcal{L}^{p,\lambda}(Q_0) \subset L^{p,\lambda}(Q_0) \). The reverse is obvious. So let \( Q = Q_\rho = \) a cube with edge length \( \rho \), then

\[
\varphi(\rho) = \int_{Q_\rho} |u|^p \leq 2^p \int_{Q} |u - u_Q|^p + 2^p \int_{Q} |u_Q|^p
\]

\[
\leq 2^p \rho^{n-\lambda} |u|_{p,\lambda;Q_0} + 2^p \frac{\rho^n}{\rho^m} \int_{Q_r} |u|^p.
\]

Then with Lemma 2.2 and \( \gamma = n - \lambda < n - \epsilon \) for some \( \epsilon > 0 \), and \( \beta = n \), it follows that

\[
\varphi(\rho) \leq C\rho^{n-\lambda} \left( |u|_{p,\lambda;Q_0} + |u|_{L^p(Q_0)}^p \right)
\]

for all \( 0 < \rho < R \).

For the case \( \lambda = 0 \), applying the J-N Lemma gives \( 1 < p < \infty \)

\[
\text{BMO}(Q_0) = \left\{ f : \sup_{Q \subset Q_0} \left( \frac{1}{|Q|} \int_Q |f - f_Q|^p \, dx \right)^{1/p} < \infty \right\},
\]

for any \( Q_0 \subset \mathbb{R}^n \).

Finally, when \( -p < \lambda < 0 \), we remark that

\[
\mathcal{L}^{p,\lambda}(Q_0) \equiv C^\alpha(Q_0), \alpha \equiv -\lambda/p > 0,
\]  

(2.3)

was the consequence of independent investigations by S. Campanato [Ca] and N.G. Meyers [M2]. See [To], VIII.5. The proof in this case is a bit more delicate than that given above for \( 0 < \lambda \leq n \), but is repeated in several sources, namely in both [Tr] and [G].
2.3 Sobolev Spaces $W^{m,p}(\Omega)$, $G_\alpha(L^p)$, $I_\alpha(L^p)$

The space $W^{m,p}(\Omega)$ consists of those weakly differentiable functions $u(x)$ on $\Omega$ for which

$$\int_{\Omega} |D^m u|^p \, dx + \int_{\Omega} |u|^p \, dx \equiv ||u||_{W^{m,p}(\Omega)}^p < \infty,$$

where $m = \text{positive integer}$, $D^m u$ is the set of all $m$th order derivatives of $u$ on $\Omega$. But for us, it will be essential to have a potential theoretic version of $W^{m,p}(\Omega)$. Here we refer to [AH] with $G_\alpha(x)$ the Bessel potential operator, $\alpha > 0$. So when $\alpha = m$, a positive integer, then $W^{m,p}(\mathbb{R}^n) = G_\alpha(L^p(\mathbb{R}^n))$, and with $u(x) = G_m f$. $||u||_{W^{m,p}(\mathbb{R}^n)} \sim ||f||_{L^p(\mathbb{R}^n)}$, with equivalent norms; $1 < p < \infty$.

Now if $I_\alpha(x) = |x|^{\alpha-n}$, $0 < \alpha < n$, and $f \in L^p(\mathbb{R}^n)$. $I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) \, dy$, the Riesz potential of $f$ of order $\alpha$ (which is finite a.e. when $\alpha p < n$), then

$$||f||_{L^p(\mathbb{R}^n)} + ||I_\alpha f||_{L^p(\mathbb{R}^n)}$$

is an equivalent Sobolev norm, $\alpha = m < n$, $1 < p < n/\alpha$. This is the result of the fact that the Calderon-Zymund singular integrals are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. See [St2] and [To].

2.4 Morrey-Sobolev Spaces $I_\alpha(L^{p,\lambda})$

From the above, we shall denote by $I_\alpha(L^{p,\lambda}(\mathbb{R}^n))$ a Morrey-Sobolev space, $0 < \lambda < n$, $1 < p < n/\alpha$, $0 < \alpha < n$. And often we will write $I_\alpha f$ when $f$ has compact support, and then from the results latter of Chapter 8, we set

$$||u||_{W^{m,p,\lambda}} = ||u||_{L^{p,\lambda}(\mathbb{R}^n)} + ||f||_{L^{p,\lambda}(\mathbb{R}^n)},$$

when $u(x) = I_m f$; i.e., $u$ and its $m$-th order derivatives belong to the Morrey Space $L^{p,\lambda}(\mathbb{R}^n)$.

2.5 Dense/non-dense subspaces, Zorko Spaces, $V^{L^{p,\lambda}}$, $VMO$

It is well known that class $C_0^\infty(\Omega) = C^\infty$ functions on $\Omega$ with compact support in $\Omega$ is dense in $W^{m,p}$ and in $G_\alpha(L^p)$ when $\Omega = \mathbb{R}^n$. What we seek here is the density subspaces for $L^{p,\lambda}(\Omega)$. For this we turn to Zorko [Z].
Our first observation is: there are \( f \in L^p \) that cannot be approximated even by continuous functions in the norm \( \| \cdot \|_{L^p} + \| \cdot \|_{L^p, VMO} \). In fact Zorko shows that \( f_{x_0}(x) = |x - x_0|^{-\lambda} \), \( x_0 \in \Omega \), is one, for every \( x_0 \). In fact

\[
\int_{|x-x_0|<\rho} |f_{x_0}(x) - g(x)|^p \, dx \geq 2^{-p} \int_{|x-x_0|<\rho} |f_{x_0}(x)|^p \, dx - \int_{|x-x_0|<\rho} |g(x)|^p \, dx
\]

\[
\geq \frac{2^{-p}w_{n-1}}{n-\lambda} \rho^{n-\lambda} - \|g\|_{L^\infty(B(x_0,\rho))} \cdot \frac{w_{n-1}}{n} \rho^n
\]

\[
= w_{n-1} \rho^{n-\lambda} \left( \frac{2^p}{n-\lambda} - \|g\|_{L^\infty(B(x_0,\rho))} \cdot \rho^\lambda \right).
\]

So for \( \rho \leq \rho_0 = \text{sufficiently small} \),

\[
\rho^{\lambda-n} \int_{|x-x_0|<\rho} |f_{x_0}(x) - g(x)|^p \, dx \geq c_0 > 0.
\]

This motivates us to set

\[
VL^p,\lambda(\mathbb{R}^n) = \left\{ f \in L^p : \| f(\cdot - y) - f(\cdot) \|_{L^p, \lambda} \to 0, \ |y| \to 0 \right\}.
\]

Here the \( V \) stands for “vanishing” honoring Sarason’s \( VMO = \text{Vanishing mean oscillation function space} \); see [To]. We also at times will refer to this space as the Zorko subspace of \( L^{p,\lambda} \), the Morrey Space on \( \mathbb{R}^n \). We now note

**Theorem 2.3.** If \( f \in VL^p,\lambda \), then \( f \) can be approximated by \( C_0^\infty(\mathbb{R}^n) \) in the norm

\[
\| \cdot \|_{VL^p,\lambda} = \| \cdot \|_{L^p(\mathbb{R}^n)} + \| \cdot \|_{L^p,\lambda(\mathbb{R}^n)}.
\]

**Proof.** Let \( \varphi \in C_0^\infty(B(0,1)) \) with \( \int \varphi(x) \, dx = 1 \), then upon setting \( \varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon) \), \( \epsilon > 0 \), and \( \varphi_\epsilon * f \) the usual convolution, we get

\[
\left( \int_{B(x_0,\rho)} |\varphi_\epsilon * f - f|^p \, dx \right)^{1/p} \leq \int \varphi_\epsilon(y) \left( \int_{B(x_0,\rho)} |f(x-y) - f(x)|^p \, dx \right)^{1/p} \, dy
\]

hence

\[
\left( \rho^{\lambda-n} \int_{B(x_0,\rho)} |\varphi_\epsilon * f - f|^p \, dx \right)^{1/p} \leq \int \varphi_\epsilon(y) \| f(\cdot - y) - f(\cdot) \|_{L^p,\lambda} \, dy
\]

or

\[
\| \varphi_\epsilon * f - f \|_{L^p,\lambda} \leq \sup_{|y|<\epsilon} \| f(\cdot - y) - f(\cdot) \|_{L^p,\lambda}.
\]
This may seem a bit disconcerting, but as it turns out, all is not lost, for we have:

**Theorem 2.4.** If $f \in L^{p,\lambda}$, then $f \in V^{\mu,\mu}_L$ for all $\mu > \lambda$, $0 < \lambda < n$.

**Proof.** Again by mollifying $f$ as above, we can write

$$r^{\mu-n} \int_{B(x_0,r)} |f - f \ast \varphi_\epsilon|^p \leq \left( r^{\lambda-n} \int_{B(x_0,r)} |f - f \ast \varphi_\epsilon|^p \, dx \right)^{1/q} \left( \int_{B(x_0,r)} |f - f \ast \varphi_\epsilon|^p \, dx \right)^{1/q'}$$

where $\lambda < \mu < n$, $q = (n - \lambda)/(n - \mu)$, and $q' = q/(q - 1)$. The first factor is bounded by $C \|f\|_{L^{p,\lambda}}$ and the second factor tends to zero as $\epsilon \to 0$ due to the density of smooth functions in the $L^p$ spaces. \qed

### 2.6 Note

1. Extensions of the idea of a Morrey Space and the study of various operators (classical or not) on these spaces are numerous. We collect a few of these that have caught our attention in the Bibliography under “Generalized Morrey Spaces and some applications.”
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