Chapter 2  
A Few Basic Concepts

For simplicity, we frame our discussions in terms of unweighted, undirected networks. When such a network is time-independent, it can be represented using a symmetric adjacency matrix $A = A^T$ with elements $A_{ij} = A_{ji}$ that are equal to 1 if nodes $i$ and $j$ are connected (or, more properly, “adjacent”) and 0 if they are not. We also assume that $A_{ii} = 0$ for all $i$, so none of our networks include self-edges. \(^1\) We denote the total number of nodes in a network (i.e., a network’s “size”) by $N$. The degree $k_i$ of node $i$ is the number of edges that are connected to it. For a large network, it is common to examine the distribution of degrees over all of its nodes. The degree distribution $P_k$ is defined as the probability that a node—chosen uniformly at random from the set of all nodes—has degree $k$, and the degree sequence is the set of all node degrees (including multiplicities). The mean degree $\zeta$ is the mean number of edges per node and is given by $\zeta = \sum_k kP_k$. For example, classical Erdős–Rényi (ER) random graphs have a Poisson degree distribution, $P_k = \frac{\zeta^k e^{-\zeta}}{k!}$, in the $N \to \infty$ limit. \(^2\) However, many real-world networks have right-skewed (i.e., heavy-tailed) degree distributions \(^5\), so the mean degree $\zeta$ only provides minimal information about the structure of a network. The most popular type of heavy-tailed distribution is a power law \(^29\), for which $P_k \sim k^{-\gamma}$ as $k \to \infty$ (where the parameter $\gamma$ is called the “power” or “exponent”). Networks with a power-law degree distribution are often called “scale-free networks” (though such networks can still have scales in them, so the monicker is misleading), and many generative mechanisms—such as de Solla Price’s model \(^6\) and the Barabási–Albert (BA) model \(^16\)—produce networks with power-law degree distributions.

\(^1\)This is a standard assumption, but it is not always desirable. For example, one may wish to investigate narcissism in people tagging themselves in pictures on Facebook, a set of coupled oscillators can include self-interactions, and so on.

\(^2\)By analogy with statistical physics, the $N \to \infty$ limit is often called a “thermodynamic limit.”
When studying dynamical processes on networks, it can be very insightful to construct networks using convenient random-graph ensembles (i.e., probability distributions on graphs), including both “realistic” ones and patently unrealistic ones. The effects of network structure on dynamics are often studied using a random-graph ensemble known as the configuration model [34, 228]. In this ensemble, one specifies the degree distribution \( P_k \) (or the degree sequence), but the network stubs (i.e., ends of edges) are then connected to each other uniformly at random. In the limit of infinite network size, one expects a network drawn from a configuration-model ensemble to have vanishingly small degree–degree correlations and local clustering. It is also important to consider computational implementations (and possible associated biases) of the configuration model and its generalizations [21]. Moreover, note that there exist multiple variants of the configuration model.

Degree–degree correlation measures the (Pearson) correlation between the degrees of nodes at each end of a randomly chosen edge of a network. (The edge is chosen uniformly at random from the set of edges.) Degree–degree correlation can be significant, for example, if high-degree nodes are connected preferentially to other high-degree nodes. This is true in a social network if popular people tend to be friends with other popular people, and one would describe the network as “homophilous” by degree. By contrast, a network for which high-degree nodes are connected preferentially to low-degree nodes is “heterophilous” by degree.

The simplest type of local clustering arises as a result of a preponderance of triangle motifs in a network. (More complicated types of clustering—which need not be local—include motifs with more than three nodes, community structure, and core–periphery structure [64, 228, 259].) Triangles are common, for example, in social networks, so the lack of local clustering in configuration-model networks (in the \( N \to \infty \) limit) is an important respect in which their structure differs significantly from that in most real networks. Investigations of dynamical systems on networks with different types of clustering is a focus of current research [129, 213, 216].

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3 Reference [212] gives one illustration of how considering a very unrealistic random-graph ensemble can be crucial for developing understanding of the behavior of a dynamical process on networks.

4 Strictly speaking, one also needs to ensure appropriate conditions on the moments of \( P_k \) as \( N \to \infty \). For example, one could demand that the second moment remains finite as \( N \to \infty \).
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