

# Chapter 2

## The First Mystery: Interference and Superpositions

### 2.1 The Spin

We will start with the simplest quantum mechanical situation, the one concerning the “spin” of a particle.<sup>1</sup> Despite its simplicity, it will allow us to explain one of the basic “mysteries” of quantum mechanics. Some particles, electrons for example, possess a property called “spin”, which is a quantity that can be measured in different directions and takes, in each direction, only two values, denoted up  $\uparrow$  and down  $\downarrow$ . We will consider here only two directions in which the spin can be measured, denoted 1 and 2, so that we can have four possibilities: spins that are up 1  $\uparrow$  or down 1  $\downarrow$  in direction 1 and up 2  $\uparrow$  or down 2  $\downarrow$  in direction 2. One should not confuse the directions 1 or 2 in which the spin is measured and the values up or down that can be the result of those measurements in each direction.

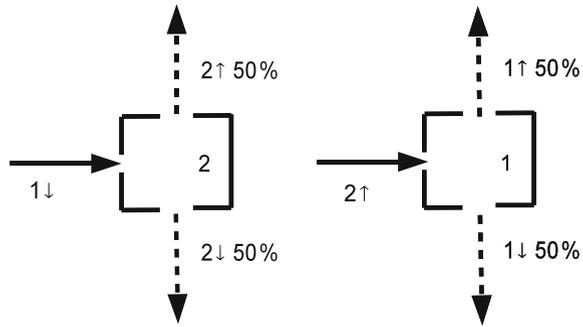
There is no need for the moment to try to understand what the property of “spin” means. We will start from a completely “phenomenological” attitude about the spin, namely, we will simply describe what happens in experiments that are “measuring” the spin, without at first trying to explain how these experiments work.

We put scare quotes here around the word “measuring” because, as we will see in Sect. 2.5 (and this will be one of the most important themes of this book), the notion that there is an intrinsic property of a particle corresponding to its spin in a given direction and that is being measured when one “measures its spin” is untenable. We will not put quotation marks everywhere, but it should be remembered that when we use the word “measurement” we do not want to suggest that some intrinsic property of a particle is being discovered.

The whole discussion using the spin may seem rather abstract, but it is easy to analyze mathematically, as we will see in Sect. 2.3. There is another, similar example,

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<sup>1</sup>The first four sections of this chapter draw heavily on David Albert’s book *Quantum Mechanics and Experience*. We emphasize that the “experiments” here are meant to illustrate the theory rather than real experiments. The latter are generally carried out with photons, whose polarization plays a role similar to the spin here. But all the experiments described below correspond to what quantum mechanics predicts.

**Fig. 2.1** Measuring the spin

the double-slit experiment, which may seem more familiar, but is less easy to analyze and which will be discussed in Appendix 2.E.

Let us see what happens in the experiments described in Fig. 2.1, where many particles are sent, so that we get statistical results. Note also that here, as in every experiment described in this book, particles are sent (in principle at least) one at a time, so that there are no possible interactions between different particles that could account for their strange behavior. So in Fig. 2.1, we have two devices that “measure the spin” of the particle in two different directions (they are unrelated to each other). The reader who wants a more realistic view of these experiments can look at Fig. 5.4.

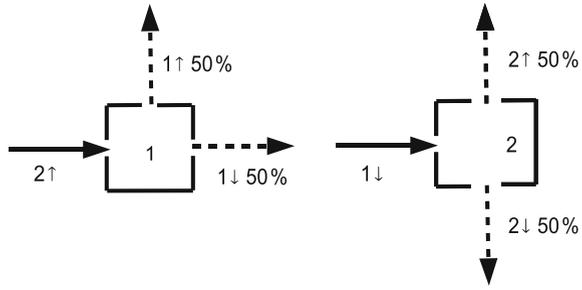
If we send a particle through one such device, the particle comes out through one of two holes, depending on the value of its spin, as shown in Fig. 2.1. We also suppose that we can select particles having a given spin value, up or down, in either of the two directions 1 or 2 (we will explain below how to do that). By “having a given spin value”, we mean that if one measures, say, in direction 1 the spin of a particle that is up in that direction 1, we will always get up.

First, we send particles that are down in direction 1 into a device that measures the spin in direction 2 (Fig. 2.1 left); if we repeat that operation many times, we will get 50%  $2 \uparrow$ , 50%  $2 \downarrow$ . If we had started with particles that were up in direction 1, we would have gotten the same result. Likewise, if we send particles that are up in direction 2 into a device that measures the spin in direction 1 (Fig. 2.1 right), we get 50%  $1 \uparrow$ , 50%  $1 \downarrow$  and we would get the same results starting with particles that are down in direction 2. So far, so good: there is no particular mystery here!

This explains also how one can select particles having a given spin value, up or down, in either of the two directions 1 or 2: just take the particle exiting, say, the device that measures the spin in direction 2 through the  $2 \uparrow$  hole and we will have selected particles that have spin up in direction 2 (i.e. if we then send those particles through a device that measures the spin in direction 2, they will always exit through the  $2 \uparrow$  hole). And likewise for the other possibilities.

Now, we may ask: can we select particles that are, say, down in direction 1 and up in direction 2? We might think that one way to do this, at least naively, is to send particles that are up in direction 2 in a device that measures the spin in direction

**Fig. 2.2** Trying to measure the spin in both directions simultaneously



1 and select those that are down in that direction. That way, the particle should be down in direction 1 and up in direction 2 (see Fig. 2.2 left).

But if we want to check that we really have particles that are down in direction 1 and up in direction 2, we might measure the spin in direction 2 of those selected particles (see Fig. 2.2 right).<sup>2</sup> However, we find that the result is again 50%  $2 \uparrow$ , 50%  $2 \downarrow$ . This is our first surprise.

The same result would occur if we tried to have particles that are, say, down in direction 2 and up in direction 1, or with any of the four possible combinations. It seems that, by measuring in direction 1, the spin of a particle that is up in direction 2, we “erase” the fact that it is up in direction 2. Indeed, the results of a later measurement of the spin in direction 2 of the particles which are up or down in direction 1 are just what they would be for such particles, independently of the fact that they had a spin up in direction 2, before the measurement in direction 1.

This is a simple example of the Heisenberg uncertainty relations or of what Bohr called “complementarity”<sup>3</sup>: we cannot measure simultaneously the spin in two different directions, because they require different devices and, applying one device in one spin direction and then a second one in another direction, destroys the result of the first device. So, one could consider two different “complementary” pictures of the particle: one describing the spin in direction 1, the other the spin in direction 2. However, one should not try to combine the two pictures (spin in directions 1 and 2)

<sup>2</sup>In Fig. 2.2 left, we put one hole to the right instead of downwards, because the particles exiting through that hole go into the box on the right of the figure.

<sup>3</sup>This is discussed mathematically in more detail in Appendix 2.C. Bohr explained the “complementary, or reciprocal, mode of description” by emphasizing [71] “the relative meaning of every concept, or rather of every word, the meaning depending upon our arbitrary choice of view point, but also that we must, in general, be prepared to accept the fact that a complete elucidation of one and the same object may require diverse points of view which defy a unique description. Indeed, strictly speaking, the conscious analysis of any concept stands in a relation of exclusion to its immediate application.” The reader may be forgiven for not understanding exactly what this means. We try here to give a plausible interpretation of that idea. See [181] for a discussion of different interpretations of Bohr’s thinking. We will return to a discussion of Bohr’s views, in relation to his debate with Einstein in Sect. 7.1.

together, to get a more complete description of the particle. One should rather choose to use one or the other picture but not both at the same time.<sup>4</sup>

As Bell stresses, the use of the word “complementary” by Bohr is not the usual one: one might say that an elephant, from the front, is “head, trunk, and two legs”, while, from the back, “she is bottom, tail, and two legs” and yet something else from top or bottom. Bell adds: “These various views are complementary in the usual sense of the word. They supplement one another, they are consistent with one another, and they are all entailed by the unifying concept ‘elephant’.” But, for Bohr, according to Bell, “complementarity” means rather the reverse of that common usage: it means contradictoriness, since the description of the spin in directions 1 and 2 exclude each other [49, p. 190].

This seems somewhat surprising, and the uncommon use of the word “complementarity” to describe the situation does not help, but in actual fact it may not be that surprising: one might think that these measuring devices, being big objects, necessarily perturb the microscopic system, the electron, being observed. By measuring in direction 1, we perturb the value of the spin in direction 2. If this view were tenable (we will see in Sect. 2.5 that it is not), there would be no big mystery in quantum mechanics, although the uncertainty principle is sometimes presented as *the* main new characteristic of the quantum world.

However, there is a far more perplexing situation to which we turn now.

## 2.2 The Mach–Zehnder Interferometer

Figure 2.3 describes an “interferometer”.<sup>5</sup> The box inside carrying the label 2 measures the spin in direction 2 with the  $2 \uparrow$  and  $2 \downarrow$  particles exiting through different holes. After passing through the box measuring the spin in direction 2, there are two possible paths for the particle, one for  $2 \uparrow$ , the other for  $2 \downarrow$ . The particles are reflected by mirrors and their paths join at the black arrow (whose functioning is not described for the time being).

The reader may think, given what was said in Chap. 1, that in quantum mechanics there is no such thing as the path of a particle. This will be discussed in detail in Chap. 5, but let us now simply use the word “path” to mean that, if we try to detect the particle along any one of those paths, we will always find it along one and only one of them.

Let us start by sending particles that are  $2 \uparrow$  in the box with the label 2 (instead of  $1 \downarrow$  as in Fig. 2.3). Then 100% of the particles will follow one path (the one indicated  $2 \uparrow$ ). If, after the black arrow, one measures the spin in direction 1, one obtains 50%  $1 \uparrow$ , 50%  $1 \downarrow$  (unlike what is represented in Fig. 2.3). If one sends particles that are  $2 \downarrow$ , one gets a similar result (with the path  $2 \downarrow$  being followed).

<sup>4</sup>Bohr did not use spin as an example, but rather the descriptions in terms of wave and particle, which we will discuss in Appendix 2.E.

<sup>5</sup>See Greenberger [239] for a more detailed discussion.

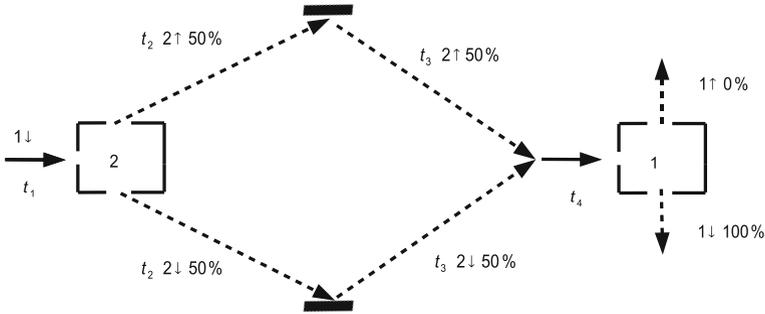


Fig. 2.3 Interference

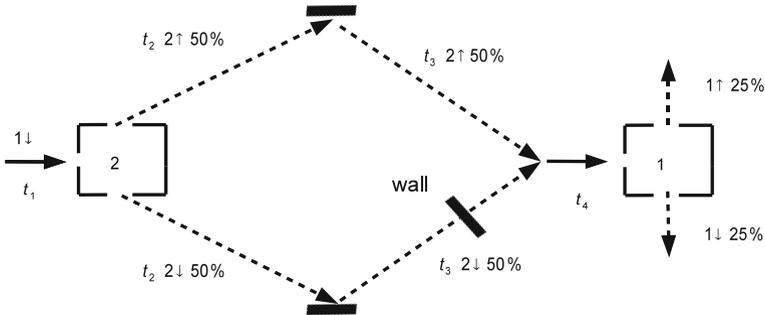


Fig. 2.4 Interference with a wall

Let us now send particles with the spin in direction 1 to be, say, down, as in Fig. 2.3. Then, 50% follow one path (2 ↑), 50% follow the other path (2 ↓). Since, in the previous setup, we had 50% 1 ↑, 50% 1 ↓, for the particles that follow either the path 2 ↑ or 2 ↓, one would expect to get 50% 1 ↑, 50% 1 ↓ here too, if one measures the spin in direction 1 after the black arrow. But one finds that, *after* the black arrow, 100% are 1 ↓, so that, whichever path they follow, the particles “remember” that they were 1 ↓ to start with. Of course, the same would happen if we did the experiment with particles that are initially 1 ↑.

This result is surprising, since we just learned from trying to measure simultaneously the spin in the directions 1 and 2 that one aspect of quantum mechanics is that measurements tend to erase the memory of past states. The first surprise is that particles seem to have no memory, after a measurement, of their pre-measurement state, but now they seem to remember it completely.

And there are more surprises. Suppose we add a wall along the path 2 ↓, as in Fig. 2.4. If the wall is inserted across the path, it blocks the particles taking that path. What will we then observe after the black arrow?

1. 50% fewer particles (which is to be expected since half of the particles follow the path 2 ↓ and are now blocked).
2. Without the wall, 100% of those that take the path 2 ↑ are found to be 1 ↓ after the black arrow. The same is true for those that followed the path 2 ↓. If one blocks the path 2 ↓, one would think that it should not affect the particles that take the path 2 ↑. Thus, one should get 100% 1 ↓ (of the remaining 50% of particles that reach the black arrow).

And here is the big surprise: one gets 25% 1 ↓ and 25% 1 ↑ (that is, half of the remaining 50% for each possibility). Therefore, one acts in a certain way on the particles that take the path 2 ↑ by blocking the path 2 ↓ *that they do not take!*

This leads to an apparent *dead end*. Let us go back to the experiment without the wall, sending particles that are 1 ↓. What does each particle do?

- Does it take path 2 ↑? No because if it did, one would have 25% 1 ↑, 25% 1 ↓ at the black arrow, as one sees when one puts a wall blocking the path 2 ↓.
- The path 2 ↓? No, for the same reason.
- Both paths? No, one always finds the particle along one of the paths if one tries to measure it.
- Neither of the paths? No, if both paths are blocked, nothing happens at the black arrow.

This phenomenon and other related phenomena are called *interference*, because whether one path is open or not seems to influence the behavior of the particles following the other path. This is the essence of the first quantum mystery!

It should be remembered that, in principle, the experiment is done by sending one particle at a time, so that no explanation can possibly be based on interactions between particles.

The way this experiment is usually described is by saying that the particle “follows both paths if they are both open” and only one path if one of them is blocked. But how does the particle know ahead of time, whether both paths are open or not?

Indeed, one might do a “delayed choice” experiment,<sup>6</sup> that is, introducing the wall *after* the passage of the particle through the first box measuring the spin in direction 2 (we can imagine both paths to be very long or put the wall just before the black arrow).

Alternatively, one could remove the black arrow while the particle is in flight and then, there would be no recombination of the paths and the particles would continue their trajectory and pass each other (see Fig. 2.5). Those following the path 2 ↑ continue downwards and those following the path 2 ↓ continue upwards. If we then measure the spin in direction 1, along any of these paths, we get 25% 1 ↑, 25% 1 ↓ in each case. Indeed, we have, along each path, particles that are only 2 ↑ or 2 ↓, and are measured in direction 1; the result is then as in Fig. 2.1 (right) and also for 2 ↓ instead of 2 ↑.

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<sup>6</sup>See [507, 509] for the theoretical proposal of such experiments by Wheeler, and [280] for experimental realizations.

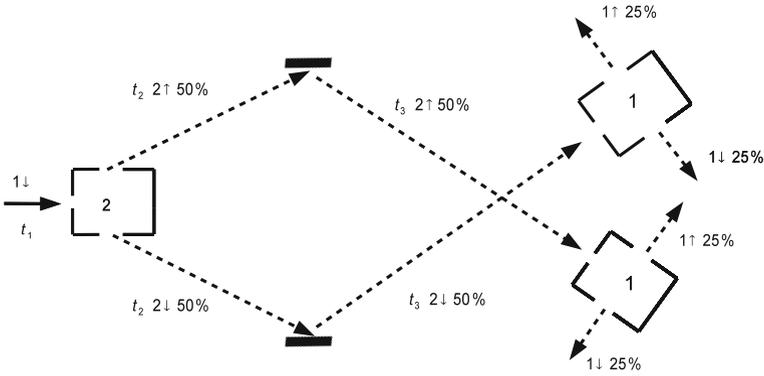


Fig. 2.5 No interference without the *black arrow*

This only deepens the mystery and is the basis of the claim by Wheeler, that “the past [meaning here whether the particle “has chosen” to follow both paths or only one] is not really the past until it has been registered” [107, p. 68]. Moreover, Wheeler invented an ingenious scheme where such “experiments” would not take place in the laboratory, but on a cosmic scale. If we accept his reasoning, this implies that we could decide *now*, by choosing which experiment to perform on the light coming from distant quasars, what happened billions of years ago<sup>7</sup> [507].

Another paradoxical consequence of the experiment described here is the Elitzur–Vaidman bomb-testing mechanism.<sup>8</sup> Suppose that we have a stock of bombs, some of which are active and some of which are duds. We want to find out which is which, but an active bomb will explode if it is hit by only one particle. On the other hand, by definition, a dud is totally insensitive to being hit by one or more particles, so that it does not affect those particles in any way. How could we tell, by classical means, which bombs are active without exploding them? There seems to be no way to do that.

But there is a trick, based on the Mach–Zehnder interferometer, that allows to identify at least a fraction of the active bombs as being active without exploding them. Let us replace the wall in Fig. 2.4 by a bomb. First, suppose that the bomb is a dud. Then, since it is insensitive to the particles, it is as if we had done nothing, i.e., as if we had not put a wall. The particle will behave as if there was no wall and therefore its spin at the black arrow will always be  $1 \downarrow$  if we measure the spin in direction 1.

On the other hand, if the bomb is active and detects the particle, it explodes and that’s it—it is lost. That happens half of the time if the bomb is active. But suppose that the bomb is active and does not explode. This means that the particle took the path  $2 \uparrow$ ; if we then measure the spin at the black arrow in direction 1, we will get

<sup>7</sup>This will be clarified in Sect. 5.1.2. See also [38], where Bell discusses the delayed-choice experiment from the viewpoint of the de Broglie–Bohm theory.

<sup>8</sup>See [173] for the theory and [300] for experiments.

1 ↓ for half of those particles and 1 ↑ for the other half. If we get 1 ↓, we cannot conclude anything since that would also happen if the bomb were a dud. *But*, if we get 1 ↑, then we can be certain that the bomb was *not* a dud since that would *never* happen if the active bomb is replaced by a dud. Since each result 1 ↓, 1 ↑ happens half of the time (among the 50% that have not exploded), we can identify 25% of our initial stock of active bombs as being active without exploding them.

Altogether, half of the active bombs explode and are lost, but a quarter are “saved” (not exploded and known to be active). For the remaining quarter, we don’t know. We can then repeat the operation (together with the duds, since we don’t know which is which) and identify as active one quarter of that remaining quarter. Repeating the operation many times, we can get as close as we like to a total of one third of the initial stock of active bombs as being known to be active and not exploded,<sup>9</sup> since  $1/3 = \sum_{n=1}^{\infty} (1/4)^n$ .

### 2.3 The Quantum Formalism

We now describe a mathematical algorithm that allows us to predict these surprising results, without worrying yet about what it “means” physically.<sup>10</sup>

We associate with each particle a “state”, which is simply a two-dimensional vector. In principle, the vector is complex, i.e., the vector space is  $\mathbb{C}^2$  rather than  $\mathbb{R}^2$ , but this will not matter here. The association is as follows (there is of course some arbitrariness in the way this association is made, but let us put that aside):

$$|1 \uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.3.1)$$

$$|1 \downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (2.3.2)$$

$$|2 \uparrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (2.3.3)$$

$$|2 \downarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (2.3.4)$$

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<sup>9</sup>If one can modify the experiment so that a fraction  $p$  of particles follow the path 2 ↓ and a fraction  $1 - p$  follow the path 2 ↑, then one can “save” a fraction  $(1 - p)/2$  of the bombs in one operation and, repeating this many times, one can eventually identify a fraction  $(1 - p)/(1 + p) = \sum_{n=1}^{\infty} [(1 - p)/2]^n$  of the active bombs, which is as close to 1 as one wants, for  $p$  small.

<sup>10</sup>For an elementary introduction to the quantum formalism, see also Susskind and Friedman [467].

We have the obvious relations

$$|2 \uparrow\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow\rangle + |1 \downarrow\rangle), \quad (2.3.5)$$

$$|2 \downarrow\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow\rangle - |1 \downarrow\rangle), \quad (2.3.6)$$

$$|1 \uparrow\rangle = \frac{1}{\sqrt{2}}(|2 \uparrow\rangle + |2 \downarrow\rangle), \quad (2.3.7)$$

$$|1 \downarrow\rangle = \frac{1}{\sqrt{2}}(|2 \uparrow\rangle - |2 \downarrow\rangle). \quad (2.3.8)$$

The states of the particles, i.e., the vectors, change according to the following rules:

1. When no measurements are made:

$$|\text{state}(t)\rangle = c_1(t) |1 \uparrow\rangle + c_2(t) |1 \downarrow\rangle = d_1(t) |2 \uparrow\rangle + d_2(t) |2 \downarrow\rangle, \quad (2.3.9)$$

where  $c_1(t)$ ,  $c_2(t)$ ,  $d_1(t)$ ,  $d_2(t)$  are related by (2.3.7) and (2.3.8), and change *continuously* in time in such a way that, at all times, we have  $|c_1(t)|^2 + |c_2(t)|^2 = 1$  and  $|d_1(t)|^2 + |d_2(t)|^2 = 1$ .

This evolution is *deterministic*, i.e., if, at some time, say 0, we give ourselves a state  $|\text{state}(0)\rangle$ , then this determines a unique state  $|\text{state}(t)\rangle$  for all times.

To say more precisely what this evolution is, one would have to write down a differential equation for  $c_i(t)$ ,  $d_i(t)$ ,  $i = 1, 2$ , which in more general situations is called the Schrödinger equation.<sup>11</sup> But we will not need to go into that for the moment (the Schrödinger equation is discussed in Appendix 2.A).

This evolution is *linear*, i.e., if, at some time, say 0, we have

$$|\text{state}(0)\rangle = c_1 |\text{state}_1(0)\rangle + c_2 |\text{state}_2(0)\rangle, \quad (2.3.10)$$

for two states  $|\text{state}_1\rangle$  and  $|\text{state}_2\rangle$  and numbers  $c_1$ ,  $c_2$ , then, at all times, we have

$$|\text{state}(t)\rangle = c_1 |\text{state}_1(t)\rangle + c_2 |\text{state}_2(t)\rangle. \quad (2.3.11)$$

2. But if a measurement is performed, the rule of evolution changes. Suppose the state is (2.3.9):

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<sup>11</sup>Or, to be precise, the Dirac or the Pauli equations in order to deal with spin.

If one measures the spin in direction 1 at time  $t$ , one finds  $\uparrow$  with probability  $|c_1(t)|^2$  and  $\downarrow$  with probability  $|c_2(t)|^2$ , bearing in mind that  $|c_1(t)|^2 + |c_2(t)|^2 = 1$  so that the probabilities add up to 1.

If one measures the spin in direction 2, one finds  $\uparrow$  with probability  $|d_1(t)|^2$  and  $\downarrow$  with probability  $|d_2(t)|^2$ , where  $|d_1(t)|^2 + |d_2(t)|^2 = 1$ .

After the measurement in direction 1, if one “sees” the result  $\uparrow$ , the state changes and becomes  $|1 \uparrow\rangle$ , and if one “sees”  $\downarrow$ , the state changes and becomes  $|1 \downarrow\rangle$ . The same thing holds for the spin in direction 2.

This second rule is called the “reduction”, or the “collapse” of the state.

Before proceeding, let us note that, a priori, the two rules are mutually *incompatible*: one rule gives rise to a *continuous in time, deterministic and linear evolution*, the other to a *discontinuous in time, non-deterministic and nonlinear* one. It is discontinuous because, when the state is reduced, it “jumps” discontinuously to one state or the other. It is non-deterministic because, in the operation of reduction, one only assigns probabilities to each possible result. Finally, it is nonlinear because, if the state is  $c_1|1 \uparrow\rangle + c_2|1 \downarrow\rangle$ , and if one then measures the spin in direction 1 and the result is  $\uparrow$ , the state is reduced to  $|1 \uparrow\rangle$ , irrespective of the values of the coefficients  $c_1$  and  $c_2$ , while if the operation were linear, the result should be a linear combination involving  $c_1$  and  $c_2$ . The coefficients  $c_1, c_2$  determine the probability of a given result, but do not affect the resulting state after the collapse.

Moreover, the theory does not tell us what a measurement is. In practice, one knows what it is, and therefore there are no practical problems here, but there is a serious problem of principle: what are the exact physical properties that define what a “measurement” is and why do we have to use one rule “between measurements”, and another “during measurements”? It is of course because of this second rule that “measurements” play a central role in the quantum theory and this explains to some extent the strange statements quoted in Chap. 1.

Note that there are other quantities to which this algorithm can be applied. In classical physics, one introduces the momentum, the energy, the angular momentum, etc. and there are quantum analogue of these variables. In Appendix 2.B we give the definition of the quantum state, and explain its time evolution and the collapse rule when one “measures” those quantities.

What we have described here is the “spin” part of the quantum state. But there is also a position part, namely a function  $\Psi(x, t)$  called the *wave function*, where  $x \in \mathbb{R}^3$  is a variable like the spin above, but taking continuous rather than discrete values.  $|\Psi(x, t)|^2$  is then the probability density for finding the particle at  $x$  at time  $t$  if one “measures” its position.

The wave function  $\Psi(x, t)$  varies with time. The time evolution is governed by a differential equation, the Schrödinger equation (see Appendix 2.A), which is such that, if one has  $\int |\Psi(x, 0)|^2 dx = 1$  at the initial time, then  $\int |\Psi(x, t)|^2 dx = 1$  at

all times.<sup>12</sup> This is analogous to the constraint  $|c_1(t)|^2 + |c_2(t)|^2 = 1$  above. The evolution is continuous in time, deterministic and linear: if the initial wave function at time 0,  $\Psi(x, 0) = c_1\Psi_1(x, 0) + c_2\Psi_2(x, 0)$ , with  $c_1, c_2$ , complex numbers, then for all times,

$$\Psi(x, t) = c_1\Psi_1(x, t) + c_2\Psi_2(x, t) ,$$

with all three terms solving the Schrödinger equation.

What happens to  $\Psi(x, t)$  if we measure the position of the particle? Suppose that we have a wave function  $\Psi(x, t) = c_1\Psi_1(x, t) + c_2\Psi_2(x, t)$ , where  $\Psi_1(x, t)$  and  $\Psi_2(x, t)$  have disjoint support,<sup>13</sup> meaning that  $\Psi_1(x, t) = 0$  for all  $x$  such that  $\Psi_2(x, t) \neq 0$  and vice-versa. Suppose also that  $\int |\Psi_1(x, t)|^2 dx = \int |\Psi_2(x, t)|^2 dx = 1$ , and  $|c_1|^2 + |c_2|^2 = 1$  which implies, since the supports of  $\Psi_1$  and  $\Psi_2$  are disjoint,  $\int |\Psi(x, t)|^2 dx = |c_1|^2 \int |\Psi_1(x, t)|^2 dx + |c_2|^2 \int |\Psi_2(x, t)|^2 dx = 1$ . Then, we will find the particle in the region where  $\Psi_1(x, t) \neq 0$  with probability  $|c_1|^2$ , and in the one where  $\Psi_2(x, t) \neq 0$  with probability  $|c_2|^2$ . After the measurement, the wave function “collapses” to either  $\Psi_1(x, t)$  or  $\Psi_2(x, t)$ , depending on the result. This change “after a measurement”, like the one for the spin measurement, is also incompatible with the continuous in time, deterministic and linear evolution given by the Schrödinger equation.

The problem posed by this duality of rules is expressed ironically by Bell:

Was the wavefunction of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system ... with a PhD?

John S. Bell [46, p. 34]

However, we will put such questions aside for the time being and show how this formalism can predict the strange experimental results described in the previous section. Let us first introduce the following terminology:

1. We will call the function  $\Psi(x, t)$  the *wave function* of the particle. We will extend this notion to a system comprising several particles at the end of Appendix 2.A and also in Chaps. 4 and 5.
2. We will call a product of the form  $\Psi(x, t)|1 \uparrow\rangle$ , or a linear combination of such products, e.g.,  $\Psi^\uparrow(x, t)|1 \uparrow\rangle + \Psi^\downarrow(x, t)|1 \downarrow\rangle$  or the corresponding expression with  $|2 \uparrow\rangle, |2 \downarrow\rangle$ , the *quantum state* [see (2.4.2) below for an example of such state]. It combines both the spatial and the spin parts discussed above. We will sometimes use the notation  $\Psi$  without the arguments  $(x, t)$  to denote the quantum state.

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<sup>12</sup>Since  $\Psi(x, t)$  is in general a complex number,  $|\Psi(x, t)|^2 = \Psi(x, t)^*\Psi(x, t)$ , where  $z^*$  is the complex conjugate of  $z \in \mathbb{C}$ .

<sup>13</sup>The support of a function is the closure of the set on which it is nonzero.

## 2.4 How Does It Work?

Let us first show how this formalism accounts for the fact that one cannot have particles with a well defined spin in directions 1 and 2 simultaneously. Indeed, if we measure the spin in direction 1 of a particle with its spin up in direction 2, as in Fig. 2.2 (left), we get either the state  $|1 \uparrow\rangle$  or the state  $|1 \downarrow\rangle$ , because of the collapse rule, and each of these states is a superposition of  $|2 \uparrow\rangle$  and  $|2 \downarrow\rangle$  with equal coefficients [see (2.3.5) and (2.3.6)]. Hence, if we measure the spin in direction 2 for the particles that are now in the state  $|1 \downarrow\rangle$  (see Fig. 2.2 right), we will get 50%  $|2 \downarrow\rangle$  and 50%  $|2 \uparrow\rangle$ . The “memory” of the particle being  $|2 \uparrow\rangle$  to start with is lost. So we see here that the collapse erases the memory.

Let us now go back to the data of Sect. 2.2 and consider Fig. 2.3. At time  $t_1$ , the state is [see (2.3.8)]

$$|1 \downarrow\rangle = \frac{1}{\sqrt{2}}(|2 \uparrow\rangle - |2 \downarrow\rangle) . \quad (2.4.1)$$

At times  $t_2$  and  $t_3$ , we have

$$\frac{1}{\sqrt{2}}(|2 \uparrow\rangle|\text{path } 2 \uparrow\rangle - |2 \downarrow\rangle|\text{path } 2 \downarrow\rangle) , \quad (2.4.2)$$

where  $|\text{path } 2 \uparrow\rangle$  and  $|\text{path } 2 \downarrow\rangle$  are the spatial parts of the quantum state  $\Psi(x, t)$ , which propagate more or less along the paths indicated  $2 \uparrow$  and  $2 \downarrow$  (meaning that, if we detect where the particle is, we will always find it along one of those paths).

Let us assume that, at time  $t_4$ , the black arrow is a device that is able to recombine the paths of the two particles and send them in the direction  $\rightarrow$ . Then the state becomes

$$\frac{1}{\sqrt{2}}(|2 \uparrow\rangle - |2 \downarrow\rangle)|\text{path } \rightarrow\rangle = |1 \downarrow\rangle|\text{path } \rightarrow\rangle . \quad (2.4.3)$$

Since the spin part of the state is now  $|1 \downarrow\rangle$ , we will get 100% “down” results if we measure the spin in direction 1 after the black arrow.

If we now put a wall on the path  $2 \downarrow$  (see Fig. 2.4), this means that we perform a measurement, since it allows us to know which path the particle follows<sup>14</sup>: if the particle is blocked by the wall, it means that it took the path  $|\text{path } 2 \downarrow\rangle$ , but if the particle is not blocked, its path must be  $|\text{path } 2 \uparrow\rangle$ . So if the particle is not blocked, the state “collapses” and becomes

$$|\text{state}\rangle \longrightarrow |2 \uparrow\rangle|\text{path } 2 \uparrow\rangle , \quad (2.4.4)$$

which becomes at time  $t_4$ , at the “black arrow”,

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<sup>14</sup>The same is true if one puts an active bomb (as opposed to a dud, which is the same as not putting anything), along the path  $2 \downarrow$ . Depending on whether the bomb explodes or not, one knows which path the particle took.

$$\frac{1}{\sqrt{2}}(|1 \uparrow\rangle + |1 \downarrow\rangle)|\text{path } \rightarrow\rangle. \quad (2.4.5)$$

Thus, if we then measure the spin in direction 1, we will get 25%  $1 \uparrow$  and 25%  $1 \downarrow$ .

Here, the strange role of “measurements” cannot be explained away by appealing to local disturbances caused by the measuring device (as we did in Sect. 2.1 when discussing the impossibility of simultaneous measurement of the spin in two different directions), since the wall has an effect on the particle even when it follows a path on which the wall is *not* inserted. Moreover, those two paths can in principle be arbitrarily far from each other, and the size of the separation has no effect on the final statistics (which always remain 25%  $1 \uparrow$  and 25%  $1 \downarrow$ ).

Finally, note that if we remove the black arrow (say, while the particles are moving past the first box, so that we are dealing with a “delayed choice” experiment), as in Fig. 2.5, there would be no recombination of the paths and the state would remain in the form

$$\frac{1}{\sqrt{2}}(|2 \uparrow\rangle|\text{path } 2 \uparrow\rangle - |2 \downarrow\rangle|\text{path } 2 \downarrow\rangle).$$

The parts  $|2 \uparrow\rangle|\text{path } 2 \uparrow\rangle$  and  $|2 \downarrow\rangle|\text{path } 2 \downarrow\rangle$  then continue along their trajectories and pass each other: the part  $|2 \uparrow\rangle|\text{path } 2 \uparrow\rangle$  continues downwards and the part  $|2 \downarrow\rangle|\text{path } 2 \downarrow\rangle$  continues upwards. If we then measure the spin in direction 1, along either of these paths, we will get 25%  $1 \uparrow$  and 25%  $1 \downarrow$  in each case, since this is what is predicted by both the states

$$|2 \uparrow\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow\rangle + |1 \downarrow\rangle) \quad \text{and} \quad |2 \downarrow\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow\rangle - |1 \downarrow\rangle).$$

So it looks as though we could affect the past behavior of the particle (whether it follows “both paths at once” or only one path) by inserting or not the black arrow, *after* they have already started their trip.

As we see, the mysterious behavior described in Sects. 2.1 and 2.2 can easily be predicted by the quantum formalism which, at least in this special case, is not mathematically complicated. However, a natural question arises: how are we supposed to understand all this talk about vectors or wave functions, and also, of course, this duality of rules?

## 2.5 What Is the Meaning of the Quantum State?

To me it seems like “quantum theory” is in a sense like a traditional herbal medicine used by “witch doctors”. We don’t REALLY understand what is happening, what the ultimate truth really is, but we have a “cook book” of procedures and rituals that can be used to obtain useful and practical calculations (independent of fundamental truth).

There are several possible reactions to what has been described in the previous sections.<sup>15</sup>

A *first reaction* is to claim that one cannot understand the microscopic world and that one must content oneself with predicting the results of measurements, which are necessarily macroscopic, and are thus described in a “classical” (i.e., understandable) language. For example, one could say that, if one prepares the particle in state  $|1 \uparrow\rangle$ , say, it will have such and such probability of ending up in a given state if we do this or that experiment. The final state is described by the result of measurement and can be described in terms of where the particle goes (through which hole it leaves our measuring device of Fig. 2.1, for example). Since we know which hole the particle goes through by detecting it, and since the detection leads to a direct observation by us, humans, one can understand why human beings seem to be put at the center of a physical theory.

Those who do not like this return to anthropocentrism may of course claim that human beings are not essential here: all that matters is a macroscopic trace left on the measuring device, which could be seen by us, but which is there whether we look at it or not. This is certainly a possible understanding of the “orthodox” view. Landau and Lifshitz wrote in their standard textbook:

[...] we emphasize that, in speaking of ‘performing a measurement’, we refer to the interaction of an electron with a classical ‘apparatus’, which in no way presupposes the presence of an external observer.

Lev Landau and Evgeny Lifshitz [302, p. 2], quoted in [46, p. 35]

A *second reaction* follows naturally from this idea, and consists in the hope that, by analyzing the measurement process in more detail, as a purely physical process (with no reference whatsoever to an outside “observer”), one may arrive at an understanding of what is going on.

A *third reaction* is to view the quantum state as representing, not an individual system, but an ensemble of systems and having thus a role similar to probabilities in classical physics,<sup>16</sup> with  $|c_i(t)|^2$ ,  $|d_i(t)|^2$ ,  $i = 1, 2$ , in (2.3.9) being the probabilities of the spin being either up or down in directions 1 and 2. According to this view, the quantum state provides us with incomplete information on individual microscopic systems. The latter do have properties such as a given spin value in all directions, or a position and a velocity, but the quantum state itself does not contain that information. We may not know how to prepare an individual system with those given properties, but they nevertheless exist. When we produce several particles in a given quantum state, we actually produce particles with different individual properties, whose statistics are encoded in the quantum state. This approach may be called the statistical interpretation of quantum mechanics (see, e.g., Ballentine [27], Blokhintsev [59], or Taylor [468] for a defense of this viewpoint).

A *fourth reaction* (sometimes motivated by the third) is to propose a more complete theory than quantum mechanics. One would not simply say, as in the third reaction,

<sup>15</sup>This section is in part an extension of [81].

<sup>16</sup>The latter are discussed in Sect. 3.4.3.

that particles do have properties not described by the quantum state, but one would try to say what these properties are and how they evolve in time. Einstein's reaction (which we will discuss in Chap. 7) was basically the third one, but he also hoped for a more complete theory.<sup>17</sup>

The first reaction is associated with the Copenhagen interpretation and is basically what was called the "official" position in Chap. 1. It was criticized there and will be discussed further in Chap. 3. The fourth reaction will be the topic of Chap. 5. The de Broglie–Bohm theory is a subtle form of the statistical interpretation, but based on a more complete theory than ordinary quantum mechanics.

We now discuss the second and the third reactions, one of which probably lies in the back of the minds of most physicists who don't see any problem with quantum mechanics. Of course, if one of those positions were tenable, they would be right not to worry, but, as we will try to show, neither of these "ways out" is compatible with either the quantum formalism (for the second reaction) or with experimental facts (for the third one).

These two reactions are different answers to the same question: does the measurement somehow create the result that is being observed or does it simply reveal some pre-existing property of the system? The word "measurement" suggests the latter meaning: if I measure the length of a table, I assume that the table has a certain length before I measure it. The same thing holds for more indirect measurements, like the distance between the Earth and the Sun. This view leads us to the statistical interpretation (since the quantum state does not assign a fixed value to the quantities being measured) and we will return to it in Sect. 2.5.2.

One natural objection to the idea that the quantum state is just like a classical probability comes from the bizarre interference effects described above (which have no equivalent in classical probabilities). In the Mach–Zehnder interferometer, if we assign a probability  $1/2$  to the event "the particle follows the path 2  $\uparrow$ " and  $1/2$  to the event "the particle follows the path 2  $\downarrow$ " when both paths are open, and a probability 1 to the event "the particle follows the path 2  $\uparrow$ " when the path 2  $\downarrow$  is blocked, why do we get different results in those two situations if we measure the spin in direction 1 at the black arrow?<sup>18</sup> If probabilities reflect our ignorance (of the path being taken), simply knowing which path is taken should not have any physical effect. Does this not show that the quantum state is a physical quantity rather than a pure expression of our ignorance?

But if the quantum state is physical, should one not think that the measurement, viewed as a physical process, perturbs the quantum state in such a way as to "create"

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<sup>17</sup>In [43], Bell considers six "possible worlds of quantum mechanics", i.e., six possible reactions with respect to the problems discussed here: the pragmatic attitude, the Bohr approach, introducing the mind into physics, the many-worlds interpretation, spontaneous collapse theories, and the de Broglie–Bohm theory. The last three theories pertain to what we call the fourth reaction and will be discussed in Chaps. 5 and 6. The pragmatic attitude and the Bohr approach both exemplify what we call the first reaction, while introducing the mind into physics could be considered as part of the fourth reaction, but we will not discuss it beyond a few words in Chap. 3.

<sup>18</sup>As we saw in Sect. 2.2, we get 100%  $1 \downarrow$  if we measure the spin after the black arrow in direction 1 when both paths are open, and 25%  $1 \uparrow$ , 25%  $1 \downarrow$ , when path 2  $\downarrow$  is blocked.

the result? There is nothing a priori irrational or even strange about this idea: the measuring device is necessarily macroscopic (otherwise we would not be able to see the result) and the object being measured is microscopic. The huge difference in size between the measuring device and the system being “measured” leaves ample room for the macroscopic object to affect the microscopic one.

One could of course declare a priori that the measuring device, being macroscopic, “collapses” the quantum state during a measurement, and that view is also sometimes associated with the Copenhagen interpretation. The two basic rules of the time evolution of a quantum state would then simply reflect the micro/macro distinction: a microstate evolves according to the Schrödinger equation, except when it interacts with a macroscopic object, in which case it may be reduced or collapsed.<sup>19</sup> However, that view assumes that there is a sharp distinction between microscopic and macroscopic; but if the Schrödinger evolution applies to one particle, two particles, ten particles, and so on, where should we stop?

A way out of this problem would exist if one could treat the measuring device in a quantum mechanical way and obtain those reduced states for the microscopic systems at the end of a measurement.<sup>20</sup> We begin by considering this possibility.

### 2.5.1 *The Measurement Process Within the Quantum Formalism*

Let us see what happens if we analyze the measurement process within the quantum formalism, an analysis that goes back to von Neumann [496]. We want to see if, within the quantum formalism, one can avoid the dual nature of the evolution law for the quantum state.

We consider a very simplified measurement process. Let

$$\Psi_0 = \varphi_0(z) \left[ c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right],$$

which describes the original state of a particle whose spin is going to be measured, viz.,

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

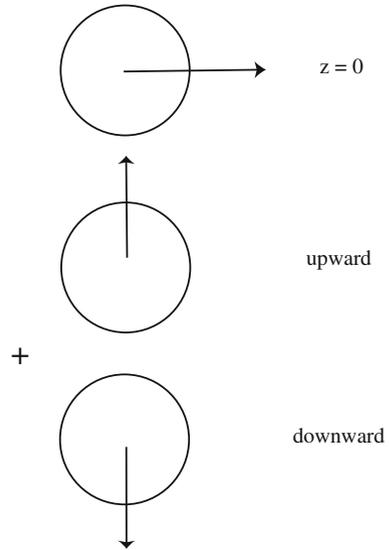
and the state  $\varphi_0(z)$  of the measuring device. Here  $z$  is a macroscopic variable, indicating the position of the measuring device (for example, the position of its center of

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<sup>19</sup>This would make the collapse rule somewhat analogous to the phenomenon of entropy increase in statistical mechanics. We will discuss this analogy further in Sect. 5.1.7.

<sup>20</sup>We will see in Sect. 5.1.6 that this is, in a sense, what happens in the de Broglie–Bohm theory; but it only works because, in that theory, one has a more complete description of the quantum system than the one in ordinary quantum mechanics.

**Fig. 2.6** Evolution of the pointer during a measurement



mass along the vertical axis), and  $\varphi_0(z)$  is centered at  $z = 0$ , meaning that the pointer is as in the first picture in Fig. 2.6. To simplify matters, we do not include here the wave function of the particle whose spin is being measured, considering only the “spin” part of its quantum state.

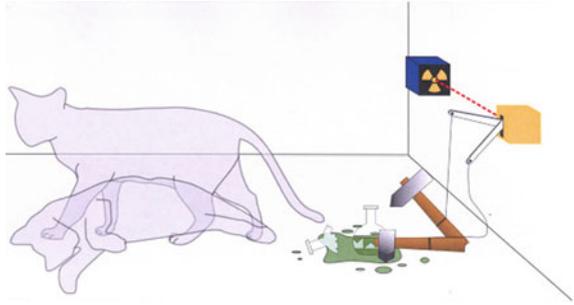
One might question the assignment of a quantum state to a macroscopic object. But this is exactly what we mean by “working within the quantum formalism”. In that formalism, by assumption, every object is describable by such a state. Of course, the pointer is composed of a great number of particles, not just the variable  $z$  introduced here. However, if we measured the positions of all the particles composing the pointer (which is of course impossible in practice, but we can at least imagine doing that), we would also know the value of  $z$ , so that one may consider that variable as being determined by all the other variables. Note that we say “if we measured” the positions. We do not assume that particles *have* positions independently of whether we measure them or not. The same holds of course for the value of  $z$ .

As we show in Appendix 2.D, the state resulting after the measurement is

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi^\uparrow(z) + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi^\downarrow(z) , \tag{2.5.1.1}$$

where  $\varphi^\uparrow(z)$  and  $\varphi^\downarrow(z)$  correspond to the last two pictures in Fig. 2.6, i.e., the pointer pointing upward or downward. Thus, the system is in a superposition of two macroscopically distinct states: one in which the pointer is pointing upward *and* one in which it is pointing downward. The problem is that we never see the pointer in such a superposed state: we see it *either* up *or* down, but not both. The ordinary quantum formalism does not correctly predict the state of the measuring device at the end

**Fig. 2.7** The cat which is both alive and dead. By Dhatfield (own work) [CC BY-SA 3.0 (<http://creativecommons.org/licenses/by-sa/3.0/>)], via Wikimedia Commons



of the experiment, since it unambiguously predicts a superposed state, and this is simply not what is observed.

Note that here we assume, somewhat naively, that we can identify a macroscopic quantum state with a physical situation in the three-dimensional world, such as the state  $\varphi^\uparrow(z)$  and a pointer pointing up. This is often implicitly assumed in all discussions about macroscopic superpositions, but we will see in Sect. 6.1 that this assumption, far from being obvious, is in fact hard to justify.<sup>21</sup>

But since the situation is now macroscopic, one may just *look* at the result. If the pointer points upward, we take the state to be  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi^\uparrow(z)$ . If the pointer points downward, the state becomes  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi^\downarrow(z)$ . One thus reduces the quantum state, which now describes a macroscopic object, just by looking at it.

One may also replace the pointer by a cat, as in Schrödinger's famous thought experiment [441]: suppose a cat is in a sealed box and there is a purely classical mechanism linking the pointer above to a hammer that will break a bottle containing some deadly poison if the pointer is up, but not if it is down. If the poison is released, it kills the cat (see Fig. 2.7). Then, following the same reasoning as above, including now the state of the cat, we get after the measurement:

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi^\uparrow(z) |\text{cat dead}\rangle + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi^\downarrow(z) |\text{cat alive}\rangle .$$

The natural interpretation of the state of the cat is that it is “both alive and dead”. Of course, we never see a cat in such a state. We do not even know what that could mean. But the cat example just dramatizes a problem that occurs already with the pointer, namely the fact that ordinary quantum mechanics predicts macroscopic superpositions that are simply not observed and that are even hard to conceive.

To better illustrate the problem, consider a coin. Once it has been thrown and has fallen on one of its sides, it is either heads or tails. If we do not look at the result, we may attribute probability one-half to each of these possibilities, but it would be

<sup>21</sup>See Maudlin [325] for a detailed discussion.

silly to say that these probabilities give a complete description of the state of the coin. The latter is heads or tails, not both, and our assignment of probabilities simply reflects our ignorance. Saying that the quantum mechanical description is complete is similar to saying that the probabilistic description of the coin is complete. While we may not know whether this is true or not for microscopic systems (about which we have no direct experience), it is manifestly absurd for a pointer or a cat.

Sometimes people seem to think that quantum mechanics has proven that the unfortunate cat is both alive and dead before anybody looks. But that was certainly not Schrödinger's idea when he introduced the cat example, which he called "quite ridiculous", as a *reductio ad absurdum* of the quantum formalism [441]. More precisely, Schrödinger wanted to show that the quantum formalism does not provide a complete description of at least *some* systems, because, at least in the case of the cat, we know that the latter is either alive or dead, and not both.

In order to turn Schrödinger's reasoning around and produce an argument in favor of the existence of macroscopic superpositions, one has to assume that the quantum formalism is absolutely true and applies to all objects, irrespective of their size. But there has never been any observable consequence of such an extension of the quantum formalism to macroscopic objects.<sup>22</sup>

One possible answer to this problem is to say that looking at the pointer or the cat changes its state and thus collapses its quantum state. But what does "looking" mean? There are many different ways to look at an object. With the help of binoculars, or with a telescope, one could look from far away. One could peek through a small hole made in the box where the cat is, etc. None of this changes anything regarding the result of course: the pointer is always up or down, the cat is alive and dead. Since all the different physical ways of looking do not make any difference, isn't it reasonable to think that looking does not have any physical effect on the system itself and that by "looking" we simply *learn* something about the state of the system, without changing it? In other words, this situation would be analogous to throwing a coin and first hiding the result; then when we later look at the coin, we see whether it is heads or tails, but of course the coin was heads or tails before we looked. This analogy is the common-sensical solution to the cat problem, and it is the one that Schrödinger had in mind.

Of course, one may also hold the view that, as long as "looking" is described in physical terms, with eyes, brains, etc., it only produces more macroscopic superpositions: in the end, the whole universe has a quantum state like (2.5.1.1), with  $\varphi^\uparrow(z)$  and  $\varphi^\downarrow(z)$  corresponding to the pointer being up or down and the observer (at least as long as it is considered to be a physical object) seeing it up or down, and everything linked to the observer being in such a superposed state. Then one arrives at a sort of infinite regress (everything physical being in a superposed state) and one has to appeal to a nonphysical consciousness to collapse the quantum state. But even if we accept the existence of a conscious mind entirely independent of the laws of physics,

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<sup>22</sup>Of course, there are macroscopic phenomena like superconductivity or superfluidity whose explanations appeal directly to quantum mechanics, but these are different from the examples of the cat or the pointer.

which consciousness should it be? Mine when I observe, someone else's when I tell them the results, a universal mind? This leads us to "theories" that are very poorly defined, assuming that they make any sense at all.<sup>23</sup>

Another line of thought is to say that the quantum state never collapses and that the two terms in (2.5.1.1) simply correspond to two different universes. This is called the "many-worlds interpretation" and will be discussed in Sect. 6.1. But in any case, it can be considered as part of the fourth reaction, since it modifies ordinary quantum mechanics, in which the collapse is part of the theory.

It is sometimes suggested that decoherence solves the cat problem: decoherence means that it is, in practice, impossible to produce interference effects between the states of the live and the dead cat or between the states of the upward and downward pointers, as was done for a single particle in the interferometer described in Sect. 2.2, between the state of the particle following one path and the state following the other path.<sup>24</sup> This is due to the large number particles composing a pointer or a cat.<sup>25</sup>

Therefore, we don't see, for cats or pointers, the strange effects that were seen in that experiment. This is something on which all sides agree: indeed, if it wasn't so, the macroscopic world would look very different from what it does, because if the cat is alive, its future behavior could interfere with the state of the dead cat, as the states following the two different paths in the Mach-Zehnder interferometer do, and that could lead to strange behavior for the live cat.

But decoherence does not change the fact that one needs to look at the result, and then "collapse" the state according to what *we see* in order for the formalism to work. Therefore, it does not remove the centrality of observations in the quantum theory. In [46], John Bell analyzes the books by Dirac [137], Landau and Lifshitz [302], Gottfried [236], and an article by van Kampen [494] and shows that in each case there is a subtle but unacknowledged transition between *and* and *or*: quantum mechanics predicts unambiguously that the cat is alive *and* dead, while we always see it either alive *or* dead. It is only by switching from *and* to *or* that one can "eliminate" that problem.

What decoherence does show is that there are no empirical consequences of quantum mechanics for macroscopic objects, at least in situations such as those described here. So for the pointer, the most natural interpretation of the state (2.5.1.1) is that of a classical probability: the pointer is up or down and the state simply reflects our ignorance: hence we say that it is up with probability one-half and down with

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<sup>23</sup>This should not be confused with the familiar "mind-body" problem: how can the material body produce mental states, and in particular, conscious ones (see [345] for a good explanation of the problem)? Even those, like Colin McGinn [329], who regard the link between the body or the brain and the qualitative aspects of consciousness (e.g., pain) as being an unsolvable mystery, given the limitations of the human mind, admit that they can be caused by, or at least correlated with, physical events in the brain. But here, when we consider the possibility that the mind collapses the quantum state, we are envisaging a direct action of the mind on matter, and this entails a radical form of dualism, since the mind would then act totally independently of the brain!.

<sup>24</sup>See the double-slit experiment described in Appendix 2.E for another example of interference between states of one particle.

<sup>25</sup>This notion of decoherence will be explained in more detail in Sect. 5.1.6 and Appendix 5.E.

the same probability. This is just like a coin which is heads or tails before we look at it and where we simply learn which of the two is the case when we do actually look at it.

But this means that, for macroscopic objects, the quantum state is not a complete description of the physical situation. There is no way within the quantum formalism to make a definite macroscopic state emerge as result of a measurement, because of the linearity of the Schrödinger equation. So that the idea, implicit in the statement of Landau and Lifshitz quoted at the beginning of this section and in many others analyzed by Bell in [46], that the interaction with a macroscopic device creates the results of measurements is simply untenable if we stick to the existing quantum formalism.

On the other hand, if we give a classical probabilistic interpretation to the state (2.5.1.1), why not do the same for microscopic objects described by quantum states like those in Sects. 2.3 and 2.4? All this leads us naturally to the third reaction, namely the ensemble or statistical interpretation of quantum mechanics.

### 2.5.2 *The “Naive” Statistical Interpretation*

The statistical approach assumes that systems of all sizes have properties, such as the spin being up or down, before they are measured and that a measurement simply reveals something pre-existing (as the word “measurement” suggests). Of course, that does not mean that we have a theory about those properties or that we can predict or control them. It simply means that we can think of them as existing prior to our measurements and as being revealed by them.

If one adopts that view, probabilities in quantum mechanics are not very different from classical probabilities: they just reflect our ignorance, and so does the quantum state. When we measure a certain physical quantity, we simply learn something about the system, and we thus modify our state (or our probabilities) accordingly. This view means that quantum mechanics is incomplete; the very definition of incomplete is that each individual system is characterized by variables other than the quantum state and that the latter has only a statistical meaning. To come back to the analogy with a coin, if we learn that it is heads or tails (a property that the coin had before looking at it), we change our probabilities for heads and tails from  $(1/2, 1/2)$  (before looking) to  $(1, 0)$  or  $(0, 1)$ , depending on the result. The collapse of the quantum state would be similar to that adjustment of probabilities.

This is the basic idea behind what are called “hidden variables”. They would be variables that characterize an individual system and whose statistical distribution would be determined by the quantum state. For example, if we prepare particles in a state such as (2.3.9), the assumption of hidden variables means that a fraction  $|c_1(t)|^2$  of such particles have spin up in direction 1 and a fraction  $|c_2(t)|^2$  of such particles have spin down in direction 1, and similarly for direction 2, with  $|c_i(t)|^2$  replaced by  $|d_i(t)|^2$ .

Of course, the time evolution of the quantum state, which allows for the interference effects, is very different from the one encountered in the application of probability theory to classical physics. But the time evolution of the state and its status are two separate issues and there is nothing *a priori* inconsistent in thinking that the spin is up or down before we measure it or that the particle goes through one slit or the other in the double-slit experiment (see Appendix 2.E). We simply have to take into account the fact that the behavior of the particle after passing through one slit is affected by whether the other slit is open or not, and likewise, in the interferometer experiment, whether the other path is blocked or not.

The uneasiness in treating the quantum state as a classical probability also follows from the fact that one cannot simultaneously measure the spin in two different directions, or the position and the velocity. But if we think that the measurement disturbs the system being measured, then there is nothing *a priori* implausible in the statistical interpretation. Note, however, that this “perturbation” view of measurements cannot be incorporated within ordinary quantum mechanics, as we just saw above, simply by analyzing measurements within quantum mechanics. But since assuming the existence of variables other than the quantum state means going beyond quantum mechanics, it is perfectly logical to assume that measurements affect those variables in ways that are not covered by the ordinary quantum theory. Then, of course, affecting those variables would modify the state and we would have to change our probabilities accordingly (and this would then be the reason behind the collapse rule).

Consider, for example, what Heisenberg wrote in his famous paper on the uncertainty relations:

At the instant of time when the position is determined — therefore, at the moment when the photon is scattered by the electron — the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength of the light employed — that is, the more exact the determination of the position. At the instant at which the position of the electron is known, its momentum therefore can be known only up to magnitudes which correspond to that discontinuous change; thus, the more precisely the position is determined, the less precisely the momentum is known, and conversely.

Werner Heisenberg [510, p. 64]. Originally published in [256]

This quote clearly expresses a “perturbation” view of measurements: if I want to measure the position of the electron, then I must perturb it in such a way that its momentum is affected. Therefore, I cannot measure both its position and the momentum it had when I did that measurement. But this statement certainly suggests that the electron *has* both a position and a momentum.

However, there is a serious problem for the statistical interpretation, namely that it is inconsistent. To see how this comes about, let us define the idea of “hidden variables” more precisely. There are many physical quantities besides spin that can in principle be measured: for example, the angular momentum, the energy, and the momentum.<sup>26</sup> The statistical interpretation means that, in each individual system,

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<sup>26</sup>These quantities are called observables and are represented mathematically by matrices or operators acting on the quantum states. The eigenvalues of these operators are the possible results of

each of these quantities will have a well defined value, which may be unknown or even unknowable, and uncontrollable, but which nevertheless *exists*.

Let us denote by  $A$  a physical quantity and by  $v(A)$  the value that this quantity has for a particular system, which of course varies from system to system, but in such a way that the quantum state gives the statistical distribution of those values: going back to the examples of Sect. 2.3, a state like (2.3.9) would mean that, if we prepare a large number of particles with that same state, then a fraction  $|c_1|^2$  of them will have the value  $v(\text{spin in direction 1}) = \uparrow$  and a fraction  $|c_2|^2$  of them will have the value  $v(\text{spin in direction 1}) = \downarrow$ , and similarly for direction 2 (the situation with more general quantities is discussed in Appendix 2.B).

To make the statistical interpretation interesting, we have to assume that  $v(A)$  exists for more than one  $A$ . For example, it would be quite arbitrary to assume that the spin values exist, but only in one direction, since our definition of directions is completely conventional. Now, if we assume that  $v(A)$  exists for a reasonable class of quantities  $A$ , and that those values agree with quantum mechanical predictions, we can derive a contradiction:

### Theorem 2.5.1 No Hidden Variables Theorem<sup>27</sup>

- (1) *There does not exist a function  $v : \mathcal{O} \rightarrow \mathbb{R}$  where  $\mathcal{O}$  is a collection of quantities related to “spin”, such that  $\forall A \in \mathcal{O}$ ,  $v(A)$  agrees with the predictions of quantum mechanics.*
- (2) *There does not exist a function  $v : \mathcal{O} \rightarrow \mathbb{R}$  where  $\mathcal{O}$  is the set of functions of the four quantities representing the positions and the momenta of two particles moving on a line, such that  $\forall A \in \mathcal{O}$ ,  $v(A)$  agrees with the predictions of quantum mechanics.*

It is important to stress that the requirement of “agreeing with the predictions of quantum mechanics” means only that certain constraints, inherent to the quantum algorithm for predicting results of measurements and discussed in Appendix 2.F, have to be satisfied. This requirement is thus totally independent of any quantum state or of the need to reproduce any particular quantum statistics.

A possible, but misleading, reaction to this theorem is to say that there is nothing new here, since it is well known that there is no quantum state that assigns a given

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(Footnote 26 continued)

the measurement of these observables, but we do not need these notions here. They are explained in Appendix 2.B.

<sup>27</sup>We refer to Appendices 2.B, 2.C and 2.F for a more precise formulation of this theorem in terms of “observables” and “operators”, and for a definition of the sets  $\mathcal{O}$ . The original version of this theorem is due to Bell [36] and to Kochen and Specker [291] for the first part (the proofs were based on a theorem of Gleason [215]), and to Clifton [98] for the second part. The version given here is simpler than the original ones and is due to Mermin (see [335] and reference therein) and Perez [392, 393] for the first part and to Myrvold for the second [344]. The proofs are given in Appendix 2.F. We will discuss another no hidden variables theorem in connection with nonlocality in Chap. 4. We will discuss other variants of this theorem in Sect. 5.3.4 and in Sect. 6.3 and Appendix 6.C. We will also discuss the famous but misleading no hidden variables theorem due to von Neumann in Sect. 7.4.

value to all the spin variables simultaneously or to both position and momentum, as we saw, at least for the spin (for the position and momentum, this is discussed in Appendix 2.C). But that misses the point: the theorem considers the possibility that there be other variables characterizing an individual system than its quantum state (in other words, that the quantum description is incomplete), variables whose values would be revealed by proper measurements. The theorem shows that, at least if the class of those variables is large enough, merely assuming the existence of those variables is impossible. Note that we are not assuming that there exists a theory about those “hidden” variables, telling us how they evolve in time for example, but merely that these variables exist and that their values agree with the quantum mechanical predictions.

Of course, this result *does not mean* that a theory introducing hidden variables cannot exist (we will discuss such a theory in Chap. 5), but it does mean that one cannot introduce such variables for all the “observables” or at least for a sufficiently large class of them at once. If we want to express precisely what the statistical interpretation of quantum mechanics means, we have to assume that individual systems possess values corresponding to a physical property  $A$ , which we denote  $v(A)$ , before being measured, and that a measurement simply reveals that value  $v(A)$ . But that leads to contradictions and therefore the “naive” statistical view is untenable.

## 2.6 Conclusions

As we already said (and it would be interesting to make a sociological study of this issue), it is probable that the creation of a definite quantum state by the interaction with a macroscopic apparatus, or the statistical interpretation, lies in the back of the mind of many physicists who are not bothered by the problems raised by quantum mechanics. It is unlikely that most physicists literally believe that the cat suddenly becomes alive or dead, simply because we look at it, especially if “looking” refers to the action of a mind independent of all physical laws.

But now we face a serious conundrum. There are two positions that would naturally justify the “no worry” attitude with respect to the meaning of the quantum state: *either* a proper quantum treatment of the measurement process would lead to a collapsed state and the need for two different laws of evolution would be eliminated, *or* the quantum state does not represent a single system but an ensemble of systems, each having its own individual properties that a measurement would simply reveal. But neither of these positions are defensible, either because the linearity of the Schrödinger equation leads necessarily to macroscopic superpositions, or because of the no hidden variables theorems.

We are left with the first and the fourth reactions. We will deal with the last one, i.e., look for a more complete theory than ordinary quantum mechanics, in Chap. 5. But before doing that, we have to discuss a host of philosophical arguments trying to

present the first reaction as not simply making the best of a bad deal, but as a necessity, independently of quantum mechanics, or even as a positive development. In the next chapter, we will examine those arguments.

## Appendices

### 2.A The Wave Function and the Schrödinger Equation

In this appendix, we will describe some of the mathematical properties of Schrödinger's equation, without discussing in detail its physical meaning, something already done in the main text of this chapter and in Chaps. 4 and 5.

#### 2.A.1 Linear Differential Equations

Let us start with the simplest differential equation<sup>28</sup>:

$$\frac{dz(t)}{dt} = az(t) , \quad (2.A.1.1)$$

where  $t \in \mathbb{R}$ ,  $z : \mathbb{R} \rightarrow \mathbb{R}$ , and  $a \in \mathbb{R}$ . By definition, a solution of this equation is a function satisfying it for all  $t$ . It is easy to see that all solutions are of the form

$$z(t) = Ae^{at} , \quad (2.A.1.2)$$

for some constant  $A$ .

We obtain a unique solution if we fix some initial condition, that is, if we fix the value of  $z(t)$  at a given time  $t$ . To simplify the notation, let  $t = 0$  and let us look for a solution such that  $z(0) = z_0$ . Then we obtain a unique solution:

$$z(t) = z_0e^{at} . \quad (2.A.1.3)$$

In this simple example, we see that (2.A.1.1) has a class of solutions (2.A.1.2) and a unique solution (2.A.1.3) once an initial condition is fixed. This is true for more general equations of the type

$$\frac{dz(t)}{dt} = f(z(t)) , \quad (2.A.1.4)$$

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<sup>28</sup>For an introduction to differential equations, see, e.g., [18, 267].

for fairly general functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , at least for short intervals of time<sup>29</sup> (but we will not use those more general equations here).

Equation (2.A.1.1) is said to be linear because, if  $z_1(t)$  and  $z_2(t)$  are solutions of (2.A.1.1), then the function  $z(t) = c_1 z_1(t) + c_2 z_2(t)$ , with  $c_1, c_2 \in \mathbb{R}$ , is also a solution.

We now generalize this simple example. First, we could replace  $z(t)$  by a complex-valued function:  $z : \mathbb{R} \rightarrow \mathbb{C}$ , with  $a \in \mathbb{C}$  in (2.A.1.1). Nothing changes except that  $A$  and  $z_0$  in (2.A.1.2) and (2.A.1.3) are also complex.

Next, we replace  $z(t)$  by an  $n$ -component complex vector:

$$\mathbf{z} : \mathbb{R} \rightarrow \mathbb{C}^n, \quad \mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix}, \quad z_i(t) \in \mathbb{C}, \quad i = 1, \dots, n.$$

Equation (2.A.1.1) is replaced by

$$\frac{d\mathbf{z}(t)}{dt} = \mathcal{A}\mathbf{z}(t), \quad (2.A.1.5)$$

where  $\mathcal{A}$  is an  $n \times n$  complex matrix. The general solution is of the form

$$\mathbf{z}(t) = e^{\mathcal{A}t} \mathbf{A}, \quad (2.A.1.6)$$

where  $\mathbf{A} \in \mathbb{C}^n$ ,

$$e^{\mathcal{A}t} = \sum_{n=0}^{\infty} \frac{\mathcal{A}^n t^n}{n!}, \quad (2.A.1.7)$$

and  $\mathcal{A}^n$  denotes the  $n$ th product of  $\mathcal{A}$  with itself. Equation (2.A.1.5) is again linear.

If we fix an initial condition  $\mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{C}^n$ , we get a unique solution:

$$\mathbf{z}(t) = e^{\mathcal{A}t} \mathbf{z}_0. \quad (2.A.1.8)$$

When  $\mathcal{A}$  possesses a basis of eigenvectors, i.e.,

$$\mathcal{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i, \quad (2.A.1.9)$$

where  $(\mathbf{e}_i)_{i=1}^n$  is an orthonormal basis of  $\mathbb{C}^n$ , the solution (2.A.1.8) can be written more explicitly. Indeed, (2.A.1.9) and (2.A.1.7) imply that

$$e^{\mathcal{A}t} \mathbf{e}_i = e^{\lambda_i t} \mathbf{e}_i, \quad (2.A.1.10)$$

and if we expand  $\mathbf{z}_0$  in the basis  $(\mathbf{e}_i)_{i=1}^n$ , i.e.,

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<sup>29</sup>See, e.g., [267, Chap. 7] for more details.

$$\mathbf{z}_0 = \sum_{i=1}^n c_i \mathbf{e}_i, \quad (2.A.1.11)$$

where

$$c_i = \langle \mathbf{e}_i | \mathbf{z}_0 \rangle, \quad (2.A.1.12)$$

and  $\langle \cdot | \cdot \rangle$  is the scalar product in  $\mathbb{C}^n$  ( $\langle z_1 | z_2 \rangle \equiv \sum_{n=1}^N z_{1n}^* z_{2n}$ ), we find by linearity that (2.A.1.8) can be written as

$$\mathbf{z}(t) = \sum_{i=1}^n c_i e^{\lambda_i t} \mathbf{e}_i. \quad (2.A.1.13)$$

So the “recipe” for solving (2.A.1.5) is to solve the eigenvalue/eigenvector problem for  $\mathcal{A}$  (assuming that  $\mathcal{A}$  has a basis of eigenvectors), compute the coefficients using (2.A.1.12), and insert the result in (2.A.1.13).

## 2.A.2 The Schrödinger Equation

Let us start with the equation for one particle in three-dimensional space:

$$i\hbar \frac{d}{dt} \Psi(x, t) = H \Psi(x, t), \quad (2.A.2.1)$$

where  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^3$ , and  $\hbar = h/2\pi$ , with  $h$  the Planck constant. The unknown here is  $\Psi$ , which is a complex-valued function of  $x$  and  $t$ .

One can think of  $\Psi$  as playing the role of  $\mathbf{z}$  in (2.A.1.5), with the index  $i = 1, \dots, n$  being replaced by a continuous variable  $x$ . The factor  $i = \sqrt{-1}$ , while essential for the physics of (2.A.2.1), does not make much difference at this stage with respect to (2.A.1.5), since  $\Psi$ , like  $\mathbf{z}$ , is complex anyway.  $H$  plays the role of  $\mathcal{A}$  in (2.A.1.5) and is a linear operator: it transforms a given function  $\Psi(x, t)$  into a new function  $(H\Psi)(x, t)$  and does it in a linear way:

$$H(\alpha\psi_1 + \beta\psi_2) = \alpha H\psi_1 + \beta H\psi_2, \quad (2.A.2.2)$$

which implies that a linear combination of solutions of (2.A.2.1) is again a solution of (2.A.2.1).

The detailed form of  $H$  or the justification of (2.A.2.1) will not matter very much and they can be found in any textbook on quantum mechanics,<sup>30</sup> but for one particle of mass  $m$  moving in  $\mathbb{R}^3$ , the operator  $H$  has the form

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<sup>30</sup>For a discussion close to our point of view, see [152, Chap. 7].

$$H = \frac{\hbar^2}{2m} \left( -\frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} - \frac{d^2}{dx_3^2} \right) + V(x) , \quad (2.A.2.3)$$

where  $x = (x_1, x_2, x_3)$  and  $V(x)$  is simply the classical potential [so that the force  $F(x)$  in classical mechanics equals  $F(x) = -\nabla V(x)$ ,  $\nabla$  denoting the gradient]. The first term, viz.,

$$\frac{\hbar^2}{2m} \left( -\frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} - \frac{d^2}{dx_3^2} \right) ,$$

is the kinetic energy term. To simplify notation, we will often consider the situation in one spatial dimension, where  $H$  is given by:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) , \quad (2.A.2.4)$$

with  $x \in \mathbb{R}$ . Classically, the Hamiltonian is (again, in one dimension)

$$H = \frac{p^2}{2m} + V(x) ,$$

and corresponds to the energy of an isolated system. The quantum version (2.A.2.4) is obtained by replacing the momentum variable  $p$  (equal to the mass times the velocity) in the classical Hamiltonian by the operator<sup>31</sup>  $P = -i\hbar d/dx$  and the variable  $x$  by the operator  $Q$  of multiplication by  $x$  [and hence  $V(x)$  by the operator of multiplication by  $V(x)$ ]. We will not justify this replacement now, but we will explain why the statistical distribution of results of measurements of momenta is related to the operator  $P$  in Appendices 2.A.3 and 2.B. For the variable  $x$ , we have already said that  $|\Psi(x, t)|^2$  is the probability density of the results of position measurements.

In the rest of these appendices, we will choose units in which  $\hbar = 1$ . Then, given an initial condition  $\Psi(x, 0) = \Psi_0(x)$ , the solution of (2.A.2.1) is (remember that  $1/i = -i$ ):

$$\Psi(x, t) = (e^{-iHt} \Psi_0)(x) , \quad (2.A.2.5)$$

where the operator  $e^{-iHt}$  can be defined through a power series as in (2.A.1.7) when the series converge, and in more subtle ways otherwise.<sup>32</sup> We will give concrete examples of what this solution looks like below, and also in Appendix 2.D and Chap. 5.

An important property of (2.A.2.5) is

$$\int_{\mathbb{R}^3} |\Psi(x, t)|^2 dx = \int_{\mathbb{R}^3} |\Psi_0(x)|^2 dx , \quad (2.A.2.6)$$

<sup>31</sup>Operators are linear functions that map “ordinary” functions into other functions. The space of functions on which they act is infinite dimensional. We will not give a rigorous or detailed treatment of these operators; see, e.g., [152, Chaps. 13–15] or [412, Chaps. 7 and 8] for such a treatment.

<sup>32</sup>See, for example, [412, 413] and [152, Chap.14].

for all  $t$ , which allows us to consider  $|\Psi(x, t)|^2$  as the probability density of finding the particle at  $x$  if one measures its position at time  $t$ , provided one normalizes  $\int_{\mathbb{R}^3} |\Psi_0(x)|^2 dx = 1$ .

What about  $H$  having a basis of eigenvectors? For that to make sense, we have to define a space of functions  $\Psi$  and explain what a basis in that space means, but a simple example is provided by Fourier series.<sup>33</sup> Let  $f(x)$ ,  $x \in \mathbb{R}$ , be a complex-valued integrable periodic function of period  $2\pi$  :

$$f(x + 2\pi) = f(x) , \quad \forall x \in \mathbb{R} . \tag{2.A.2.7}$$

Then  $f(x)$  can be written as

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}} , \tag{2.A.2.8}$$

with

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx , \tag{2.A.2.9}$$

at least when the series converge, which happens, in different senses, given certain properties of  $f(x)$ . If  $f(x)$  is square integrable over  $[0, 2\pi]$ , i.e.,  $\int_0^{2\pi} |f(x)|^2 dx < \infty$ , then

$$\lim_{N \rightarrow \infty} \int_0^{2\pi} \left| f(x) - \sum_{n=-N}^{n=N} c_n \frac{e^{inx}}{\sqrt{2\pi}} \right|^2 dx = 0 \tag{2.A.2.10}$$

and

$$\sum_{n=-\infty}^{n=\infty} |c_n|^2 < \infty , \tag{2.A.2.11}$$

which means, by definition, that the family of functions  $(e^{inx}/\sqrt{2\pi})_{n=-\infty}^{+\infty}$  is a basis of the space<sup>34</sup> of square integrable functions over  $[0, 2\pi]$ .

These relations are similar to those in spaces of  $N$  dimensions (with  $N < \infty$ ), the main difference being that in (2.A.2.8), (2.A.2.10), and (2.A.2.11) one has to take a limit  $N \rightarrow \infty$  and not simply write algebraic identities.

Now if  $H$  has a basis of eigenvectors, viz.,

$$H|e_n(x)\rangle = \lambda_n|e_n(x)\rangle , \tag{2.A.2.12}$$

<sup>33</sup>See, for example, Dym and McKean [155] for the properties of Fourier series and integrals used here.

<sup>34</sup>This space, denoted  $L^2([0, 2\pi], dx)$ , is a Hilbert space and the family  $(e^{inx}/\sqrt{2\pi})_{n=-\infty}^{+\infty}$  is a Hilbert basis, but we will not need any detailed property of such spaces in this book. The basis here is orthonormal, which will be implicit when we use the word basis.

with  $n \in \mathbb{N}$  (in general, the family of eigenvectors will be infinite but countable, as in the example of the Fourier series, so that it can be indexed by  $\mathbb{N}$ ), then one can apply the same recipe that led to (2.A.1.13). We thus write

$$\Psi(x, 0) = \Psi_0(x) = \sum_{n=0}^{\infty} c_n |e_n(x)\rangle, \quad (2.A.2.13)$$

and the solution of (2.A.2.1) is

$$\Psi(x, t) = \sum_{n=0}^{\infty} c_n \exp(-i\lambda_n t) |e_n(x)\rangle. \quad (2.A.2.14)$$

Since  $|e^{-i\lambda_n t}| = 1$ , one can show that the solution converges for all times, provided that we have  $\int_{\mathbb{R}^3} |\Psi_0(x)|^2 dx < \infty$  (which implies, as for the Fourier series,  $\sum_{n=0}^{\infty} |c_n|^2 < \infty$ ).

To illustrate what precedes with a simple example, consider a free particle, i.e., with  $V(x) = 0$  in (2.A.2.4), on a circle of radius 1. This means that we take  $\Psi(x, t)$  to be periodic of period  $2\pi$  in  $x \in \mathbb{R}$  [see (2.A.2.7)], for all  $t$ . The operator  $H$  being given by  $H = -(1/2m)d^2/dx^2$ , the eigenvalue/eigenvector problem is easy to solve. We have the following periodic eigenvectors:

$$H \frac{e^{inx}}{\sqrt{2\pi}} = -\frac{1}{2m} \frac{d^2}{dx^2} \frac{e^{inx}}{\sqrt{2\pi}} = \frac{1}{2m} n^2 \frac{e^{inx}}{\sqrt{2\pi}}, \quad (2.A.2.15)$$

and, applying what we just said about Fourier series (2.A.2.7) and using (2.A.2.13) and (2.A.2.14), we get

$$\Psi(x, t) = \sum_{n=-\infty}^{+\infty} c_n \exp\left(-\frac{in^2 t}{2m}\right) \frac{e^{inx}}{\sqrt{2\pi}}, \quad (2.A.2.16)$$

where the coefficients  $c_n$  come from (2.A.2.8) for  $f(x) = \Psi(x, 0)$ .

Sometimes the operator  $H$  does not have a basis of eigenvectors but the Schrödinger equation nevertheless has a more or less explicit solution. One example that we will refer to is given by a free particle [ $V(x) = 0$  in (2.A.2.3)] in the whole  $d$ -dimensional space  $\mathbb{R}^d$ , instead of the circle (but we will set  $d = 1$  for simplicity). We want to solve the Schrödinger equation (2.A.2.1), with  $H = -(1/2m)d^2/dx^2$ , so we want to solve:

$$i \frac{d}{dt} \Psi(x, t) = -\frac{1}{2m} \frac{d^2}{dx^2} \Psi(x, t). \quad (2.A.2.17)$$

It is convenient to introduce the *Fourier transform* of  $\Psi(x, t)$ :

$$\hat{\Psi}(p, t) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \Psi(x, t) e^{-ipx} dx. \quad (2.A.2.18)$$

This is an invertible operation (for suitable functions  $\Psi(x, t)$ , for example those satisfying  $\int_{\mathbb{R}} |\Psi(x, t)|^2 dx < \infty$ ):

$$\Psi(x, t) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \hat{\Psi}(p, t) e^{ipx} dp . \quad (2.A.2.19)$$

This last formula defines the *inverse Fourier transform*. Inserting (2.A.2.19) into (2.A.2.17), we see that  $\hat{\Psi}(p, t)$  satisfies the equation

$$i \frac{d}{dt} \hat{\Psi}(p, t) = \frac{p^2}{2m} \hat{\Psi}(p, t) , \quad (2.A.2.20)$$

whose solution is  $\hat{\Psi}(p, t) = \exp(-itp^2/2m) \hat{\Psi}(p, 0)$ . So the solution of (2.A.2.17) is

$$\Psi(x, t) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{itp^2}{2m}\right) \hat{\Psi}(p, 0) e^{ipx} dp , \quad (2.A.2.21)$$

where  $\hat{\Psi}(p, 0)$  is given in terms of the initial wave function by

$$\hat{\Psi}(p, 0) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \Psi(x, 0) e^{-ipx} dx .$$

To see what happens in a concrete example, let us start with a Gaussian wave function, in  $d = 1$ :

$$\Psi_0(x) = \Psi(x, 0) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2}\right) ,$$

which is normalized so that  $\int_{\mathbb{R}} |\Psi_0(x)|^2 dx = 1$ . Then one easily computes that

$$\hat{\Psi}(p, 0) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{p^2}{2}\right) .$$

Inserting this in (2.A.2.21), one gets

$$\Psi(x, t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi^{1/4}} \int_{\mathbb{R}} \exp\left(-\frac{itp^2}{2m}\right) \exp\left(-\frac{p^2}{2}\right) e^{ipx} dp , \quad (2.A.2.22)$$

and the integral can again be computed to yield

$$\Psi(x, t) = \frac{1}{(1 + it/m)^{1/2}} \frac{1}{\pi^{1/4}} \exp\left[-\frac{x^2}{2(1 + it/m)}\right] . \quad (2.A.2.23)$$

The important property of  $\Psi(x, t)$  is its spreading:

$$|\Psi(x, t)|^2 = \frac{1}{\sqrt{\pi[1 + (t/m)^2]}} \exp\left[-\frac{x^2}{1 + (t/m)^2}\right]. \quad (2.A.2.24)$$

Note that we have  $\int_{\mathbb{R}} |\Psi(x, t)|^2 = 1$ , in conformity with (2.A.2.6).

This means that the variance<sup>35</sup> of the Gaussian  $|\Psi(x, t)|^2$  which was equal to  $1/2$  at  $t = 0$ , becomes equal to  $[1 + (t/m)^2]/2$  as time goes by. So the Gaussian becomes more and more “flat”, which means that, if  $|\Psi(x, t)|^2$  represents the probability density of finding the particle in some region of space, then that probability becomes less and less localized as time increases, and in a sense more and more “uncertain”.

We will end this appendix with a remark which, although well known, is at the root of the most revolutionary aspect of quantum mechanics, as we will see in Chap. 4.

Suppose we have a system of  $N$  particles, each of them in  $\mathbb{R}^3$ . Then the wave function<sup>36</sup> is a function  $\Psi(x_1, \dots, x_{3N}, t)$  of  $\mathbb{R}^{3N} \times \mathbb{R}$  with values in  $\mathbb{C}$ . It still satisfies the Schrödinger equation (2.A.2.1), but  $H$  now has the form

$$H = -\frac{1}{2} \sum_{i=1}^N \frac{1}{m_i} \Delta_i + V(x_1, \dots, x_N), \quad (2.A.2.25)$$

where

$$\Delta_i = \frac{d^2}{dx_{i_1}^2} + \frac{d^2}{dx_{i_2}^2} + \frac{d^2}{dx_{i_3}^2}, \quad (2.A.2.26)$$

$m_i$  is the mass of the  $i$ th particle, and  $V$  is again the classical potential. What is “revolutionary” or at least has revolutionary consequences, is that  $\Psi$  is defined on what is called the *configuration space* of the system, i.e., the set of all possible positions of all the  $N$  particles, where  $N$  is arbitrary and could in principle include all the particles in the universe.

So there is a sense (although not very precise at this stage) in which all the particles of the universe are linked with one another. What this implies will be clarified in Chaps. 4 and 5.

### 2.A.3 The Probability Distribution for Results of Momentum Measurements

We want to show here that the results of measurements of the momentum  $p$  (which classically is just the mass times the velocity of the particle) are distributed with a

<sup>35</sup>See the definition of variance in Appendix 2.C, (2.C.1.1)–(2.C.1.3).

<sup>36</sup>Ignoring here the issue of symmetry or antisymmetry, for bosons and fermions.

probability density given by  $|\hat{\Psi}(p)|^2$ , where  $\hat{\Psi}$  is the Fourier transform of  $\Psi$ , defined by (2.A.2.18) (without the time variable).

More precisely,

$$\int_A |\hat{\Psi}(p)|^2 dp \tag{2.A.3.1}$$

is the probability that the value obtained by a measurement of momentum will belong to  $A \subset \mathbb{R}$ .

In order to prove (2.A.3.1), we will measure  $p$  by measuring  $x(t)$  at time  $t$ , using  $p = mx(t)/t$ , since  $p$  is the mass times the velocity. Since we want the result to be independent of  $t$ , we will consider the asymptotic position, which means letting  $t \rightarrow \infty$ . We will set  $m = 1$  here and consider one dimension for simplicity.

We already know that the probability density of finding  $x(t) = x$ , when one measures the position at time  $t$ , is given by  $|\Psi(x, t)|^2$ . Then, the probability of the momentum being observed to belong to a subset  $A \subset \mathbb{R}$  is  $\int_{At} |\Psi(x, t)|^2 dx$ . Now, by a change of variable  $x = pt$ , we get

$$\int_{At} |\Psi(x, t)|^2 dx = t \int_A |\Psi(pt, t)|^2 dp . \tag{2.A.3.2}$$

Suppose that we have an initial wave function  $\Psi_0(x) = \Psi(x, 0)$  supported in a bounded region  $B \subset \mathbb{R}$ . We will prove that,  $\forall A \subset \mathbb{R}$ ,

$$\lim_{t \rightarrow \infty} t \int_A |\Psi(pt, t)|^2 dp = \int_A |\hat{\Psi}(p, 0)|^2 dp . \tag{2.A.3.3}$$

Combining with (2.A.3.2), this means that, if we measure the asymptotic position  $x$  as  $t \rightarrow \infty$ , we will obtain the quantum mechanical predictions (2.A.3.1).

To prove<sup>37</sup> (2.A.3.3), we consider the free evolution, which should hold for  $t$  large, and use (2.A.2.21). Since the inverse Fourier transform of a product of functions is the convolution of their inverse Fourier transforms, divided by  $\sqrt{2\pi}$ , we get

$$\Psi(x, t) = \left(\frac{1}{2\pi it}\right)^{1/2} \int_{\mathbb{R}} \exp\left[\frac{i(x-y)^2}{2t}\right] \Psi(y, 0) dy , \tag{2.A.3.4}$$

using the fact that  $\sqrt{1/it} \exp(ix^2/2t)$  is the inverse Fourier transform of  $\exp(-itp^2/2)$ . Set  $x = pt$  in (2.A.3.4) and write it as

$$\Psi(pt, t) = \left(\frac{1}{2\pi it}\right)^{1/2} \exp\left(\frac{ip^2 t}{2}\right) \int_{\mathbb{R}} \exp\left(-ipy + i\frac{y^2}{2t}\right) \Psi(y, 0) dy . \tag{2.A.3.5}$$

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<sup>37</sup>We follow here [268, Sect. 8.6] and proceed informally; for a more rigorous treatment, see [152, pp. 306–310].

Since  $\Psi(y, 0)$  vanishes outside a bounded region  $B$ , we have,  $\forall y \in B$ ,  $\lim_{t \rightarrow \infty} \exp(iy^2/2t) = 1$ , which implies

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\frac{1}{2\pi}\right)^{1/2} \int_B \exp(-ipy + \frac{iy^2}{2t}) \Psi(y, 0) dy &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{\mathbb{R}} \exp(-ipy) \Psi(y, 0) dy \\ &= \widehat{\Psi}(p, 0), \end{aligned} \tag{2.A.3.6}$$

where, in the last equality, we use the fact that  $\Psi(y, 0)$  is supported in  $B$ . Obviously,

$$\left| \left(\frac{1}{it}\right)^{1/2} \exp\left(\frac{ip^2t}{2}\right) \right|^2 = t^{-1}. \tag{2.A.3.7}$$

Inserting (2.A.3.5)–(2.A.3.7) in the left-hand side of (2.A.3.3) proves (2.A.3.3).

## 2.B Quantum States, “Observables” and the “Collapse” Rule

We have already encountered in Sect. 2.3 the special role of measurements within the quantum formalism. As we saw, we can have two different bases in  $\mathbb{C}^2$ , ( $|1 \uparrow\rangle$ ,  $|1 \downarrow\rangle$ ) or ( $|2 \uparrow\rangle$ ,  $|2 \downarrow\rangle$ ), and using (2.3.5)–(2.3.8), we can write any given state in terms of those different bases. A measurement of the spin in direction 1 or 2 is associated with a given basis, and after a measurement, the state collapses onto one vector of the basis, depending on the result. Let us explain now the general quantum formalism.

In quantum mechanics, the space of states is a complex vector space, of finite dimension,  $\mathbb{C}^N$ , or of infinite dimension (we will discuss that situation below). The finite dimensional case generalizes the states associated with spin of Sect. 2.3. The state is endowed with a scalar product  $\langle z_1 | z_2 \rangle \equiv \sum_{n=1}^N z_{1n}^* z_{2n}$ , where, for  $z \in \mathbb{C}$ ,  $z^*$  denotes its complex conjugate.

The state is also endowed with a norm associated with that scalar product:  $\|z\|^2 = \langle z | z \rangle$ . The quantum state  $|\text{state}(t)\rangle$  is a vector in that space and evolves in time, when no measurements are made, according to a deterministic equation: a given state at time 0,  $|\text{state}(0)\rangle$ , determines a unique state at time  $t$ ,  $|\text{state}(t)\rangle$ , for all times. This evolution is continuous in time and linear, see (2.3.10), (2.3.11). The norm of that vector  $\|\text{state}(t)\|$  is constant in time.

In classical physics, one introduces various physical quantities such as angular momentum, energy, etc. (all of which are functions of the positions and the velocities). In quantum mechanics, one associates with each such physical quantity a basis of vectors ( $|e_n\rangle$ ) of the state space and a set of numbers ( $\lambda_n$ ), where  $n$  runs over  $\{1, \dots, N\}$ . The choice of these numbers  $\lambda_n$  is conventional. When there is a measurement of the quantity associated with those vectors and numbers at a certain time  $t$ , one writes the state as a linear combination of the basis vectors:

$$|\text{state}(t)\rangle = \sum_{n=1}^N c_n(t) |e_n\rangle, \tag{2.B.1}$$

where  $c_n(t) = \langle e_n | \text{state}(t) \rangle$ .

The recipe for computing probabilities of results of measurements, which generalizes what we discussed in Sect. 2.3, is that a measurement at time  $t$  yields a value  $\lambda_k$  with probability  $|c_k(t)|^2$ . Since  $\|\text{state}(t)\|^2 = \sum_{n=1}^N |c_n(t)|^2$  is constant in time,<sup>38</sup> if we normalize  $\|\text{state}(0)\| = 1$ , we have  $\sum_{n=1}^N |c_n(t)|^2 = 1$  for all times, so that the sum of the probabilities of all the results equals 1.

This assignment of probabilities to results of measurements is called *Born's rule*. Moreover, after the measurement, the quantum state collapses to  $|e_k\rangle$ . As we explained in Sect. 2.3, that collapse is neither continuous in time, nor deterministic nor linear, contrary to the time evolution when no measurements are made.

To simplify matters, we assume here that each eigenvalue is non-degenerate, i.e., it corresponds to a unique eigenvector. In general, if there are several eigenvectors with the same eigenvalue  $\lambda_k$ , the collapsed state is the projection of the original state on the subspace spanned by those eigenvectors, and the probability of occurrence of  $\lambda_k$  is the norm of that projected vector.

A correspondence can be made with the example of the spin measurement by associating  $\lambda = +1$  with the up result and  $\lambda = -1$  with the down result, but other conventions could be chosen.

The more advanced reader may find the above presentation somewhat unusual. Indeed, the standard approach is to associate a matrix with any physical quantity when  $N$  is finite, these having a basis of eigenvectors, viz.,

$$A|e_n\rangle = \lambda_n|e_n\rangle, \tag{2.B.2}$$

where the  $\lambda_n$  are real.<sup>39</sup> But this is just a way to repeat what we said above: what matters is the basis of vectors ( $|e_n\rangle$ ), while the choice of the numbers  $\lambda_n$  as the real eigenvalues of a self-adjoint matrix  $A$  is a matter of convenience.<sup>40</sup>

In the spin example, for direction 1, we could introduce the matrix<sup>41</sup>

<sup>38</sup>That last formula comes from:

$$\|\text{state}(t)\|^2 = \langle \text{state}(t) | \text{state}(t) \rangle = \sum_{n,m=1}^N c_n^*(t) c_m(t) \langle e_n | e_m \rangle = \sum_{n=1}^N |c_n(t)|^2,$$

since, by orthonormality of the basis vectors,  $\langle e_n | e_m \rangle = 0$  if  $n \neq m$  and equals 1 if  $n = m$ .

<sup>39</sup>This is automatic if we assume that  $A$  is self-adjoint. For matrices, this means that its matrix elements satisfy  $A_{ij} = A_{ji}^*$ .

<sup>40</sup>There is a more general notion associated with measurements, namely, the positive operator-valued measure (POVM), discussed further in [147] and [152, Chap. 12].

<sup>41</sup>These are the usual Pauli matrices:  $\sigma_1 = \sigma_x$ ,  $\sigma_2 = \sigma_y$ , while  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.B.3)$$

and for direction 2 the matrix

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.B.4)$$

It is easy to check, using the definitions (2.3.1)–(2.3.4), that

$$\sigma_1|1 \uparrow\rangle = |1 \uparrow\rangle, \quad \sigma_1|1 \downarrow\rangle = -|1 \downarrow\rangle, \quad \sigma_2|2 \uparrow\rangle = |2 \uparrow\rangle, \quad \sigma_2|2 \downarrow\rangle = -|2 \downarrow\rangle, \quad (2.B.5)$$

so that our basis vectors are indeed eigenvectors of the corresponding matrices with eigenvalues  $+1$  and  $-1$ . But all that we really need conceptually are the eigenvectors and the associated numbers, even though the language of operators is very useful in practice.

Now we must also consider the spaces of wave functions, that are infinite dimensional.<sup>42</sup> One introduces (let's say for a physical system consisting of one particle in one dimension)<sup>43</sup> the space of complex-valued functions  $\Psi : \mathbb{R} \rightarrow \mathbb{C}$  that are square-integrable:  $\int_{\mathbb{R}} |\Psi(x)|^2 dx < \infty$ .

One can define a scalar product on that space:  $\langle \Psi | \Phi \rangle = \int_{\mathbb{R}} \Psi^*(x) \Phi(x) dx$ , and, therefore, one can also define the notion of orthonormal sets of vectors and a norm associated to the scalar product:  $\|\Psi\|^2 = \langle \Psi | \Psi \rangle = \int_{\mathbb{R}} |\Psi(x)|^2 dx$ .

The wave function is a vector in that space that depends on time,  $\Psi(x, t)$ . When no measurements are made, that vector evolves according to a deterministic equation, like Schrödinger's equation, the evolution is continuous in time and linear. Moreover,  $\|\Psi(t)\|^2 = \int_{\mathbb{R}} |\Psi(x, t)|^2 dx$  is constant in time, as in (2.A.2.6).

Again, one associates to physical quantities linear operators (see (2.A.2.2)) that act on functions, like matrices act on vectors.<sup>44</sup> If a quantity is associated to an operator  $A$  satisfying (2.B.2), with  $n$  running now over  $\mathbb{N}$ , we have the same rule as above when one measures  $A$ , except that the sum (2.B.1) has to be replaced by a limit, as in (2.A.2.10). We have again  $\int_{\mathbb{R}} |\Psi(x, t)|^2 dx = \sum_{n \in \mathbb{N}} |c_n(t)|^2$ , and if we normalize  $\|\Psi(0)\|^2 = \int_{\mathbb{R}} |\Psi(x, 0)|^2 dx = 1$ , we have  $\sum_{n \in \mathbb{N}} |c_n(t)|^2 = 1$  for all times.

For example, suppose that we measure the quantity associated with  $H$ , defined in (2.A.2.3), (2.A.2.4) (this quantity corresponds classically to the energy). Suppose also that  $H$  has a basis of eigenvectors, see (2.A.2.12), and that the state is of the form

<sup>42</sup>This fact is intuitively understandable since a set of functions defined on  $\mathbb{R}$  cannot be characterized by a finite set of parameters, which would be the case if the space was finite dimensional (the parameters would be the coefficients of the expansion of a function in a basis of the space).

<sup>43</sup>The extension to more dimensions or more particles is straightforward: for  $M$  particles in a physical space of  $k$  dimensions, the wave functions are functions  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}$ , where  $N = kM$ , and the integrals are over  $\mathbb{R}^N$ .

<sup>44</sup>We proceed formally here; see, e.g., [152, Chap. 15] or [412] for more details on the definition of operators.

(2.A.2.14). Then we get the result  $\lambda_k$  with probability  $|c_k|^2$ , and after the measurement the wave function becomes  $|e_k(x)\rangle$ .<sup>45</sup>

We will also need, but only in Appendix 2.F, operators that do not have a basis of eigenvectors. We introduced these operators  $Q$  and  $P$  in Appendix 2.A. The operator  $Q$  is called the position operator, and acts as

$$Q\Psi(x) = x\Psi(x) , \tag{2.B.6}$$

and its eigenvectors are formally Dirac delta functions  $\delta(q - x)$ .<sup>46</sup> We have

$$Q\delta(q - x) = q\delta(q - x) , \tag{2.B.7}$$

with eigenvalue  $q$ . If we write  $\Psi(x, t) = \int \delta(q - x)\Psi(q, t)dq$ , we can see this as a sort of continuous version of (2.B.1), and the interpretation of  $|c_k|^2$  as the probability of finding the eigenvalue  $\lambda_k$  upon measurement of  $A$ , translates here into considering  $|\Psi(q, t)|^2$  as the probability density of finding the particle at  $q$ , upon measurement of its position.

The momentum operator  $P$  is defined by:

$$P\Psi(x) = -i\frac{d}{dx}\Psi(x) , \tag{2.B.8}$$

and we have the eigenvectors<sup>47</sup>

$$\frac{1}{(2\pi)^{1/2}} \exp(ipx) ,$$

with eigenvalue  $p$ . Indeed, one checks that

$$P\frac{1}{(2\pi)^{1/2}} \exp(ipx) = p\frac{1}{(2\pi)^{1/2}} \exp(ipx) . \tag{2.B.9}$$

If we consider the inverse Fourier transform formula (2.A.2.19), we can see it as the continuous version of (2.B.1), with eigenvectors  $[1/(2\pi)^{1/2}] \exp(ipx)$ , and the interpretation of  $|c_k|^2$  as the probability of finding the eigenvalue  $\lambda_k$  upon measurement of  $A$ , translates here into considering  $|\hat{\Psi}(p, t)|^2$  as the probability density of

<sup>45</sup>In the concrete example (2.A.2.15), (2.A.2.16), we get the result  $k^2/2m$  with probability  $|c_k|^2$  and, after the measurement, the wave function becomes  $e^{ikx}/\sqrt{2\pi}$  (the factor of  $1/\sqrt{2\pi}$  coming from the requirement that  $\int_0^{2\pi} |\Psi(x, t)|^2 dx = 1$  at all times).

<sup>46</sup>These are not real functions but can be thought as limits of functions whose integrals are always equal to one and that tend to 0 for all  $x \neq q$ , for example the sequence  $f_n(x) = \sqrt{\frac{n}{2\pi}} \exp(-\frac{n(x-q)^2}{2})$ , as  $n \rightarrow \infty$ . In that limit, the function becomes more and more concentrated on  $x = q$ , and tends to 0 elsewhere. This explains Eq.(2.B.7) below.

<sup>47</sup>The factor  $1/(2\pi)^{1/2}$  plays the role of a normalization factor.

finding the value  $p$  upon measurement of its momentum, see (2.A.3.1), derived in Appendix 2.A.3.

We see that, for both  $Q$  and  $P$ , the set of possible results of measurements is the set  $\mathbb{R}$  of real numbers. This set plays the same role here as the one played by the eigenvalues for matrices.<sup>48</sup>

The “collapse rule” in the case of measurements of  $Q$  and  $P$  works as follows: since a measurement whose result can be any real number is never infinitely precise, but is rather an interval of real numbers, the collapsed wave function will be the original wave function restricted to that interval and normalized so that  $\int_{\mathbb{R}} |\Psi(x, t)|^2 dx = 1$  holds after the collapse.

All this may sound terribly abstract and “unphysical”, but the goal of this presentation is precisely to emphasize how much the quantum algorithm is an unambiguous method for accurately predicting results of measurements, and nothing else. In particular, it should not be associated with any mental picture of what is “really” going on. The main issue of course is whether one should consider this algorithm as satisfactory or as being, in some sense, the “end of physics”, or whether one should try to go beyond it.

## 2.C “Uncertainty” Relations and “Complementarity”

An easy remark about the uncertainty relations is that there is a great deal of uncertainty about what exactly they mean: indeed, are they uncertainty relations or indeterminacy relations, and what are the differences between these two terms?

The first derivation of these relations by Heisenberg [256], which was more a heuristic argument than a real derivation,<sup>49</sup> was entirely compatible with a disturbance view of measurement, as expressed, for example, in the statement by Heisenberg [256] quoted in Sect. 2.5.2. This way of speaking assumes that electrons have a position and a velocity, even when they are not measured. It only shows that there are limits to how much we can know about one of these quantities without disturbing the other.

However, more radical conclusions are sometimes drawn, namely, that those uncertainty relations are really indeterminacy relations, i.e., that the positions and the velocities are indeterminate or do not exist before we measure them, or even that it does not make sense to speak of quantities that we cannot measure simultaneously. Here, we will leave aside these issues, which ultimately depend on our views about the meaning of the quantum state, discussed in Sect. 2.5, and simply give some precise versions of those relations.

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<sup>48</sup>The set  $\mathbb{R}$  is called the spectrum of the operators  $Q$  and  $P$  and is also called “continuous”. See, e.g., [152, Chap. 15] or [412] for more details on the spectra of operators.

<sup>49</sup>The first real derivation is due to Kennard [287], see, e.g., [266] for the history of the uncertainty principle.

### 2.C.1 A Statistical Relation

Consider a random variable  $x$  that can take values  $a_1, \dots, a_n$  with respective probabilities  $p_i, i = 1, \dots, n$ . The variance of  $x$ ,  $\text{Var}(x)$ , is a way to measure how much the distribution of  $x$  is spread around its mean. For  $f : \{a_1, \dots, a_n\} \rightarrow \mathbb{R}$ , we define the mean or the average of  $f(x)$  by

$$\langle f(x) \rangle = \sum_{i=1}^n f(a_i) p_i . \tag{2.C.1.1}$$

Then  $\text{Var}(x)$  is defined as

$$\text{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \langle (x - \langle x \rangle)^2 \rangle , \tag{2.C.1.2}$$

where the second equality is checked by expanding the binomial. The quantity  $|x - \langle x \rangle|$  expresses the deviation of the variable  $x$  from its mean, so (2.C.1.2) gives a measure of the size of that deviation.

If  $x$  is a continuous random variable on  $\mathbb{R}$  (we work in one dimension for simplicity), with probability density  $p(x)$ , then the definition (2.C.1.2) is still valid, with (2.C.1.1) replaced by

$$\langle f(x) \rangle = \int_{\mathbb{R}} f(x) p(x) dx . \tag{2.C.1.3}$$

A precise statement of the uncertainty relations is as follows. Given a wave function  $\Psi(x)$ , we know that the probability distribution density of results of measurements of the position  $x$  is  $|\Psi(x)|^2$ , meaning that  $\int_A |\Psi(x)|^2 dx$  is the probability that, when the position of the particle is measured, the result belongs to  $A \subset \mathbb{R}$ . We also showed in Appendix 2.A.3 that the results of measurements of the momentum  $p$  (which classically is just the mass times the velocity of the particle) are distributed with a probability density given by  $|\hat{\Psi}(p)|^2$ , where  $\hat{\Psi}$  is the Fourier transform of  $\Psi$ , defined by (2.A.2.18) (without the time variable), see (2.A.3.1).

We note that, since  $\int_{\mathbb{R}} |\Psi(x)|^2 dx = 1$ , then by Plancherel's theorem  $\int_{\mathbb{R}} |\hat{\Psi}(p)|^2 dp = 1$ .

Given this, we have a variance  $\text{Var}(x)$  for the distribution of  $x$  and a variance  $\text{Var}(p)$  for the distribution of  $p$ . Their product satisfies a lower bound:

$$\text{Var}(x)\text{Var}(p) \geq \frac{1}{4} , \tag{2.C.1.4}$$

bearing in mind that we choose units where  $\hbar = 1$ . The bound (2.C.1.4) is a rather simple mathematical relation between a function and its Fourier transform and its proof can be found in many textbooks on Fourier transforms (see, e.g., [155]), as well as those on quantum mechanics.

One can give a concrete example of Heisenberg's inequality (2.C.1.4) by considering Gaussian wave functions. For  $d = 1$ , let  $\Psi(x) = (a/\pi)^{1/4} \exp(-ax^2/2)$ , which is normalized so that  $\int_{\mathbb{R}} |\Psi(x)|^2 dx = 1$ . Then, using (2.A.2.18), it is easy to show that

$$\hat{\Psi}(p) = \frac{1}{(\pi a)^{1/4}} \exp\left(-\frac{p^2}{2a}\right).$$

If one computes the respective variances, one obtains:  $\text{Var}(x) = 1/2a$ ,  $\text{Var}(p) = a/2$ , whose product is  $1/4$ , namely the lower bound in (2.C.1.4).

This illustrates the impossibility of “measuring both the position and the momentum” simultaneously with arbitrary precision. Indeed, assume that, after a position measurement, the “collapsed” wave function is a “narrow” one (assumed to be Gaussian for simplicity),  $\Psi(x) = (a/\pi)^{1/4} \exp(-ax^2/2)$ , with  $a$  large, which means that the position measurement is precise, since  $\text{Var}(x) = 1/2a$  is small. Then, the variance of the distribution of future measurements of momenta,  $\text{Var}(p) = a/2$ , will necessarily be large.

Since (2.C.1.4) is a lower bound on variances of results of measurement, it implies nothing whatsoever about the intrinsic properties of quantum particles. One could perfectly think, in accordance with the statistical interpretation, that each individual particle has a well-defined position and momentum; but, when we prepare a large number of particles having the same quantum state, then the positions and momenta of those particles vary and have certain statistical distributions whose variances satisfy (2.C.1.4).

This statistical view is untenable, but not because of the uncertainty relations. The problem for that view comes, as we saw in Sect. 2.5.2, from the no hidden variables theorems.

However there is another, more qualitative, version of “uncertainty” in quantum mechanics.

### 2.C.2 A Qualitative Argument and Its Relation to “Complementarity”

Let us consider finite-dimensional systems for simplicity. As we saw in Appendix 2.B, a physical quantity (such as the spin) is associated with a self-adjoint matrix. Consider two such matrices  $A$  and  $B$ . Let us define their commutator:

$$[A, B] = AB - BA, \tag{2.C.2.1}$$

where  $AB$  is the matrix product. Suppose  $[A, B] = 0$ . If  $|e\rangle$  is an eigenvector of  $A$ , i.e.,

$$A|e\rangle = \lambda|e\rangle,$$

then it is easy to see that  $B|e\rangle$  is also an eigenvector of  $A$ , with the same eigenvalue:

$$AB|e\rangle = BA|e\rangle = \lambda B|e\rangle . \tag{2.C.2.2}$$

This holds also if we exchange  $A$  and  $B$ . Using this remark, one shows that, if  $[A, B] = 0$ , then  $A$  and  $B$  have a common basis of eigenvectors (with different eigenvalues).<sup>50</sup>

Conversely, if  $A$  and  $B$  have a common basis of eigenvectors, then  $[A, B] = 0$ . Since the only physically meaningful quantities are the basis vectors (and the associated numbers) corresponding to a physical quantity, if  $[A, B] = 0$ ,  $A$  and  $B$  just associate different numbers to the same basis.

Measuring  $A$  will reduce the quantum state to one of the eigenvectors of  $A$ . But if we then measure  $B$ , we will reduce the state to one of the eigenvectors of  $B$ , which is also an eigenvector of  $A$  if  $A$  and  $B$  commute. Hence, if we remeasure  $A$  after having measured  $B$ , the result will be with certainty the same eigenvalue of  $A$  as before and the state will not change, unlike when one tries to measure the spin in two different directions (see Fig. 2.2). It is in this sense that, if  $[A, B] = 0$ , one can measure  $A$  and  $B$  simultaneously (and also the products  $AB$  or  $BA$ ).

But if  $[A, B] \neq 0$ , there will be some eigenvector of  $A$  that is not an eigenvector of  $B$ . Suppose that one measures  $A$  when the state is an eigenvector of  $B$  with eigenvalue  $b$ . If, after the measurement of  $A$ , the state is an eigenvector of  $A$  that is not an eigenvector of  $B$ , then the result of a later measurement of  $B$  will not give back the original value  $b$ , since the state produced by the measurement of  $A$  is no longer an eigenvector of  $B$ .

This is what happened with the spin in directions 1 and 2, as was observed phenomenologically in Sect. 2.1 and described by the quantum formalism in Sect. 2.3. If we start with an eigenstate of the spin in direction 1 and then measure it in direction 2, we “lose” the memory of what value the spin had in direction 1, since the result of the spin measurement in direction 2 is an eigenvector of the matrix  $\sigma_2$  (2.B.4) and hence a superposition of states in direction 1 [see (2.3.5) and (2.3.6)].

This is one possible meaning of the word “complementarity” which was so fundamental to Niels Bohr. The measurement of  $A$  or  $B$  gives us a “classical” description of reality where “classical” does not refer to classical physics but means “expressible in ordinary language” or “representable” or “macroscopic”. But since the two quantities cannot be measured simultaneously (i.e., without the measurement of one quantity disturbing the measurement of the other), one cannot “combine” the picture coming from the measurement of  $A$  and the one coming from the measurement of  $B$  into a coherent picture.

One can check that the operators  $Q$  and  $P$ , introduced in Appendices 2.A.2 and 2.B, do not commute:

$$\begin{aligned} (PQ\Psi)(x) &= -i\hbar \frac{d}{dx} [x\Psi(x)] = -i\hbar \left[ \Psi(x) + x \frac{d}{dx} \Psi(x) \right] \neq -i\hbar x \frac{d}{dx} \Psi(x) \\ &= (QP\Psi)(x). \end{aligned}$$

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<sup>50</sup>Typically, some of these eigenvalues will be degenerate for  $A$  or  $B$  (or both).

This can then be interpreted in terms of “complementarity” between a “picture” based on positions and one based on momenta. But what this means depends on how we understand (2.C.1.4), and therefore how we understand the quantum formalism. The non-commutation of  $Q$  and  $P$  does not have an obvious meaning.

Let us remark finally that there is also a generalization of (2.C.1.4) that expresses quantitatively this incompatibility between  $A$  and  $B$ . Given a quantum state  $\Psi$ , one obtains a probability distribution for the results of the measurements of  $A$  and of  $B$  (described in Appendix 2.B: an eigenvalue  $\lambda_k$  occurs with probability  $|c_k|^2$ ). Thus we can define the variances  $\text{Var}_\Psi(A)$ ,  $\text{Var}_\Psi(B)$ , associated with those probability distributions. The generalization of (2.C.1.4) is<sup>51</sup>

$$\text{Var}_\Psi(A)\text{Var}_\Psi(B) \geq \frac{1}{4} \left| \langle \Psi | [A, B] | \Psi \rangle \right|^2, \quad (2.C.2.3)$$

which is similar to (2.C.1.4).<sup>52</sup>

## 2.D The Quantum Mechanical Description of Measurements

Let us consider a very simple measurement of the spin.<sup>53</sup> We start with a quantum state for the combined system composed of the particle and the measuring device:

$$\Psi_0 = \varphi_0(z) \left[ c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right], \quad (2.D.1)$$

where  $z$  denotes a macroscopic variable, namely the position of the center of mass of the measuring device, and  $\varphi_0(z)$  is centered at  $z = 0$ , meaning that the pointer is as in the first picture of Fig. 2.6. We leave aside here the spatial part of the quantum state of the particle, since we are only interested in what happens to the measuring device.

Let the Hamiltonian be

$$H = -i\sigma \frac{\partial}{\partial z}, \quad \text{where } \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which corresponds to the introduction of an inhomogeneous magnetic field. One neglects here the kinetic energy term (corresponding to the free evolution)  $-(1/2m)\partial^2 \Psi(z, t)/\partial z^2$ . With these simplifications, the Schrödinger equation is

<sup>51</sup>For a proof, see, e.g., [236, Sect. 24] or [447, Chap. 9].

<sup>52</sup>Note that, for states that are eigenstates of  $A$  or of  $B$ , both sides of (2.C.2.3) vanish [in contrast to (2.C.1.4)], but the impossibility of a simultaneous measurement of  $A$  and  $B$  holds nevertheless.

<sup>53</sup>We follow here Bell [49, p. 130]. See also Bohm and Hiley [70, Chap. 6].

$$i \frac{\partial}{\partial t} \Psi = -i\sigma \frac{\partial}{\partial z} \Psi ,$$

and one can easily check that its solution is

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_0(z - t) + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_0(z + t) . \quad (2.D.2)$$

Since  $\varphi_0(z)$  is centered at  $z = 0$ ,  $\varphi_0(z \pm t)$  is centered at  $z = \mp t$ , corresponding to the last two pictures in Fig. 2.6 (for a suitable  $t$ ), which is the result mentioned in Sect. 2.5.1, where we wrote  $\varphi^\uparrow(z)$  for  $\varphi_0(z - t)$  and  $\varphi^\downarrow(z)$  for  $\varphi_0(z + t)$ .

We can discuss the Mach–Zehnder interferometer in the presence of a wall in a similar way. Once the wall is inserted as in Fig. 2.4 and the state of the particle is (2.4.2), we get, for the combined system particle plus wall, the state

$$\frac{1}{\sqrt{2}} \left[ |2 \uparrow\rangle |\text{path2 } \uparrow\rangle \varphi_0(z) - |2 \downarrow\rangle |\text{path2 } \downarrow\rangle \varphi_1(z) \right] , \quad (2.D.3)$$

where  $\varphi_0(z)$  denotes the wave function of the wall not having absorbed the particle and  $\varphi_1(z)$  that of the wall having absorbed the particle. If we replace the wall by an active bomb, as in the Elitzur–Vaidman bomb testing mechanism,  $\varphi_0(z)$  will be the wave function of the unexploded bomb and  $\varphi_1(z)$  that of the bomb having exploded. In both cases, we have a macroscopic object (the wall or the bomb) that plays the same role as the pointer in (2.D.2).

Consider now the more general situation described in Appendix 2.B, where the operator  $A$  is associated with a given physical quantity having a basis of eigenvectors:

$$A|e_n\rangle = \lambda_n|e_n\rangle , \quad (2.D.4)$$

and the state of the system to be measured is

$$|\text{state}\rangle = \sum_n c_n |e_n\rangle , \quad (2.D.5)$$

where  $n$  runs over a finite set or over  $\mathbb{N}$ . Consider a quantum state for the combined system plus measuring device:

$$\Psi_0 = \varphi_0(z) \sum_n c_n |e_n\rangle , \quad (2.D.6)$$

where  $z$  and  $\varphi_0(z)$  are as above, i.e.,  $\varphi_0(z)$  is localized around 0.

Introducing a coupling between the system and the measuring device of the form  $H = -iA\partial/\partial z$  ( $A$  being the matrix  $\sigma$  in the example of the spin measurement above), one gets, neglecting again the kinetic energy term, the Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi = -iA \frac{\partial}{\partial z} \Psi, \quad (2.D.7)$$

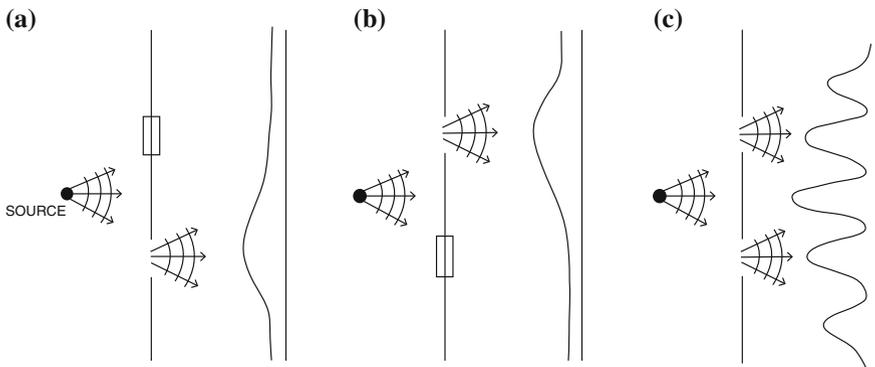
whose solution is

$$\sum_n c_n \varphi_0(z - \lambda_n t) |e_n\rangle, \quad (2.D.8)$$

which generalizes (2.D.2), with  $\varphi_0(z - \lambda_n t)$  having macroscopically disjoint supports for different  $\lambda_n$  when  $t$  is not too small, since  $\varphi_0(z)$  is localized around 0. One obtains a situation similar to the pointer in the last two pictures in Fig. 2.6, but now with more possible positions [one for each  $n$  in the sum (2.D.8)].

## 2.E The Double-Slit Experiment

A standard way to introduce interference effects in quantum mechanics, such as the ones we saw in the Mach–Zehnder interferometer, is via the double-slit experiment [184]: particles are sent (one by one) through slits in a wall and the pictures below show how the particles are distributed when they are detected on another wall somewhere behind the slits. If only one slit is open, one gets the curves (a) and (b) of Fig. 2.8, representing the densities of particles being detected behind the slits (which is not surprising), while if both slits are open, one gets the interference effects shown in the last picture (c). One might expect that, with both slits open, the distribution of the particles would be the sum of those detected when only one slit is open. But instead we get the wavy line of Fig. 2.8c, with *fewer* particles at some places than there would be with only one slit open. So opening or closing one slit seems to influ-



**Fig. 2.8** The double-slit experiment. In all three figures, there is a source of particles going towards a screen in which one or two slits are open. There is second screen behind the first one on which particles are detected. In **a** the curve represents the density of particles detected on the second screen when one slit is open, and in **b** likewise, when the other slit is open. **c** The result when both slits are open. This is clearly not the sum of the first two results

ence the particles going through the other slit. And this remains true, qualitatively, even if the open slit is very far from the closed one.

This experiment illustrates once again the role of measurements in quantum mechanics: it is often described by saying that, if we close one slit, then we *know* which slit the particle went through, hence its behavior will be affected by our measurement. The same phenomenon (suppression of interference) would occur if we put a small light behind one of the slits that would allow us to detect the slit the particle went through.

This double-slit experiment is similar to the experiment with the Mach–Zehnder interferometer described in Sect. 2.2, but in the latter we dealt with sharper figures (100 % vs. 50 %) rather than the interference patterns. The calculus with spin (i.e., vectors in two dimensions) is also easier than it would be for the double-slit experiment, where one would have to solve the Schrödinger equation with initial conditions located around each of the slits in order to deduce the interference pattern of Fig. 2.8c. Figure 5.1 in Chap. 5 shows a numerical solution yielding the interference pattern (within the de Broglie–Bohm theory).

This experiment is often considered as the essence of the quantum mechanical mystery. On the basis of this experiment, one often denies that it makes sense to speak of particles going through one slit or the other. One also sometimes says that, if both slits are open, quantum objects behave as waves, and if only one slit is open, they behave as particles, which is another instance of Bohr’s “complementarity”: one can have a “wave picture” or a “particle picture”, but not both simultaneously.

After describing the double-slit phenomenon, Feynman wrote:

Nobody knows any machinery. Nobody can give you a deeper explanation of this phenomenon than I have given; that is, a description of it.

Richard Feynman, [185, p. 145]

And in a well known classical textbook on quantum mechanics, Landau and Lifshitz said:

It is clear that [the results of the double-slit experiment] can in no way be reconciled with the idea that electrons move in paths. [...] In quantum mechanics there is no such concept as the path of a particle.

Lev Landau and Evgeny Lifshitz [302, p. 2]

We will discuss these statements in Chap. 5 in the light of the de Broglie–Bohm theory.

## 2.F Proof of the No Hidden Variables Theorem

We will now state more precisely and prove the theorem given at the the end of Sect. 2.5. The theorem is divided into two parts, and so are the proofs, which are

similar, but using different background notions. We first state each part of the theorem precisely and then give its proof.

### Precise Statement of Part 1

Let  $\mathcal{O}$  be the set of self-adjoint matrices on a complex vector space of dimension four. Then, there does not exist a function  $v$  :

$$v : \mathcal{O} \rightarrow \mathbb{R} \tag{2.F.1}$$

such that:

$$(1) \quad \forall A \in \mathcal{O}, \quad v(A) \in \{\text{eigenvalues of } A\}, \tag{2.F.2}$$

$$(2) \quad \forall A, B \in \mathcal{O}, \quad \text{with } [A, B] = AB - BA = 0, \quad v(AB) = v(A)v(B). \tag{2.F.3}$$

### Remarks

We use here the formulation of quantum mechanical “measurements” in terms of matrices and eigenvalues, see Appendix 2.B. The first condition is natural if a measurement is supposed to reveal a pre-existing value corresponding to the quantity  $A$ . However, it should be stressed that we do not use the first condition very much in the proof. In fact, we *only* use it for  $A = -\mathbf{1}, \mathbf{1}$  being the unit matrix, in the form  $v(-\mathbf{1}) = -1$ .

The second condition is necessary if the values  $v(A)$  are supposed to be in agreement with the quantum predictions, since, when  $A$  and  $B$  commute (i.e., when  $AB - BA = 0$ ), it is in principle possible to measure  $A$ ,  $B$ , and  $AB$  simultaneously, and the product of the results of the first two measurements must be equal to the result of the last one, i.e., they must satisfy (2.F.3) (see Appendix 2.C.2).<sup>54</sup> This condition, unlike the first one, will be used repeatedly in the proof. Indeed, by choosing suitable pairs of commuting matrices, and applying (2.F.3) to each pair, we will derive a contradiction.

There are similar no hidden variables theorems in any space of dimension at least 3, see Bell [36], Kochen and Specker [291], and Mermin [335], but the proof given here works only in a four dimensional space (or in any space whose dimension is a multiple of four, by considering matrices that are direct sums of copies of the matrices used here).

It should be emphasized that, even though the set  $\mathcal{O}$  contains matrices that do not commute with each other, we use relation (2.F.3) *only* for commuting matrices, so that the only assumptions of the theorem are the quantum mechanical predictions for the results of possible measurements.

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<sup>54</sup>As Mermin suggests [335, pp. 811], if the eigenvalues of the matrices were all 0 or 1 (unlike the situation here, but one could easily adapt the argument), then measuring the “observable”  $A + 2B + 4AB$  alone would give the values of all three quantities,  $A$ ,  $B$ , and  $AB$ , and they would have to satisfy  $v(AB) = v(A)v(B)$ .

Sometimes people think that this theorem rules out only “non-contextual” hidden variables: what this means is that, if we consider three matrices,  $A$ ,  $B$  and  $C$ , where  $A$  commutes with  $B$  and  $C$ , but  $B$  and  $C$  do not commute, then we are assuming that the result of measuring  $A$  does not depend on whether we choose to measure  $B$  or  $C$  simultaneously with  $A$ .<sup>55</sup> To be precise, we could write  $v(AB) = v(A)v(B)$  or  $v(AC) = v(A)v(C)$ , since  $A$  commutes with both  $B$  and  $C$ , and we assume here that one has the same value  $v(A)$  in both equations.

Hidden variables would be called contextual if they depended on that choice (so, here, the hidden variables are non-contextual). But this is not a way to “save” the possibility of hidden variables, at least those considered here: if measuring  $A$  is supposed to reveal an intrinsic property of the particle pre-existing to the measurement (and this is what is meant here by hidden variables), then it cannot possibly depend on whether I choose to measure  $B$  or  $C$  simultaneously with  $A$ , since I could measure  $A$  and nothing else. If someone has an age, a height and a weight (those being intrinsic properties of that person), then how could the result of measuring one of those properties depend on whether I measure or not another property together with that one, or on which property I would choose to measure?

The second condition (2.F.3) is necessary to derive a proof of the Theorem, but it does not affect its meaning.<sup>56</sup>

**Proof**

We use the standard Pauli matrices  $\sigma_x$  [equal to  $\sigma_2$  in (2.B.4)],  $\sigma_y$ , and  $\sigma_z$  [equal to  $\sigma_1$  in (2.B.4)]:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider a couple of each of those matrices,  $\sigma_x^i, \sigma_y^i, i = 1, 2$ , where tensor products are implicit:  $\sigma_x^1 \equiv \sigma_x^1 \otimes \mathbf{1}, \sigma_x^2 \equiv \mathbf{1} \otimes \sigma_x^2$ , etc., with  $\mathbf{1}$  the unit matrix. These operators act on  $\mathbf{C}^4$ . The following identities are well known and easy to check:

(i)

$$(\sigma_x^i)^2 = (\sigma_y^i)^2 = (\sigma_z^i)^2 = \mathbf{1}, \tag{2.F.4}$$

for  $i = 1, 2$ .

(ii) Different Pauli matrices anticommute:

$$\sigma_\alpha^i \sigma_\beta^i = -\sigma_\beta^i \sigma_\alpha^i, \tag{2.F.5}$$

---

<sup>55</sup>This is discussed by Bell [49, pp. 8–9] and Mermin [335, pp. 811–812].

<sup>56</sup>To avoid creating some later confusion in the reader’s mind, we should already mention here that the de Broglie-Bohm theory, discussed in Chap. 5, is, in some sense, a “contextual” hidden variables theory. This is explained in Sects. 5.1.4, 5.1.5 and 5.3.4. But in that theory, one does *not* introduce the hidden variables ruled out by the no hidden variables theorems (otherwise the theory would be inconsistent!).

for  $i = 1, 2$ , and  $\alpha, \beta = x, y, z$ ,  $\alpha \neq \beta$ . And they have the following commutation relations:

$$[\sigma_\alpha^i, \sigma_\beta^i] = 2i\sigma_\gamma^i, \quad (2.F.6)$$

for  $i = 1, 2$ , and  $\alpha, \beta, \gamma$  a cyclic permutation of  $x, y, z$ .

(iii) Finally,

$$[\sigma_\alpha^1, \sigma_\beta^2] = \sigma_\alpha^1\sigma_\beta^2 - \sigma_\beta^2\sigma_\alpha^1 = \mathbf{0}, \quad (2.F.7)$$

where  $\alpha, \beta = x, y, z$  and  $\mathbf{0}$  is the matrix with all entries equal to zero.

Consider now the identity

$$\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2 = -\mathbf{1}, \quad (2.F.8)$$

which follows, using first (ii) and (iii) above to move  $\sigma_x^1$  in the product from the first place (starting from the left) to the fourth place, a move that involves one anticommutation (2.F.5) and two commutations (2.F.7), viz.,

$$\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2 = -\sigma_y^2\sigma_y^1\sigma_x^2\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2, \quad (2.F.9)$$

and then using (i) repeatedly, to see that the right-hand side of (2.F.9) equals  $-\mathbf{1}$ .

We now define the operators

$$A = \sigma_x^1\sigma_y^2, \quad B = \sigma_y^1\sigma_x^2, \quad C = \sigma_x^1\sigma_x^2, \quad D = \sigma_y^1\sigma_y^2, \quad X = AB, \quad Y = CD.$$

Using (ii) and (iii), we observe:

$$(\alpha) [A, B] = 0$$

$$(\beta) [C, D] = 0$$

$$(\gamma) [X, Y] = 0$$

The identity (2.F.9) can be rewritten as

$$XY = -\mathbf{1}. \quad (2.F.10)$$

But, using (2.F.3),  $(\alpha)$ ,  $(\beta)$ ,  $(\gamma)$ , and (2.F.7), we get:

$$(a) v(XY) = v(X)v(Y) = v(AB)v(CD)$$

$$(b) v(AB) = v(A)v(B)$$

$$(c) v(CD) = v(C)v(D)$$

$$(d) v(A) = v(\sigma_x^1)v(\sigma_y^2)$$

$$(e) v(B) = v(\sigma_y^1)v(\sigma_x^2)$$

$$(f) v(C) = v(\sigma_x^1)v(\sigma_x^2)$$

$$(g) v(D) = v(\sigma_y^1)v(\sigma_y^2)$$

Since the only eigenvalue of the matrix  $-\mathbf{1}$  is  $-1$ , by combining (2.F.10) with (2.F.2) in the theorem and (a)–(g), we get

$$v(XY) = -1 = v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_y^1)v(\sigma_y^2), \quad (2.F.11)$$

where the right-hand side equals  $v(\sigma_x^1)^2v(\sigma_y^2)^2v(\sigma_y^1)^2v(\sigma_x^2)^2$ , since all the factors in the product appear twice. But this last expression, being the square of a real number, is positive, and so cannot equal  $-1$ . ■

### Part (2) of the Theorem

The proof of part (2) of the theorem is very similar to the proof of part (1) and is taken from a paper by Wayne Myrvold [344], which is a simplified version of a result due to Robert Clifton [98]. We need to introduce here operators  $Q_1, Q_2$  that act as multiplication on functions<sup>57</sup>:

$$Q_j\Psi(x_1, x_2) = x_j\Psi(x_1, x_2), \quad j = 1, 2, \quad (2.F.12)$$

and operators  $P_1, P_2$  that act by differentiation:

$$P_j\Psi(x_1, x_2) = -i\frac{\partial}{\partial x_j}\Psi(x_1, x_2), \quad j = 1, 2. \quad (2.F.13)$$

We already mentioned these operators, for one variable, in our discussion of Schrödinger's equation in Appendices 2.A.2 and 2.B.

We will also need the operators  $U_j(b) = \exp(-ibQ_j)$ ,  $V_j(c) = \exp(-icP_j)$ , with  $Q_j, P_j$  defined by (2.F.12), (2.F.13), and  $b, c \in \mathbb{R}$ . They act as

$$U_j(b)\Psi(x_1, x_2) = \exp(-ibx_j)\Psi(x_1, x_2), \quad j = 1, 2, \quad (2.F.14)$$

which follows trivially from (2.F.12), and

$$V_1(c)\Psi(x_1, x_2) = \Psi(x_1 - c, x_2), \quad (2.F.15)$$

and similarly for  $V_2(c)$ . Equation (2.F.15) follows from (2.F.13) by expanding both sides in a Taylor series, for functions  $\Psi$  such that the series converge, and by extending the unitary operator  $V_2(b)$  to more general functions  $\Psi$  (see, e.g., [412, Chap. 8] for an explanation of that extension).

### Precise Statement of Part 2

Let  $\mathcal{O}$  be the set of functions of the operators  $Q_1, Q_2, P_1$ , or  $P_2$ . Then, there does not exist a function

$$v : \mathcal{O} \rightarrow \mathbb{R} \quad (2.F.16)$$

---

<sup>57</sup>This proof is the only place in this book where we use operators that act on an infinite-dimensional space of functions and that are not simply reduced to matrices. See, e.g., [152, Chaps. 13–15] or [412, Chaps. 7 and 8] for a rigorous treatment of operators.

such that

$$(1) \quad \forall A \in \mathcal{O}, \quad v(A) \in \{\text{eigenvalues of } A\}, \quad (2.F.17)$$

$$(2) \quad \forall A, B \in \mathcal{O}, \quad \text{with } [A, B] = AB - BA = 0, \quad v(AB) = v(A)v(B), \quad (2.F.18)$$

where  $AB$  is the operator product.

### Remark

Since, for  $A \in \mathcal{O}$ , the set of possible results of measurements of  $A$  is  $\mathbb{R}$  and the function  $v : \mathcal{O} \rightarrow \mathbb{R}$ , we do not need to specify a condition like (2.F.2) in part 1 of the theorem for all  $A \in \mathcal{O}$  (that is why the condition only refers to eigenvalues). And, as in the proof of the first part of the theorem, the first condition (2.F.17) is only used for  $A = -\mathbf{1}$ ,  $\mathbf{1}$  being the unit operator, in the form  $v(-\mathbf{1}) = -1$ .

### Proof

We choose the following functions of the operators  $Q_i, P_i$ :

$$A_1 = \cos(aQ_1), \quad A_2 = \cos(aQ_2), \quad B_1 = \cos \frac{\pi P_1}{a}, \quad B_2 = \cos \frac{\pi P_2}{a},$$

where  $a$  is an arbitrary constant, and the functions are defined by (2.F.14), (2.F.15), and the Euler relations:

$$\begin{aligned} \cos(aQ_j) &= \frac{\exp(iaQ_j) + \exp(-iaQ_j)}{2}, \\ \cos \frac{\pi P_j}{a} &= \frac{\exp(i\pi P_j/a) + \exp(-i\pi P_j/a)}{2}, \end{aligned} \quad (2.F.19)$$

for  $j = 1, 2$ . By applying (2.F.18) several times to pairs of commuting operators, we will derive a contradiction.

We have the relations

$$[A_1, A_2] = [B_1, B_2] = [A_1, B_2] = [A_2, B_1] = 0, \quad (2.F.20)$$

since these operators act on different variables, and

$$A_1 B_1 = -B_1 A_1, \quad A_2 B_2 = -B_2 A_2. \quad (2.F.21)$$

To prove (2.F.21), note that, from (2.F.14) and (2.F.15), one gets

$$U_j(b)V_j(c) = \exp(-ibc)V_j(c)U_j(b), \quad (2.F.22)$$

for  $j = 1, 2$ , which, for  $bc = \pm\pi$ , means

$$U_j(b)V_j(c) = -V_j(c)U_j(b) . \quad (2.F.23)$$

Now use (2.F.19) to expand the product  $\cos(aQ_j)\cos(\pi P_j/a)$  into a sum of four terms; each term will have the form of the left-hand side of (2.F.22) with  $b = \pm a$ ,  $c = \pm\pi/a$ , whence  $bc = \pm\pi$ . Then applying (2.F.23) to each term proves (2.F.21).

The relations (2.F.20) and (2.F.21) imply

$$A_1A_2B_1B_2 = B_1B_2A_1A_2 , \quad A_1B_2A_2B_1 = A_2B_1A_1B_2 . \quad (2.F.24)$$

Let  $v(Q_1) = q_1$ ,  $v(Q_2) = q_2$ ,  $v(P_1) = p_1$ , and  $v(P_2) = p_2$ . Since the functions  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  can be defined by their Taylor series and we have  $v(Q_1^n) = v(Q_1)^n = q_1^n$  by (2.F.18) ( $Q_1$  commutes with itself), and similarly for  $Q_2$ ,  $P_1$ ,  $P_2$ , it follows that

$$v(A_1) = \cos(aq_1) , \quad v(A_2) = \cos(aq_2) , \quad v(B_1) = \cos \frac{\pi p_1}{a} , \quad v(B_2) = \cos \frac{\pi p_2}{a} . \quad (2.F.25)$$

Since  $A_1$  and  $A_2$  commute, we get from (2.F.18),

$$v(A_1A_2) = v(A_1)v(A_2) ,$$

and similarly,

$$v(B_1B_2) = v(B_1)v(B_2) , \quad v(A_1B_2) = v(A_1)v(B_2) , \quad v(A_2B_1) = v(A_2)v(B_1) . \quad (2.F.26)$$

Consider now the operators  $X = A_1A_2B_1B_2$  and  $Y = A_1B_2A_2B_1$ . Using  $B_2A_2 = -A_2B_2$ , from (2.F.21), we get

$$X = -Y . \quad (2.F.27)$$

On the other hand, since by (2.F.24)  $A_1A_2$  commutes with  $B_1B_2$ , we have from (2.F.18),

$$v(X) = v(A_1A_2B_1B_2) = v(A_1A_2)v(B_1B_2) = v(A_1)v(A_2)v(B_1)v(B_2) , \quad (2.F.28)$$

where, in the last equality, we use (2.F.26). Similarly, since by (2.F.24)  $A_1B_2$  commutes with  $A_2B_1$ ,

$$v(Y) = v(A_1B_1)v(A_2B_2) = v(A_1)v(B_2)v(A_2)v(B_1) . \quad (2.F.29)$$

Comparing (2.F.28) and (2.F.29), we see that

$$v(X) = v(Y) ,$$

while (2.F.27) implies  $v(X) = v(-Y) = v(-\mathbf{1}Y) = v(-\mathbf{1})v(Y) = -v(Y)$ . This means that  $v(X) = v(Y) = 0$  and hence that one of the four quantities  $v(A_1)$ ,  $v(A_2)$ ,  $v(B_1)$ , or  $v(B_2)$  vanishes, and this obviously cannot hold for all values of  $a$  in (2.F.25) and given values of  $q_1, q_2, p_1, p_2$ . ■



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