Preface

This book is intended as a comprehensive introduction to Riemannian geometry. The reader is assumed to have basic knowledge of standard manifold theory, including the theory of tensors, forms, and Lie groups. At times it is also necessary to have some familiarity with algebraic topology and de Rham cohomology. Specifically, we recommend that the reader be familiar with texts such as [15, 72] or [97, vol. 1]. On my web page, there are links to lecture notes on these topics as well as classical differential geometry (see [90] and [89]). It is also helpful if the reader has a nodding acquaintance with ordinary differential equations. For this, a text such as [74] is more than sufficient. More basic prerequisites are real analysis, linear algebra, and some abstract algebra. Differential geometry is and always has been an "applied discipline" within mathematics that uses many other parts of mathematics for its own purposes.

Most of the material generally taught in basic Riemannian geometry as well as several more advanced topics is presented in this text. The approach we have taken occasionally deviates from the standard path. Alongside the usual variational approach, we have also developed a more function-oriented methodology that likewise uses standard calculus together with techniques from differential equations. Our motivation for this treatment has been that examples become a natural and integral part of the text rather than a separate item that is sometimes minimized. Another desirable by-product has been that one actually gets the feeling that Hessians and Laplacians are intimately related to curvatures.

The book is divided into four parts:

Part I: Tensor geometry, consisting of chapters 1, 2, 3, and 4

Part II: Geodesic and distance geometry, consisting of chapters 5, 6, and 7

Part III: Geometry à la Bochner and Cartan, consisting of chapters 8, 9, and 10

Part IV: Comparison geometry, consisting of chapters 11 and 12

There are significant structural changes and enhancements in the third edition, so chapters no longer correspond to those of the first two editions. We offer a brief outline of each chapter below.

Chapter 1 introduces Riemannian manifolds, isometries, immersions, and submersions. Homogeneous spaces and covering maps are also briefly mentioned.

There is a discussion on various types of warped products. This allows us to give both analytic and geometric definitions of the basic constant curvature geometries. The Hopf fibration as a Riemannian submersion is also discussed in several places. Finally, there is a section on tensor notation.

Chapter 2 discusses both Lie and covariant derivatives and how they can be used to define several basic concepts such as the classical notions of Hessian, Laplacian, and divergence on Riemannian manifolds. Iterated derivatives and abstract derivations are discussed toward the end and used later in the text.

Chapter 3 develops all of the important curvature concepts and discusses a few simple properties. We also develop several important formulas that relate curvature and the underlying metric. These formulas can be used in many places as a replacement for the second variation formula.

Chapter 4 is devoted to calculating curvatures in several concrete situations such as spheres, product spheres, warped products, and doubly warped products. This is used to exhibit several interesting examples. In particular, we explain how the Riemannian analogue of the Schwarzschild metric can be constructed. There is a new section that explains warped products in general and how they are characterized. This is an important section for later developments as it leads to an interesting characterization of both local and global constant curvature geometries from both the warped product and conformal view point. We have a section on Lie groups. Here two important examples of left invariant metrics are discussed as well as the general formulas for the curvatures of biinvariant metrics. It is also explained how submersions can be used to create new examples with special focus on complex projective space. There are also some general comments on how submersions can be constructed using isometric group actions.

Chapter 5 further develops the foundational topics for Riemannian manifolds. These include the first variation formula, geodesics, Riemannian manifolds as metric spaces, exponential maps, geodesic completeness versus metric completeness, and maximal domains on which the exponential map is an embedding. The chapter includes a detailed discussion of the properties of isometries. This naturally leads to the classification of simply connected space forms. At a more basic level, we obtain metric characterizations of Riemannian isometries and submersions. These are used to show that the isometry group is a Lie group and to give a proof of the slice theorem for isometric group actions.

Chapter 6 contains three more foundational topics: parallel translation, Jacobi fields, and the second variation formula. Some of the classical results we prove here are the Hadamard-Cartan theorem, Cartan's center of mass construction in nonpositive curvature and why it shows that the fundamental group of such spaces is torsion-free, Preissman's theorem, Bonnet's diameter estimate, and Synge's lemma. At the end of the chapter, we cover the ingredients needed for the classical quarter pinched sphere theorem including Klingenberg's injectivity radius estimates and Berger's proof of this theorem. Sphere theorems are revisited in chapter 12.

Chapter 7 focuses on manifolds with lower Ricci curvature bounds. We discuss volume comparison and its uses. These include proofs of how Poincaré and Sobolev constants can be bounded and theorems about restrictions on fundamental groups for manifolds with lower Ricci curvature bounds. The strong maximum principle for continuous functions is developed. This result is first used in a warm-up exercise to prove Cheng's maximal diameter theorem. We then proceed to cover the Cheeger-Gromoll splitting theorem and its consequences for manifolds with nonnegative Ricci curvature.

Chapter 8 covers various aspects of symmetries on manifolds with emphasis on Killing fields. Here there is a further discussion on why the isometry group is a Lie group. The Bochner formulas for Killing fields are covered as well as a discussion on how the presence of Killing fields in positive sectional curvature can lead to topological restrictions. The latter is a fairly new area in Riemannian geometry.

Chapter 9 explains both the classical and more recent results that arise from the Bochner technique. We start with harmonic 1-forms as Bochner did and move on to general forms and other tensors such as the curvature tensor. We use an approach that considerably simplifies many of the tensor calculations in this subject (see, e.g., the first and second editions of this book). The idea is to consistently use how derivations act on tensors instead of using Clifford representations. The Bochner technique gives many optimal bounds on the topology of closed manifolds with nonnegative curvature. In the spirit of comparison geometry, we show how Betti numbers of nonnegatively curved spaces are bounded by the prototypical compact flat manifold: the torus. More generally, we also show how the Bochner technique can be used to control the topology with more general curvature bounds. This requires a little more analysis, but is a fascinating approach that has not been presented in book form yet.

The importance of the Bochner technique in Riemannian geometry cannot be sufficiently emphasized. It seems that time and again, when people least expect it, new important developments come out of this philosophy.

Chapter 10 develops part of the theory of symmetric spaces and holonomy. The standard representations of symmetric spaces as homogeneous spaces or via Lie algebras are explained. There are several concrete calculations both specific and more general examples to get a feel for how curvatures behave. Having done this, we define holonomy for general manifolds and discuss the de Rham decomposition theorem and several corollaries of it. In particular, we show that holonomy irreducible symmetric spaces are Einstein and that their curvatures have the same sign as the Einstein constant. This theorem and the examples are used to indicate how one can classify symmetric spaces. Finally, we present a brief overview of how holonomy and symmetric spaces are related to the classification of holonomy groups. This is used, together with most of what has been learned up to this point, to give the Gallot and Meyer classification of compact manifolds with nonnegative curvature operator.

Chapter 11 focuses on the convergence theory of metric spaces and manifolds. First, we introduce the most general form of convergence: Gromov-Hausdorff convergence. This concept is often useful in many contexts as a way of getting a weak form of convergence. The real object here is to figure out what weak convergence implies in the presence of stronger side conditions. There is a section with a quick overview of Hölder spaces, Schauder's elliptic estimates, and harmonic coordinates.

To facilitate the treatment of the stronger convergence ideas, we have introduced a norm concept for Riemannian manifolds. The main focus of the chapter is to prove the Cheeger-Gromov convergence theorem, which is called the Convergence Theorem of Riemannian Geometry, as well as Anderson's generalizations of this theorem to manifolds with bounded Ricci curvature.

Chapter 12 proves some of the more general finiteness theorems that do not fall into the philosophy developed in Chapter 11. To begin, we discuss generalized critical point theory and Toponogov's theorem. These two techniques are used throughout the chapter to establish all of the important theorems. First, we probe the mysteries of sphere theorems. These results, while often unappreciated by a larger audience, have been instrumental in developing most of the new ideas in the subject. Comparison theory, injectivity radius estimates, and Toponogov's theorem were first used in a highly nontrivial way to prove the classical quarter pinched sphere theorem of Rauch, Berger, and Klingenberg. Critical point theory was introduced by Grove and Shiohama to prove the diameter sphere theorem. Following the sphere theorems, we go through some of the major results of comparison geometry: Gromov's Betti number estimate, the Soul theorem of Cheeger and Gromoll, and the Grove-Petersen homotopy finiteness theorem.

At the end of most chapters, there is a short list of books and papers that cover and often expand on the material in the chapter. We have whenever possible attempted to refer just to books and survey articles. The reader is strongly urged to go from those sources back to the original papers as ideas are often lost in the modernization of most subjects. For more recent works, we also give journal references if the corresponding books or surveys do not cover all aspects of the original paper. One particularly exhaustive treatment of Riemannian Geometry for the reader who is interested in learning more is [12]. Other valuable texts that expand or complement much of the material covered here are [77, 97] and [99]. There is also a historical survey by Berger (see [11]) that complements this text very well.

Each chapter ends with a collection of exercises that are designed to reinforce the material covered, to establish some simple results that will be needed later, and also to offer alternative proofs of several results. The first six chapters have about 30 exercises each and there are 300+ in all. The reader should at least read and think about all of the exercises, if not actually solve all of them. There are several exercises that might be considered very challenging. These have been broken up into more reasonable steps and with occasional hints. Some instructors might want to cover some of the exercises in class.

A first course should definitely cover Chapters 3, 5, and 6 together with whatever one feels is necessary from Chapters 1, 2, and 4. I would definitely not recommend teaching every single topic covered in Chapters 1, 2, and 4. A more advanced course could consist of going through Chapter 7 and parts III or IV as defined earlier. These two parts do not depend in a serious way on each other. One can probably not cover the entire book in two semesters, but it should be possible to cover parts I, II, and III or alternatively I, II, and IV depending on one's inclination.

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