Chapter 2
Metrics on Modular Spaces

Abstract In this chapter, we address the metrizability of modular spaces.

2.1 Modular Spaces

A pseudomodular \( w \) on \( X \) (cf. Fig. 1.2 on p. 5) induces an equivalence relation \( \sim \) on \( X \) as follows: given \( x, y \in X \),

\[ x \sim y \quad \text{iff} \quad w^{x,y} \neq \infty \quad \text{iff} \quad w_{\lambda}(x, y) < \infty \quad \text{for some} \quad \lambda > 0, \]

where \( \lambda = \lambda(x, y) \), possibly, depends on \( x \) and \( y \). A modular space is any equivalence class with respect to \( \sim \). More explicitly, let us fix an element \( x^o \in X \). The set

\[ X_w^* \equiv X_w^*(x^o) = \{ x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } w_{\lambda}(x, x^o) < \infty \} \]

is called a modular space (around \( x^o \)), and \( x^o \) is called the center of \( X_w^* \) (\( x^o \) is a representative of the equivalence class \( X_w^* \)). Note that \( w^{x,y} \neq \infty \) for all \( x, y \in X_w^* \).

If \( w_{+0} \) and \( w_{-0} \) are the right and left regularizations of \( w \), then (1.2.4) imply

\[ X_{w_{+0}}^* = X_{w_{-0}}^* = X_w^*. \]

Two more modular spaces (around \( x^o \)) can be defined making use of other equivalence relations on \( X \):

\[ X_w^0 \equiv X_w^0(x^o) = \{ x \in X : w_{\lambda}(x, x^o) \to 0 \text{ as } \lambda \to \infty \} \]

and

\[ X_w^{\infty} \equiv X_w^{\infty}(x^o) = \{ x \in X : w_{\lambda}(x, x^o) < \infty \text{ for all } \lambda > 0 \}. \]

As above, \( X_{w_{+0}}^0 = X_{w_{-0}}^0 = X_w^0 \) and \( X_{w_{+0}}^{\infty} = X_{w_{-0}}^{\infty} = X_w^{\infty} \).

Clearly, \( X_w^0 \subset X_w^* \) and \( X_w^{\infty} \subset X_w^* \) (with proper inclusions in general). However, if \( w \) is convex, then \( X_w^0 = X_w^* \) (see Proposition 1.2.3(c)); moreover, note that this property is independent of the center \( x^o \), i.e., \( X_w^0(x^o) = X_w^*(x^o) \) for all \( x^o \in X \).
Example 2.1.1. The inclusion relations between the three modular spaces are illustrated by the modular \( w_\lambda(x, y) = g(\lambda)d(x, y) \) on a metric space \((X, d)\) from (1.3.1):

\[
X^*_w = \begin{cases} 
\{\infty\} & \text{if } g \equiv \infty, \\
X & \text{if } g \not\equiv \infty,
\end{cases}
X^0_w = \begin{cases} 
\{\infty\} & \text{if } \lim_{\lambda \to \infty} g(\lambda) \neq 0, \\
X & \text{if } \lim_{\lambda \to \infty} g(\lambda) = 0,
\end{cases}
\]

and

\[
X^\text{fin}_w = \begin{cases} 
\{\infty\} & \text{if } g(\lambda) = \infty \text{ for some } \lambda > 0, \\
X & \text{if } g(\lambda) < \infty \text{ for all } \lambda > 0.
\end{cases}
\]

In particular, for modulars \( w_\lambda(x, y) = d(x, y) \) (nonconvex) and \( w_\lambda(x, y) = d(x, y)/\lambda \) (convex) from Example 1.3.2(a), we have

\[
X^0_w = \{\infty\} \subset X^*_w = X^\text{fin}_w = X = X^0_w = X^*_w = X^\text{fin}_w.
\]

In the sequel, by the modular space we mean the set \( X^*_w \) (the largest among the three) if not explicitly stated otherwise.

2.2 The Basic Metric

We begin by introducing the basic (pseudo)metric \( d^0_w \) on the modular space \( X^*_w \).

Theorem 2.2.1. Let \( w \) be a (pseudo)modular on \( X \). Set

\[
d^0_w(x, y) = \inf \{ \lambda > 0 : w_\lambda(x, y) \leq \lambda \}, \quad x, y \in X \quad (\inf \emptyset = \infty).
\]

Then \( d^0_w \) is an extended (pseudo)metric on \( X \). Furthermore, if \( x, y \in X \), \( d^0_w(x, y) < \infty \) is equivalent to \( x \sim y \), and so, \( d^0_w \) is a (pseudo)metric on \( X^*_w = X^\text{fin}_w(\{\infty\}) \) (for any \( x^\circ \in X \)).

Proof. 1. Clearly, \( d^0_w(x, y) \in [0, \infty] \), \( d^0_w(x, x) = 0 \), and \( d^0_w(x, y) = d^0_w(y, x) \) for all \( x, y \in X \). Now, suppose \( w \) is a modular on \( X \), and \( x, y \in X \) are such that \( d^0_w(x, y) = 0 \). The definition of \( d^0_w \) implies \( w_\mu(x, y) \leq \mu \) for all \( \mu > 0 \). So, for all \( \lambda > 0 \) and \( 0 < \mu < \lambda \), we have from (1.2.1): \( w_\lambda(x, y) \leq w_\mu(x, y) \leq \mu \to 0 \) as \( \mu \to +0 \). Thus \( w_\lambda(x, y) = 0 \) for all \( \lambda > 0 \), and so, by axiom (i), \( x = y \).

In order to prove the triangle inequality \( d^0_w(x, y) \leq d^0_w(x, z) + d^0_w(z, y) \) for all \( x, y, z \in X \), we assume that \( d^0_w(x, z) \) and \( d^0_w(z, y) \) are finite (otherwise, the inequality is obvious). By the definition of \( d^0_w \), given \( \lambda > d^0_w(x, z) \) and \( \mu > d^0_w(z, y) \), we find \( w_\lambda(x, z) \leq \lambda \) and \( w_\mu(z, y) \leq \mu \), and so, axiom (iii) implies

\[
w_{\lambda + \mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y) \leq \lambda + \mu.
\]

It follows that \( d^0_w(x, y) \leq \lambda + \mu \), and it remains to take into account the arbitrariness of \( \lambda \) and \( \mu \) as above.
2. If \( d_w^0(x, y) < \infty \), then, for any \( \lambda > d_w^0(x, y) \), we have \( w_\lambda(x, y) \leq \lambda < \infty \), which means that \( x \sim y \). Conversely, suppose \( x \sim y \), i.e., \( w_\mu(x, y) < \infty \) for some \( \mu > 0 \). We set \( \lambda = \max\{\mu, w_\mu(x, y)\} \). Since \( \lambda \geq \mu \), the monotonicity (1.2.1) of \( w \) implies \( w_\lambda(x, y) \leq w_\mu(x, y) \leq \lambda \), and so, \( d_w^0(x, y) \leq \lambda < \infty \).

3. Given \( x, y \in X^*_w \), we have \( x \sim y \), and so, \( d_w^0(x, y) < \infty \). By step 1, this means that \( d_w^0 \) is a (pseudo)metric on \( X^*_w \).

The pair \((X^*_w, d_w^0)\), being a (pseudo)metric space generated by the (pseudo)modular \( w \), is called a (pseudo)metric modular space, and we will apply this terminology if we are interested in metric properties of \( X^*_w \) with respect to \( d_w^0 \) (or some other metric induced by \( w \)). We call \( X^*_w \) the modular space if the main concern is its modular properties (Sects. 4.2 and 4.3), which are outside the scope of metric properties.

Example 2.2.2. Suppose \( w_\lambda(x, y) = g(\lambda)d(x, y) \) is the modular from (1.3.1), where \( g : (0, \infty) \to [0, \infty] \) is a nonincreasing function, \( g \not\equiv 0 \), and \( g \not\equiv \infty \). In the examples 1–6 below, we have \( X^*_w = X \), and \( x, y \in X \) and \( \lambda_0 > 0 \) are given.

1. If \( g(\lambda) = 1/\lambda^p \) (\( p \geq 0 \)), then \( d_w^0(x, y) = (d(x, y))^{1/(p+1)} \).
2. Let \( g(\lambda) = 1 \) if \( 0 < \lambda < \lambda_0 \), and \( g(\lambda) = 0 \) if \( \lambda \geq \lambda_0 \). Then \( w \) is nonstrict and nonconvex, and \( d_w^0(x, y) = \min\{\lambda_0, d(x, y)\} \).
3. If \( g(\lambda) = 1/\lambda \) for \( 0 < \lambda < \lambda_0 \), and \( g(\lambda) = 0 \) for \( \lambda \geq \lambda_0 \), then \( w \) is nonstrict and convex, and \( d_w^0(x, y) = \min\{\lambda_0, \sqrt{d(x, y)}\} \).
4. For \( g(\lambda) = \max\{1, 1/\lambda\} \), we have: \( w \) is strict and convex, and \( d_w^0 \) is given by \( d_w^0(x, y) = \max\{d(x, y), \sqrt{d(x, y)}\} \).
5. If \( g(\lambda) = \infty \) for \( 0 < \lambda < \lambda_0 \), and \( g(\lambda) = 0 \) for \( \lambda \geq \lambda_0 \), then \( w \) is nonstrict and convex, and \( d_w^0(x, y) = \lambda_0\delta(x, y) \), where \( \delta \) is the discrete metric on \( X \).
6. Putting \( d = \delta \), for any function \( g \) as above, we have \( d_w^0(x, y) = g^0\delta(x, y) \) with \( g^0 = \inf \{\lambda > 0 : g(\lambda) \leq \lambda\} \).

Remark 2.2.3. 1. If \( \rho \) is a classical modular on a real linear space \( X \) (cf. Sect. 1.3.3), the set \( X_\rho = \{x \in X : \lim_{\alpha \to +0} \rho(\alpha x) = 0\} \) is called the modular space (with zero as its center). The modular space \( X_\rho \) is a linear subspace of \( X \), and the functional \( | \cdot |_\rho : X_\rho \to [0, \infty) \), given by \( |x|_\rho = \inf \{\varepsilon > 0 : \rho(x/\varepsilon) \leq \varepsilon\} \), is an \( F \)-norm on \( X_\rho \), i.e., given \( x, y \in X_\rho \), it satisfies the conditions: (F.1) \( |x|_\rho = 0 \) iff \( x = 0 \); (F.2) \( |x|_\rho = |x|_\rho \); (F.3) \( |x+y|_\rho \leq |x|_\rho + |y|_\rho \); and (F.4) \( |c_\mu x_n - c\mu x|_\rho \to 0 \) as \( n \to \infty \) whenever \( c_n \to c \) in \( \mathbb{R} \) and \( |x_n - x|_\rho \to 0 \) as \( n \to \infty \) (where \( x_n \in X_\rho \) for \( n \in \mathbb{N} \)). The modular space \( X_\rho^0 \), which is a counterpart of \( X_\rho \), does not play that significant role in our theory as \( X_\rho \) does in the classical theory of modulars (see also Remark 2.4.3(3)).

2. Under the assumptions of Proposition 1.3.5, where \( X \) is a real linear space and \( \rho(x) = w_1(x, 0) \), we also have: \( X_\rho = X_\rho^0(0) \) is a linear subspace of \( X \), and the functional \( |x|_\rho = d_w^0(x, 0), x \in X_\rho \), is an \( F \)-norm on \( X_\rho \).
In Theorem 2.2.1 (and Example 2.2.2(6)), we have encountered the quantity
\[ g^0 = \inf \{ \lambda > 0 : g(\lambda) \leq \lambda \}, \quad (2.2.1) \]
evaluated at the nonincreasing function \( g = w^{x,y} : (0, \infty) \to [0, \infty] \), which we denoted by \( d_w^0(x,y) = (w^{x,y})^0 \). This quantity is worth a more detailed study.

**Lemma 2.2.4.** If \( g : (0, \infty) \to [0, \infty] \) is a nonincreasing function, then \( g^0 \in [0, \infty], \) and

(a) \( g^0 = \inf_{\lambda > 0} \max \{ \lambda, g(\lambda) \} \) (where \( \max \{ \lambda, \infty \} = \infty \) for \( \lambda > 0 \));
(b) \( g^0 < \infty \) if and only if \( g \not\equiv \infty \) (so, \( g^0 = \infty \iff g \equiv \infty \));
(c) \( g^0 \neq 0 \) if and only if \( g \not\equiv 0 \) (so, \( g^0 = 0 \iff g \equiv 0 \)).

**Proof.** 1. Let us prove inequality (\( \leq \)) in (a) and implication (\( \Leftarrow \)) in (b). We may assume \( g \not\equiv \infty \) (otherwise, (a) reads \( \inf \emptyset = \infty \) and holds trivially). For each \( \lambda > 0 \) such that \( g(\lambda) < \infty \), we set \( \lambda_1 = \max \{ \lambda, g(\lambda) \} \). Then \( \lambda_1 \in (0, \infty) \), \( g(\lambda_1) \leq \lambda_1 \), and since \( \lambda \leq \lambda_1 \) and \( g \) is nonincreasing, \( g(\lambda_1) \leq g(\lambda) \). So, \( g(\lambda_1) \leq \lambda_1 \). It follows that \( g^0 \leq \lambda_1 = \max \{ \lambda, g(\lambda) \} \). This proves (b)(\( \Leftarrow \)). Taking the infimum over all \( \lambda > 0 \) such that \( g(\lambda) < \infty \) (or over all \( \lambda > 0 \)), we establish the inequality \( g^0 \leq \ldots \) in (a).

2. Let us prove inequality (\( \geq \)) in (a) and implication (\( \Rightarrow \)) in (b). Suppose \( g^0 \) is finite. Given \( \lambda_1 > g^0 \), we have \( g(\lambda_1) \leq \lambda_1 \), and so, \( g \not\equiv \infty \). This establishes (b)(\( \Rightarrow \)). Moreover (note that the monotonicity of \( g \) is not used),
\[ \inf_{\lambda > 0} \max \{ \lambda, g(\lambda) \} \leq \inf_{\lambda > 0; g(\lambda) < \infty} \max \{ \lambda, g(\lambda) \} \leq \max \{ \lambda_1, g(\lambda_1) \} = \lambda_1. \]

Passing to the limit as \( \lambda_1 \to g^0 \), we obtain the inequality \( g^0 \geq \ldots \) in (a).

3. (c)(\( \Rightarrow \)) If \( g \equiv 0 \), then \( g^0 = \inf \{ 0, \infty \} = 0 \) (equivalently, if \( g^0 \neq 0 \), then \( g \not\equiv 0 \)).
\[ (c)(\Leftarrow) \text{ Let } g^0 = 0. \text{ Then } g(\mu) \leq \mu \text{ for all } \mu > 0. \text{ Given } \lambda > 0, \text{ for any } 0 < \mu < \lambda, \text{ by virtue of the monotonicity of } g, \text{ we get } 0 \leq g(\lambda) \leq g(\mu) \leq \mu. \text{ Letting } \mu \to +0, \text{ we find } g(\lambda) = 0 \text{ for all } \lambda > 0, \text{ i.e., } g \equiv 0. \text{ In other words, we have shown that } g \not\equiv 0 \text{ implies } g^0 \neq 0. \]

\[ \square \]

**Remark 2.2.5.** It is seen from the proof of Lemma 2.2.4(a) that
\[ g^0 = \inf \{ \max \{ \lambda, g(\lambda) \} : \lambda > 0 \text{ such that } g(\lambda) < \infty \} \in [0, \infty) \text{ if } g \not\equiv \infty. \]

Following the same lines as in the proof of Lemma 2.2.4, it may be shown that \( g^0 = \sup \{ \lambda > 0 : g(\lambda) \geq \lambda \} \) (sup \( \emptyset = 0 \)) and \( g^0 = \sup_{\lambda > 0} \min \{ \lambda, g(\lambda) \} \).

As a consequence of Theorem 2.2.1 and Lemma 2.2.4, we get the following

**Corollary 2.2.6.** \( d_w^0(x,y) = \inf_{\lambda > 0} \max \{ \lambda, w_\lambda(x,y) \}, x,y \in X. \)

Given a nonincreasing function \( g : (0, \infty) \to [0, \infty] \), we denote by \( g_{+0} \) and \( g_{-0} \) the right and left regularizations of \( g \), defined (as in (1.2.2) and (1.2.3)) by:
\[ g_+ (\lambda) = g(\lambda + 0) \] and \[ g_- (\lambda) = g(\lambda - 0) \] for all \( \lambda > 0 \). Functions \( g_+ \) and \( g_- \) map \( (0, \infty) \) into \([0, \infty]\) and are nonincreasing on \((0, \infty)\). Furthermore, \( g_+ \) is continuous from the right and \( g_- \) is continuous from the left on \((0, \infty)\), and inequalities similar to (1.2.4) hold:

\[
g(\lambda) \leq g(\lambda - 0) \leq g(\mu + 0) \leq g(\mu) \quad \text{in } [0, \infty] \text{ for all } 0 < \mu < \lambda.
\] (2.2.2)

Taking the above and (2.2.1) into account, we have

**Lemma 2.2.7.** If \( g : (0, \infty) \rightarrow [0, \infty] \) is nonincreasing, then \( (g_+)^0 = g^0 = (g_-)^0 \).

**Proof.** Inequalities \((g_+)^0 \leq g^0 \leq (g_-)^0\) are consequences of the inclusions

\[
\{ \lambda > 0 : g(\lambda - 0) \leq \lambda \} \subset \{ \lambda > 0 : g(\lambda) \leq \lambda \} \subset \{ \lambda > 0 : g(\lambda + 0) \leq \lambda \},
\]

which follow from (2.2.2). Now, we may assume that \( g \neq \infty \). Then \( g_+ \neq \infty \) and \( g_- \neq \infty \), which ensures that \( g^0, (g_+)^0, \) and \( (g_-)^0 \) are finite.

Let us show that \( g^0 \leq (g_+)^0 \). Given \( \lambda > (g_+)^0 \), choose \( \mu \) such that \( (g_+)^0 < \mu < \lambda \). By (2.2.2) and definition of \( (g_+)^0 \), we get

\[
g(\lambda) \leq g(\mu + 0) = g_+ (\mu) \leq \mu < \lambda.
\]

Hence \( g^0 \leq \lambda \). Since \( \lambda > (g_+)^0 \) is arbitrary, we find \( g^0 \leq (g_+)^0 \).

In order to show that \( (g_-)^0 \leq g^0 \), we let \( \lambda > g^0 \). Then, for any \( \mu > 0 \) such that \( g^0 < \mu < \lambda \), inequalities (2.2.2) and definition of \( g^0 \) imply

\[
(g_-)^0 (\lambda) = g(\lambda - 0) \leq g(\mu) \leq \mu < \lambda.
\]

Therefore \( (g_-)^0 \leq \lambda \). Letting \( \lambda \rightarrow g^0 \), we get \( (g_-)^0 \leq g^0 \).

\[\square\]

Putting, for a (pseudo)modular \( w \) on \( X \), \( g = w^{x, y} \) in Lemma 2.2.7 and noting that \( g_{\pm 0} = (w_{\pm 0})^{x, y} \) and \( d^0_{w_{\pm 0}} (x, y) = (g_{\pm 0})^0 \), we have

**Corollary 2.2.8.** \( d^0_{w_{+0}} (x, y) = d^0_{w_{-0}} (x, y) = d^0_w (x, y) \) for all \( x, y \in X \).

In particular, if \( w \) and \( W \) are (pseudo)modulars on \( X \) such that \( w_{+0} = W_{+0} \) or \( w_{-0} = W_{-0} \), then \( d^0_w = d^0_W \) on \( X \times X \).

We conclude that the right and left regularizations of a (pseudo)modular \( w \) on \( X \) provide no new modular spaces as compared to \( X^*_w, X^0_w \) and \( X^{\text{fin}}_w \) (cf. Sect. 2.1) and no new (pseudo)metrics as compared to \( d^0_w \).

Yet, in Sect. 2.5, we establish the existence of continuum many (equivalent) metrics on the modular space \( X^*_w \).

This section is continued by studying the basic metric \( d^0_w (x, y) \) at the level of the map \( g \mapsto g^0 \), applied later to nonincreasing functions \( g = w^{x, y} \). Our next lemma clarifies the definition of \( g^0 \) and Lemma 2.2.7 and, along with (2.2.1), gives a method for evaluating \( g^0 \) in terms of solutions of certain inequalities.
Lemma 2.2.9 (inequalities for $g^0$). Let $g : (0, \infty) \to [0, \infty]$ be a nonincreasing function with $0 < g^0 < \infty$ (i.e., $g \not\equiv 0$ and $g \not\equiv \infty$), and $\lambda > 0$. We have:

(a) $g^0 < \lambda$ if and only if $g(\lambda - 0) < \lambda$;
(b) $g^0 > \lambda$ if and only if $g(\lambda + 0) > \lambda$;
(c) $g^0 = \lambda$ if and only if $g(\lambda + 0) \leq \lambda \leq g(\lambda - 0)$.

Proof. (a) Suppose $g^0 < \lambda$. Given $\lambda_1$ and $\lambda_2$ such that $g^0 < \lambda_1 < \lambda_2 < \lambda$, by the monotonicity of $g$, $g(\lambda_2) \leq g(\lambda_1)$, and the definition of $g^0$ implies $g(\lambda_1) \leq \lambda_1$. Hence $g(\lambda_2) \leq \lambda_1$. Passing to the limits as $\lambda_1 \to g^0$ and $\lambda_2 \to \lambda$, we get $g(\lambda - 0) \leq g^0$, where $g^0 < \lambda$, and so, $g(\lambda - 0) < \lambda$.

(b) By the assumption, $g(\lambda - 0) < \lambda$, where $g(\lambda - 0) = \lim_{\mu \to \lambda -0} g(\mu)$ and $\lambda = \lim_{\mu \to \lambda -0} \mu$. So, there exists $\mu_0$ with $0 < \mu_0 < \lambda$ such that $g(\mu) < \mu$ for all $\mu$ with $\mu_0 \leq \mu < \lambda$. By the definition of $g^0$, we find $g^0 \leq \mu$, which implies $g^0 < \lambda$.

(c) The statement in (a) is equivalent to the following:

\[ g^0 \geq \lambda \text{ if and only if } g(\lambda - 0) \geq \lambda, \]  

(2.2.3)

and the one in (b) is equivalent to the assertion:

\[ g^0 \leq \lambda \text{ if and only if } g(\lambda + 0) \leq \lambda. \]  

(2.2.4)

From these two observations, (c) follows.

Remark 2.2.10. (a) Actually, a little bit more is shown in the proof of Lemma 2.2.9: $g^0 < \lambda \Rightarrow g(\lambda - 0) \leq g^0 < \lambda$ in (a), and $g^0 > \lambda \Rightarrow g(\lambda + 0) \geq g^0 > \lambda$ in (b).

(b) We have $g^0 = \inf \{\lambda > 0 : g(\lambda) < \lambda\} \equiv g^{0''}$ (cf. (2.2.1) and Lemma 2.2.4).

In fact, this is clear if $g \equiv 0$ or $g \equiv \infty$, so let $0 < g^0 < \infty$. Since \{\lambda > 0 : g(\lambda) < \lambda\} \subset \{\lambda > 0 : g(\lambda) \leq \lambda\}, we get $g^0 \leq g^{0''}$. Now, given $\lambda > g^0$, inequalities (2.2.2) and Lemma 2.2.9(a) imply $g(\lambda) \leq g(\lambda - 0) < \lambda$, and so, $g^{0''} \leq \lambda$, which yields $g^{0''} \leq g^0$.

(c) Assuming one-sided continuity of $g$ on $(0, \infty)$, in view of (2.2.4) and (2.2.3), we get some useful particular cases of Lemma 2.2.9:

\[ g^0 \leq \lambda \Leftrightarrow g(\lambda) \leq \lambda, \text{ provided } g \text{ is continuous from the right}; \]
\[ g^0 < \lambda \Leftrightarrow g(\lambda) < \lambda, \text{ provided } g \text{ is continuous from the left}; \]
\[ g^0 = \lambda \Leftrightarrow g(\lambda) = \lambda \text{ (i.e., } \lambda \text{ is a fixed point of } g), \text{ provided } g \text{ is continuous}. \]
To illustrate Lemma 2.2.9, consider $g : (0, \infty) \rightarrow (0, \infty)$ defined by: $g(\lambda) = 3$ if $0 < \lambda < 1$, $g(\lambda) = 2$ if $\lambda = 1$, and $g(\lambda) = 0$ if $\lambda > 1$. Clearly, $g$ is nonincreasing and $g^0 = \inf(1, \infty) = 1$. Inequalities in Lemma 2.2.9(c) are of the form:

$$g(1 + 0) = 0 < g^0 = 1 < 3 = g(1 - 0).$$

Although strict inequality $g(1 - 0) = 3 > 1 = \lambda$ holds in (2.2.3), we have $g^0 = \lambda = 1$. Similarly, $g(1 + 0) = 0 < 1 = \lambda$ in (2.2.4) and $g^0 = 1 = \lambda$.

Setting $g = w^{x,y}$ in Lemma 2.2.9 (for $x, y \in X_\ast^\mu$), we obtain the following important result for modulares $w$ on $X$ (cf. also Remark 2.2.10(a), (c)).

**Theorem 2.2.11.** Let $w$ be a (pseudo)modular on the set $X$, $X_\ast^\mu$ be the modular space, $\lambda > 0$, and $x, y \in X_\ast^\mu$. Then we have:

(a) condition $d^0_w(x, y) < \lambda$ implies $w_{\lambda-0}(x, y) \leq d^0_w(x, y) < \lambda$, and conversely,

(b) inequality $d^0_w(x, y) > \lambda$ implies $w_{\lambda+0}(x, y) > d^0_w(x, y) > \lambda$.

Under the continuity assumptions on $w$, additional equivalences hold:

(d) if $w$ is continuous from the right, then $d^0_w(x, y) \leq \lambda \Leftrightarrow w_{\lambda}(x, y) \leq \lambda$;

(e) if $w$ is continuous from the left, then $d^0_w(x, y) < \lambda \Leftrightarrow w_{\lambda}(x, y) < \lambda$;

(f) if $w$ is continuous on $(0, \infty)$, then $d^0_w(x, y) = \lambda \Leftrightarrow w_{\lambda}(x, y) = \lambda$.

The conclusions of Theorem 2.2.11 are sharp (cf. Remark 2.2.10(d) and (1.3.1)).

**Example 2.2.12.** Let $w$ be given by (1.3.2) with $h(\lambda) = \lambda^p$ ($p > 0$). Since $w$ is continuous on $(0, \infty)$, by virtue of Theorem 2.2.11(f), the value $\lambda = d^0_w(x, y)$ with $x \neq y$ satisfies the equation $w_{\lambda}(x, y) = \lambda$, that is,

$$\lambda^{p+1} + d(x, y)\lambda - d(x, y) = 0. \quad (2.2.5)$$

If $p = 1$, then solving the corresponding quadratic equation, we get

$$d^0_w(x, y) = \frac{\sqrt{(d(x, y))^2 + 4d(x, y) - d(x, y)}}{2}. \quad (2.2.6)$$

For $p = 2$, the solution $\lambda$ of the corresponding cubic equation (2.2.5) is given by Cardano’s formula:

$$d^0_w(x, y) = \frac{1}{3}\left[\frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{a^3}{3}}\right] - \frac{1}{3}\left[\frac{a}{2} - \sqrt{\frac{a^2}{4} - \frac{a^3}{3}}\right], \quad (2.2.7)$$
where \( a = d(x, y) \), and the square and cube roots of positive numbers have uniquely determined positive values. The solution by radicals of the fourth-order equation (for \( p = 3 \)) can be obtained by Ferrari’s method, and is left to the interested reader.

Note that, for any function \( h \) from (1.3.2), we have \( d^0_w(x, y) < 1 \).

In fact, if \( h \) is continuous on \((0, \infty)\), equality \( w_\lambda(x, y) = \lambda \) is of the form \( f(\lambda) = 0 \), where \( f(\lambda) = \lambda h(\lambda) - (1 - \lambda)d(x, y) \), and \( \lambda h(\lambda) \to 0 \) as \( \lambda \to +0 \). Setting \( \lambda h(\lambda) = 0 \) if \( \lambda = 0 \), we find that \( f \) is continuous on \([0, \infty)\), \( f(0) = -d(x, y) < 0 \) (if \( x \neq y \)), and \( f(1) = h(1) > 0 \). By the Intermediate Value Theorem, \( f(\lambda) = 0 \) for some \( 0 < \lambda < 1 \), and so, \( d^0_w(x, y) = \lambda < 1 \).

In the general case, we first show that if there exists \( \mu > 0 \) such that

\[
  w_{\lambda-0}(x, y) < \mu \quad \text{for all } \lambda > 0 \text{ and } x, y \in X, \text{ then } d^0_w(x, y) < \mu \quad \text{for all } x, y \in X.
\]

Since \( w_{\lambda}(x, y) \leq w_{\lambda-0}(x, y) < \mu \), and this holds for \( \lambda = \mu \), we find \( d^0_w(x, y) \leq \mu \).

If we assume that \( d^0_w(x, y) = \mu \) (for some \( x \neq y \)), then, by Theorem 2.2.11(b), we have \( w_{\lambda}(x, y) \geq w_{\lambda+0}(x, y) > \lambda \) for all \( 0 < \lambda < d^0_w(x, y) = \mu \), and so, \( w_{\lambda-0}(x, y) \) is equal to \( \lim_{\lambda \to \mu-0} w_{\lambda}(x, y) \geq \mu \), which contradicts the assumption. It remains to note that \( w_{\lambda-0}(x, y) < 1 = \mu \) for our modular \( w \) from (1.3.2).

One more example of a (pseudo)metric from Theorem 2.2.1 is given by the quantity \( d^0_w \) on the power set \( \mathcal{P}(X) \) of \( X \), where \( W \) is the Hausdorff pseudomodular on \( \mathcal{P}(X) \) induced by a (pseudo)modular \( w \) on \( X \). There are two ways of obtaining a distance function on \( \mathcal{P}(X) \) starting from \( w \) on \( X \), namely

\[
  w \text{ on } X \xrightarrow{\text{Theorem 2.2.1}} d^0_w \text{ on } X \xrightarrow{\text{Appendix A.1}} D_{d^0_w} \text{ on } \mathcal{P}(X)
\]

and

\[
  w \text{ on } X \xrightarrow{\text{Section 1.3.5}} W \text{ on } \mathcal{P}(X) \xrightarrow{\text{Theorem 2.2.1}} d^0_W \text{ on } \mathcal{P}(X).
\]

Fortunately, the resulting distance functions \( D_{d^0_w} \) and \( d^0_W \) coincide on \( \mathcal{P}(X) \) as the following theorem asserts.

**Theorem 2.2.13.** Let \( w \) be a (pseudo)modular on \( X \), \( D = D_{d^0_w} \) be the Hausdorff distance on \( \mathcal{P}(X) \) generated by the extended (pseudo)metric \( d^0_w \) on \( X \), and \( W \) be the Hausdorff pseudomodular on \( \mathcal{P}(X) \) induced by \( w \). Then

\[
  d^0_W(A, B) = D(A, B) \quad \text{for all } A, B \in \mathcal{P}(X).
\]

**Proof.** Since \( d^0_W(\emptyset, \emptyset) = 0 = D(\emptyset, \emptyset) \), and \( d^0_W(A, \emptyset) = \infty = D(A, \emptyset) \) for all \( A \neq \emptyset \), we may assume that \( A \neq \emptyset \) and \( B \neq \emptyset \).

\((\geq)\) Suppose \( d^0_W(A, B) = \inf \{ \lambda > 0 : W_\lambda(A, B) \leq \lambda \} \) is finite, and \( \lambda > d^0_W(A, B) \).

Applying (1.2.4) and Theorem 2.2.11(a) (cf. also Remark 2.2.10(b)), we get...
2.3 The Basic Metric in the Convex Case

\[ W_\lambda(A, B) = \max\{E_\lambda(A, B), E_\lambda(B, A)\} < \lambda, \]

and so, \( E_\lambda(A, B) < \lambda \) and \( E_\lambda(B, A) < \lambda \). By (1.3.12), we have \( \inf_{y \in B} w_\lambda(x, y) < \lambda \) for all \( x \in A \). So, for each \( x \in A \) there exists \( y_x \in B \) (depending also on \( \lambda \)) such that \( w_\lambda(x, y_x) < \lambda \). The definition of \( d_0^w \) gives \( d_0^w(x, y_x) \leq \lambda \). Since

\[ \inf_{y \in B} d_0^w(x, y) \leq d_0^w(x, y_x) \leq \lambda \quad \text{for all } x \in A, \]

we get \( e(A, B) = \sup_{x \in A} \inf_{y \in B} d_0^w(x, y) \leq \lambda \). Similarly, \( E_\lambda(B, A) < \lambda \) implies inequality \( e(B, A) \leq \lambda \). Therefore \( D(A, B) = \max\{e(A, B), e(B, A)\} \leq \lambda \) for all \( \lambda > d_0^w(A, B) \), and so, \( D(A, B) \leq d_0^w(A, B) < \infty \).

\((\leq)\) Let \( D(A, B) < \infty \), and \( \lambda > D(A, B) \) be arbitrary. Then \( \lambda > e(A, B) \) as well as \( \lambda > e(B, A) = \sup_{x \in A} \inf_{y \in B} d_0^w(x, y) \) implies that, given \( x \in A \), \( \lambda > \inf_{x \in B} d_0^w(x, y) \). So, for every \( x \in A \) there exists \( y_x \in B \) (also depending on \( \lambda \)) such that \( \lambda > d_0^w(x, y_x) \). By the definition of \( d_0^w \), we have \( w_\lambda(x, y_x) \leq \lambda \). Since

\[ \inf_{y \in B} w_\lambda(x, y) \leq w_\lambda(x, y_x) \leq \lambda \quad \text{for all } x \in A, \]

we find \( E_\lambda(A, B) = \sup_{x \in A} \inf_{y \in B} w_\lambda(x, y) \leq \lambda \). Similarly, inequality \( \lambda > e(B, A) \) implies \( E_\lambda(B, A) \leq \lambda \). Hence \( W_\lambda(A, B) = \max\{E_\lambda(A, B), E_\lambda(B, A)\} \leq \lambda \). The definition of \( d_0^w \) yields \( d_0^w(A, B) \leq \lambda \) for all \( \lambda > D(A, B) \), and so, \( d_0^w(A, B) \leq D(A, B) < \infty \).

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Now we treat the case when a (pseudo)modular \( w \) on \( X \) is convex: \( w \) gives rise to an additional (pseudo)metric on the modular space \( X_w^* \) to be studied below.

We make use of the following observation. As we have seen in Remark 1.2.2(d), the convexity of a (pseudo)modular \( w \) on \( X \) is equivalent to the fact that the function \( \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y) \) is a (pseudo)modular on \( X \). On the other hand, if a function \( \hat{w} \) on \((0, \infty) \times X \times X \) is initially given, then we have: \( \hat{w} \) is a (pseudo)modular on \( X \) if and only if \( \hat{w}_\lambda(x, y) = \hat{w}_\lambda(x, y)/\lambda \) is a convex (pseudo)modular on \( X \).

From Sect. 2.1, we find

\[ X_w^0 \subset X_w^0 = X_w^* = X_w^* \quad \text{and} \quad X_w^{\text{fin}} \subset X_w^{\text{fin}} \subset X_w^* = X_w^*. \quad (2.3.1) \]

By Theorem 2.2.1, \( \hat{w} \) generates a (pseudo)metric on \( X_w^* \) of the form

\[ d_0^w(x, y) = \inf\{\lambda > 0 : \hat{w}_\lambda(x, y) \leq \lambda\} = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}. \quad (2.3.2) \]

The last expression is given in terms of \( w \) and is denoted by \( d_w^*(x, y) \).
Properties of \( d_w^* \) are gathered in the following theorem, where Theorem 2.2.1 and Corollary 2.2.6 are applied to \( \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y) \) and expressed via \( w \).

**Theorem 2.3.1.** Let \( w \) be a convex (pseudo)modular on \( X \). Then

\[
d_w^*(x, y) \equiv \inf \{ \lambda > 0 : w_\lambda(x, y) \leq 1 \} = \inf_{\lambda > 0} \max\{\lambda, \lambda w_\lambda(x, y)\}, \quad x, y \in X,
\]

(2.3.3)
is an extended (pseudo)metric on \( X \) (with \( d_w^*(x, y) < \infty \) \( \Leftrightarrow \) \( x \sim y \)), whose restriction to the modular space \( X_w^* \) is a (pseudo)metric on \( X_w^* \).

Furthermore, \( d_w^0 \) and \( d_w^* \) are nonlinearly equivalent in the following sense: given \( x, y \in X_w^* \), we have

\[
\min\{d_w^*(x, y), \sqrt{d_w^*(x, y)}\} \leq d_w^0(x, y) \leq \max\{d_w^*(x, y), \sqrt{d_w^*(x, y)}\},
\]

(2.3.4)
or, equivalently (written in a different way),

\[
d_w^0(x, y) \cdot \min\{1, d_w^0(x, y)\} \leq d_w^*(x, y) \leq d_w^0(x, y) \cdot \max\{1, d_w^0(x, y)\}.
\]

(2.3.5)

Only the second part of Theorem 2.3.1 is to be verified. For this, we need some precise inequalities for \( d_w^* = d_w^0 \), which are reformulated from Theorem 2.2.11 (applied to \( \hat{w} \)) in terms of \( w \) and stated, for ease of reference, as

**Theorem 2.3.2.** Let \( w \) be a convex (pseudo)modular on \( X \), \( \lambda > 0 \), and \( x, y \in X_w^* \).

Then we have:

1. \( d_w^*(x, y) < \lambda \) implies \( w_{\lambda-0}(x, y) \leq d_w^*(x, y)/\lambda < 1 \), and conversely,
2. \( w_{\lambda-0}(x, y) < 1 \) implies \( d_w^*(x, y) < \lambda \);
3. \( d_w^*(x, y) > \lambda \) implies \( w_{\lambda+0}(x, y) \geq d_w^*(x, y)/\lambda > 1 \), and conversely,
4. \( w_{\lambda+0}(x, y) > 1 \) implies \( d_w^*(x, y) > \lambda \);
5. \( d_w^*(x, y) = \lambda \) is equivalent to \( w_{\lambda+0}(x, y) \leq 1 \leq w_{\lambda-0}(x, y) \).

In addition, under the continuity assumptions on \( w \), we get:

1. \( d_w^*(x, y) \leq \lambda \Leftrightarrow w_{\lambda}(x, y) \leq 1 \), provided \( w \) is continuous from the right;
2. \( d_w^*(x, y) < \lambda \Leftrightarrow w_{\lambda}(x, y) < 1 \), provided \( w \) is continuous from the left;
3. \( d_w^*(x, y) = \lambda \Leftrightarrow w_{\lambda}(x, y) = 1 \), provided \( w \) is continuous on \((0, \infty)\).

**Proof (of Theorem 2.3.1 (second part)).** In steps 1 and 2, we show that inequalities \( d_w^0(x, y) < 1 \) and \( d_w^*(x, y) < 1 \) are equivalent, and if one of them holds, then

\[
d_w^*(x, y) \leq d_w^0(x, y) \leq \sqrt{d_w^*(x, y)}.
\]

(2.3.6)

Since \( d_w^*(x, y) < 1 \) implies \( d_w^*(x, y) \leq \sqrt{d_w^*(x, y)} \), inequality (2.3.6) proves (2.3.4).

1. Suppose \( d_w^0(x, y) < 1 \). Let us show that \( d_w^*(x, y) \leq d_w^0(x, y) \) (and so, \( d_w^*(x, y) < 1 \)).

In fact, for any number \( \lambda \) such that \( d_w^0(x, y) < \lambda < 1 \), the definition of \( d_w^0 \) gives
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2. Assume that \(d_w^*(x, y) < 1\). Let us prove that \(d_w^0(x, y) \leq \sqrt{d_w^*(x, y)}\), which is the right-hand side inequality in (2.3.6) (and so, \(d_w^0(x, y) < 1\)). Since \(d_w^*(x, y) \leq \sqrt{d_w^*(x, y)} < 1\), for any \(\lambda\) such that \(\sqrt{d_w^*(x, y)} < \lambda < 1\), inequalities (1.2.4) and, by virtue of convexity of \(w\), Theorem 2.3.2(a) imply

\[
w^*_\lambda(x, y) \leq w^*_{\lambda, 0}(x, y) \leq \frac{d_w^*(x, y)}{\lambda} < \frac{\lambda^2}{\lambda} = \lambda.
\]

By the definition of \(d_w^0\), \(d_w^0(x, y) \leq \lambda\). Letting \(\lambda\) tend to \(\sqrt{d_w^*(x, y)}\), we obtain the desired inequality.

As a consequence of steps 1 and 2, inequalities \(d_w^0(x, y) \geq 1\) and \(d_w^*(x, y) \geq 1\) are equivalent, as well. In steps 3 and 4, we show that if one of these inequalities holds, then

\[
\sqrt{d_w^*(x, y)} \leq d_w^0(x, y) \leq d_w^*(x, y).
\]

(2.3.7)

Since \(d_w^*(x, y) \geq 1\) implies \(d_w^*(x, y) \geq \sqrt{d_w^*(x, y)}\), (2.3.7) establishes (2.3.4).

3. Inequality \(d_w^0(x, y) \geq 1\) implies \(d_w^0(x, y) \leq d_w^*(x, y)\): in fact, by the definition of \(d_w^*\), \(w^*_{\lambda}(x, y) \leq 1\) for all \(\lambda > d_w^*(x, y)\), and since \(\lambda > 1\), \(w^*_{1}(x, y) < \lambda\). From the definition of \(d_w^0\), we get \(d_w^0(x, y) \leq \lambda\). The assertion follows thanks to the arbitrariness of \(\lambda > d_w^*(x, y)\).

4. Suppose \(d_w^0(x, y) \geq 1\), and let us show that \(\sqrt{d_w^*(x, y)} \leq d_w^0(x, y)\), which is the left-hand side inequality in (2.3.7). Given \(\lambda > d_w^0(x, y)\), we have \(w^*_{\lambda}(x, y) \leq \lambda\), and since \(\lambda > 1\), \(\lambda^2 > \lambda\). The convexity of \(w\) and (1.2.5) imply

\[
w^*_\lambda(x, y) \leq \frac{\lambda}{\lambda^2} w^*_{\lambda}(x, y) \leq \frac{\lambda}{\lambda^2} \cdot \lambda = 1,
\]

whence \(d_w^*(x, y) \leq \lambda^2\). Letting \(\lambda\) go to \(d_w^0(x, y)\), we get \(d_w^*(x, y) \leq (d_w^0(x, y))^2\).

\[\square\]

Remark 2.3.3. 1. If \(w\) is nonconvex, the quantity \(d_w^*(x, y) \in [0, \infty]\) from (2.3.3) has only two properties: \(d_w^*(x, x) = 0\), and \(d_w^*(x, y) = d_w^*(y, x)\). It follows from (2) in this Remark that \(d_w^*(x, y) = 0 \not\Rightarrow x = y\), and from (4)—that the triangle inequality may not hold for \(d_w^*\).

2. The convexity of \(w\) is essential for inequalities (2.3.4) and (2.3.5): modular (1.3.2) is nonconvex, and \(d_w^0\) is a well-defined metric on \(X\) (e.g., (2.2.6) and (2.2.7)), but, since \(w^*_\lambda(x, y) < 1\) for all \(\lambda > 0\), we have \(d_w^*(x, y) = 0\) for all \(x, y \in X\) (and, in particular, \(d_w^*\) is not a metric on \(X\)).

3. In the proof of Theorem 2.3.1, the implications in steps 1 and 3, which are of the form \(d_w^0(x, y) < 1 \Rightarrow d_w^*(x, y) \leq d_w^0(x, y)\), and \(d_w^*(x, y) \geq 1 \Rightarrow d_w^0(x, y) \leq d_w^*(x, y)\), do not rely on the convexity of \(w\) and are valid for those (pseudo)modulars \(w\), for which the quantity \(d_w^*(x, y)\) is well-defined. The example in (2) above is consistent with the former implication.
4. For the modular \(w_\lambda(x,y) = d(x,y)/\lambda^p\) (\(p > 0\)) from Example 2.2.2(1), we have \(d^0_w(x,y) = (d(x,y))^{1/(p+1)}\) and \(d^*_w(x,y) = (d(x,y))^{1/p}\), where we note that \(d^*_w\) is a metric on \(X\) if and only if \(w\) is convex, i.e., \(p \geq 1\). So, for \(p \geq 1\), setting \(a = d(x,y)\), inequalities (2.3.6) and (2.3.7) assume the form:

\[
\frac{a^p}{p+1} \leq a^\frac{1}{p+1} \leq a^\frac{1}{p} \quad \text{if} \quad 0 \leq a < 1, \quad \text{and} \quad a^\frac{1}{p} \leq a^\frac{1}{p+1} \leq a^\frac{p}{p+1} \quad \text{if} \quad a \geq 1.
\]

5. Inequalities (2.3.4) are the best possible: see Example 2.3.5(1).

Remark 2.3.4. 1. If \(\rho\) is a classical convex modular on a real linear space \(X\) (cf. Sect. 1.3.3 and Remark 2.2.3), then the modular space \(X_\rho\) coincides with the set \(X_\rho^* = \{x \in X : \rho(\alpha x) < \infty \text{ for some } \alpha > 0\}\), and the functional \(\|x\| = \inf \{\varepsilon > 0 : \rho(x/\varepsilon) \leq 1\} (x \in X_\rho^*)\) is a norm on \(X_\rho = X_\rho^*\), which is nonlinearly equivalent to the \(F\)-norm \(|x|\) in the same sense as in Theorem 2.3.1. Moreover, under the assumptions of Proposition 1.3.5, where \(X\) is a linear space and \(\rho(x) = w_1(x,0)\), we have: \(X_\rho^* = X_\rho^*(0) = X_\rho\) is a linear subspace of \(X\), and the functional \(\|x\| = d^*_w(x,0), x \in X_\rho^*,\) is a norm on \(X_\rho^*\).

2. Similar to Corollary 2.2.8, if \(w\) is convex, then \(d_{w^0}^* = d_{w^0}^* = d^*_w\) on \(X \times X\). In fact, \((w^\lambda)_\lambda(x,y) \equiv \hat{\omega}_\lambda(x,y) = \lambda w_\lambda(x,y)\) is also a (pseudo)modular on \(X\), and \((w^\lambda)_{\lambda=0}^\lambda = \hat{(w)}_{\lambda=0}^\lambda = (w^\lambda)_{\lambda=0}^\lambda\), which can be seen as follows. Given \(\lambda > 0\) and \(x,y \in X, (1.2.2)\) and (1.2.3) imply

\[
(\hat{(w^\lambda)})_\lambda(x,y) = \lambda(w^\lambda)_{\lambda=0}^\lambda(x,y) = \lambda w_{\lambda=0}^\lambda(x,y) = \lim_{\mu \to \lambda=0} \mu w_{\mu}(x,y)
\]

\[
= \lim_{\mu \to \lambda=0} (w^\lambda)_{\mu}(x,y) = (w^\lambda)_{\lambda=0}^\lambda(x,y) = ((w^\lambda)_{\lambda=0}^\lambda)(x,y).
\]

By virtue of (2.3.3) and (2.3.2), \(d^*_w = d_{w^\lambda}^*\), and Corollary 2.2.8 yields

\[
d_{w^\lambda=0}^* = d_{w^0}^* = d_{w^\lambda=0}^* = d_{w^\lambda}^* = d^*_w.
\]

Example 2.3.5. Consider the modular \(w_\lambda(x,y) = \varphi(d(x,y)/\lambda)\) from (1.3.5), where the function \(\varphi : [0, \infty) \to [0, \infty]\) is nondecreasing and such that \(\varphi(0) = 0, \varphi \not= 0,\) and \(\varphi \not= \infty\), \((X,d)\) is a metric space, \(x,y \in X_w^* = X,\) and \(\lambda > 0\).

1. Let \(\varphi(u) = u^p\) (\(p > 0\)). Then \(w\) is strict, convex if \(p \geq 1\), and nonconvex if \(0 < p < 1\). For any \(p > 0\), we have

\[
d^0_w(x,y) = (d(x,y))^{p/(p+1)} \quad \text{and} \quad d^*_w(x,y) = d(x,y).
\]

To show that inequalities (2.3.4) are the best possible, we note that if \(p = 1\), then \(d^0_w(x,y) = \sqrt{d^*_w(x,y)}\), and if \(p > 1\), then \((w\) is convex and) we find

\[
d^0_w(x,y) = (d^*_w(x,y))^{p/(p+1)} \to d^*_w(x,y) \quad \text{as} \quad p \to \infty.
\]
2. Let \( w \) be the \((a, 0)\)-modular from (1.3.9). If \( a = \infty \), then \( w \) is nonstrict and convex, and we have: \( d_w^0(x, y) = d_w^*(x, y) = d(x, y) \). Now, if \( a > 0 \), then \( w \) is nonstrict and nonconvex, and we have: \( d_w^0(x, y) = \min\{a, d(x, y)\} \), \( d_w^*(x, y) = 0 \) if \( a \leq 1 \), and \( d_w^*(x, y) = d(x, y) \) if \( a > 1 \).

3. If \( \varphi(u) = u \) for \( 0 \leq u \leq 1 \), and \( \varphi(u) = 1 \) for \( u > 1 \), then the modular

\[
  w_\lambda(x, y) = 1 \text{ if } 0 < \lambda < d(x, y), \quad \text{and } w_\lambda(x, y) = \frac{d(x, y)}{\lambda} \text{ if } \lambda \geq d(x, y),
\]

is strict and nonconvex, and \( d_w^0(x, y) = \min\{1, \sqrt{d(x, y)}\} \).

4. Let \( \varphi(u) = 0 \) for \( 0 \leq u \leq 1 \), and \( \varphi(u) = u - 1 \) for \( u > 1 \). We have:

\[
  w_\lambda(x, y) = \frac{d(x, y)}{\lambda} - 1 \text{ if } 0 < \lambda < d(x, y), \quad \text{and } w_\lambda(x, y) = 0 \text{ if } \lambda \geq d(x, y),
\]

is nonstrict and convex, and (note that \( d_w^0(x, y) < d(x, y) \) if \( x \neq y \))

\[
  d_w^0(x, y) = \frac{\sqrt{1 + 4d(x, y)} - 1}{2} \quad \text{and} \quad d_w^*(x, y) = \frac{d(x, y)}{2}.
\]

5. Suppose \( \varphi(0) = 0 \), \( \varphi(u) = 1 \) if \( 0 < u < 1 \), and \( \varphi(u) = u \) if \( u \geq 1 \). Given \( \lambda > 0 \) and \( x, y \in X \), we have: \( w_\lambda(x, y) = 0 \) if \( x = y \), and if \( x \neq y \),

\[
  w_\lambda(x, y) = \frac{d(x, y)}{\lambda} \text{ if } 0 < \lambda \leq d(x, y), \quad \text{and } w_\lambda(x, y) = 1 \text{ if } \lambda > d(x, y).
\]

Then the modular \( w \) is strict and nonconvex, \( d_w^0(x, y) = \max\{1, \sqrt{d(x, y)}\} \) if \( x \neq y \), and \( d_w^0(x, y) = 0 \) if \( x = y \).

6. Suppose \( \varphi \) is given by: \( \varphi(u) = u \) if \( 0 \leq u \leq 1 \), \( \varphi(u) = 1 \) if \( 1 < u < 2 \), and \( \varphi(u) = u - 1 \) if \( u \geq 2 \). The corresponding modular \( w \) is strict and nonconvex, and we have: \( d_w^0(x, y) = \sqrt{d(x, y)} \) if \( d(x, y) \leq 1 \), \( d_w^0(x, y) = 1 \) if \( 1 < d(x, y) < 2 \), and \( d_w^0(x, y) = \frac{1}{2}(\sqrt{1 + 4d(x, y)} - 1) \) if \( d(x, y) \geq 2 \).

### 2.4 Moduliars and Metrics on Sequence Spaces

Let \((M, d)\) be a metric space, \(X = M^\mathbb{N}\)—the set of all sequences \( x = \{x_n\} \) from \( M \), and \( x^0 = \{x_n^0\} \subseteq M\)—a given sequence (the center of a modular space). In this section, we study two special moduliars defined on \( X \).

1. The modular \( w \) from (1.3.10) with \( \varphi(u) = u^p \) (\( p > 0 \)) and \( h(\lambda) = \lambda^q \) (\( q \geq 1 \)) is strict and continuous, and it is convex if \( p \geq 1 \). The modular spaces (around \( x^0 \)) are given by
\[ X^*_w = X^0_w = X^\text{fin}_w = \left\{ x = \{x_n\} \in X : \sum_{n=1}^{\infty} (d(x_n, x^o_n))^p < \infty \right\} \]

(if \( M = \mathbb{R} \) with metric \( d(x, y) = |x - y| \) and \( x^o = 0 = \{0\}_{n=1}^{\infty} \), then \( X^*_w(0) \) is the usual space \( \ell_p \) of all real \( p \)-summable sequences).

Let \( H(\lambda) = \lambda(h(\lambda))^p = \lambda^{pq+1} \). The metric \( d^*_w \) on \( X^*_w \) is of the form:

\[ d^*_w(x, y) = H^{-1}\left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right)^{1/(pq+1)} = \left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right)^{1/(pq+1)}, \]

where \( H^{-1}(\mu) = \mu^{1/(pq+1)} \) is the inverse function of \( H \) on \([0, \infty)\).

If \( p \geq 1 \), then \( w \) is convex, and we also have metric \( d^*_w \) on \( X^*_w \) of the form:

\[ d^*_w(x, y) = h^{-1}\left( \left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right)^{1/p} \right)^{1/pq} = \left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right)^{1/pq}, \]

where \( h^{-1} : [0, \infty) \rightarrow [0, \infty) \) is the inverse function of \( h \) (see Example 1.3.10, and Appendix A.1 concerning general superadditive functions \( h \)).

2. Given \( \lambda > 0 \) and \( x = \{x_n\}, y = \{y_n\} \in X = M^\mathbb{N} \), we set

\[ w_\lambda(x, y) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, y_n)}{\lambda} \right)^{1/n}. \quad (2.4.1) \]

**Proposition 2.4.1.** \( w = \{w_\lambda\}_{\lambda > 0} \) is a strict nonconvex continuous modular on \( X \).

**Proof.** Axioms (i), (ii), and (iii) are clear, and axiom (iii) follows from inequalities (1.3.11) with \( \varphi(u) = u^{1/n} \) and \( h(\lambda) = \lambda \).

In order to see that \( w \) is nonconvex, we show that \( X^0_w(x^o) \neq X^*_w(x^o) \) for some \( x^o \in X \) (cf. Sect. 2.1). Choose any \( x^o \in M \) and \( x \in M, x \neq x^o \), and let \( x^\infty = \{x^o\}_{n=1}^{\infty} \) and \( x = \{x\}_{n=1}^{\infty} \) also denote the corresponding constant sequences from \( X \). Given \( \lambda > d(x, x^o) > 0 \), we find

\[ w_\lambda(x, x^o) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, x^o_n)}{\lambda} \right)^{1/n} = \lim_{n \to \infty} \left( \frac{d(x_n, x^o_n)}{\lambda} \right)^{1/n} = 1, \]

and so, \( x \in X^*_w(x^o) \setminus X^0_w(x^o) \).

Let us show that \( w_\lambda(x, y) \leq w_{\lambda+0}(x, y) \) and \( w_{\lambda-0}(x, y) \leq w_\lambda(x, y) \) for all \( \lambda > 0 \) and \( x, y \in X \), which, by virtue of inequalities (1.2.4), establish the continuity property of \( w \). For any \( n \in \mathbb{N} \) and \( \mu > \lambda \), the definition of \( w \) implies

\[ \left( \frac{d(x_n, y_n)}{\mu} \right)^{1/n} \leq w_\mu(x, y), \]
and so, as $\mu \to \lambda + 0$, we get
\[
\left( \frac{d(x_n, y_n)}{\lambda} \right)^{1/n} \leq w_{\lambda+0}(x, y).
\]
Taking the supremum over all $n \in \mathbb{N}$, we obtain the first inequality above. Now, given $\lambda, \mu > 0$, we have
\[
w_{\mu}(x, y) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, y_n)}{\lambda} \right)^{1/n} \cdot \left( \frac{\lambda}{\mu} \right)^{1/n} \leq w_{\lambda}(x, y) \cdot \sup_{n \in \mathbb{N}} \left( \frac{\lambda}{\mu} \right)^{1/n}
\]
\[= w_{\lambda}(x, y) \cdot \max\{1, \lambda/\mu\}, \quad x, y \in X. \tag{2.4.2}
\]
It follows that if $0 < \mu < \lambda$, then $w_{\mu}(x, y) \leq w_{\lambda}(x, y) \cdot \lambda/\mu$, and so, passing to the limit as $\mu \to \lambda - 0$, we get $w_{\lambda-0}(x, y) \leq w_{\lambda}(x, y)$. \qed

Note that (2.4.2) with $y = x^o$ proves that $X^w_{\text{fin}}(x^o) = X^*_w(x^o)$, and establishes the following characterization of this modular space in terms of sequences $x = \{x_n\}$ and $x^o = \{x^o_n\}$ themselves:
\[
x \in X^*_w(x^o) \quad \text{if and only if} \quad w_1(x, x^o) = \sup_{n \in \mathbb{N}} \left( d(x_n, x^o_n) \right)^{1/n} < \infty. \tag{2.4.3}
\]

The modular space $X^0_w(x^o)$ is characterized in the following way.

**Proposition 2.4.2.** Given $x \in X$, $x \in X^0_w(x^o)$ if and only if $\lim_{n \to \infty} \left( d(x_n, x^o_n) \right)^{1/n} = 0$.

**Proof.** Suppose $x \in X^0_w(x^o)$. Then $w_{\lambda}(x, x^o) \to 0$ as $\lambda \to \infty$, and so, for each $\varepsilon > 0$ there exists $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that
\[
w_{\lambda_0}(x, x^o) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, x^o_n)}{\lambda_0} \right)^{1/n} \leq \varepsilon. \tag{2.4.4}
\]
This inequality is equivalent to
\[
\left( d(x_n, x^o_n) \right)^{1/n} \leq \left( \lambda_0 \right)^{1/n} \cdot \varepsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.4.5}
\]
Passing to the limit superior as $n \to \infty$, we get
\[
\limsup_{n \to \infty} \left( d(x_n, x^o_n) \right)^{1/n} \leq \varepsilon.
\]
Due to the arbitrariness of $\varepsilon > 0$, $(d(x_n, x^o_n))^{1/n} \to 0$ as $n \to \infty$. 
Now, assume that \((d(x_n, x^o_n))^{1/n} \to 0\) as \(n \to \infty\). Given \(\varepsilon > 0\), there exists a number \(n_0 = n_0(\varepsilon) \in \mathbb{N}\) such that \((d(x_n, x^o_n))^{1/n} < \varepsilon\) for all \(n > n_0\). Setting

\[
\lambda_1(\varepsilon) = \max\{1, 1/\varepsilon^{n_0}\} \cdot \max_{1 \leq n \leq n_0} d(x_n, x^o_n)
\]

and noting that

\[
d(x_n, x^o_n) = \frac{d(x_n, x^o_n)}{\varepsilon^n} \leq \lambda_1(\varepsilon) \cdot \varepsilon^n \quad \text{for all} \quad 1 \leq n \leq n_0,
\]

we obtain (2.4.5) with \(\lambda_0 = \lambda_0(\varepsilon) = \max\{1, \lambda_1(\varepsilon)\}\). It follows that inequality (2.4.4) holds, whence, by virtue of (1.2.1), \(w_\lambda(x, x^o) \leq w_{\lambda_0}(x, x^o) \leq \varepsilon\) for all \(\lambda \geq \lambda_0\). This means that \(w_\lambda(x, x^o) \to 0\) as \(\lambda \to \infty\), i.e., \(x \in X^s_w(x^o)\).

The metric \(d^0_w\) on the modular space \(X^0_w(x^o)\) is given by

\[
d^0_w(x, y) = \sup_{n \in \mathbb{N}} (d(x_n, y_n))^{1/(n+1)}, \quad x, y \in X^s_w(x^o).
\]

Recalling that \(w\) is nonconvex, we note that \(d^*_w(x, y) = \sup_{n \in \mathbb{N}} d(x_n, y_n)\) is only an extended metric on \(X^*_w(x^o)\) and \(X\) (however, \(d^*_w\) is a metric on the set of all bounded sequences in \(M\); see Remark 2.4.3 below).

Writing \(x = \{x_n\} \in c(x^o)\) if \(\lim_{n \to \infty} d(x_n, x^o_n) = 0\), and \(x = \{x_n\} \in \ell_\infty(x^o)\) if \(\sup_{n \in \mathbb{N}} d(x_n, x^o_n) < \infty\), we have the following (proper) inclusion relations:

\[
X^0_w(x^o) \subset c(x^o) \subset \ell_\infty(x^o) \subset X^\infty_w(x^o) = X^*_w(x^o).
\]

(Here \(c(x^o)\) is the set of all sequences in \(M\), which are metrically equivalent to \(x^o = \{x^o_n\}\), and \(\ell_\infty(x^o)\) is the set of all sequences in \(M\), which are bounded relative to \(x^o\).) The first inclusion is a consequence of Proposition 2.4.2, and the third one is established as follows: if \(b = \sup_{n \in \mathbb{N}} d(x_n, x^o_n) < \infty\), then, for all \(\lambda > 0\), we have:

\[
w_\lambda(x, x^o) = \sup_{n \in \mathbb{N}} \left(\frac{d(x_n, x^o_n)}{\lambda}\right)^{1/n} \leq \sup_{n \in \mathbb{N}} \left(\frac{b}{\lambda}\right)^{1/n} = \max\{1, b/\lambda\} < \infty.
\]

**Remark 2.4.3.**

1. If \(x^o = \{x^o_n\}\) is a convergent sequence in \(M\), then every sequence \(x = \{x_n\} \in c(x^o)\) is also convergent in \(M\) (to the limit of \(x^o\)), and if \(x^o\) is bounded in \(M\) (i.e., \(\sup_{n, m \in \mathbb{N}} d(x^o_n, x^o_m) < \infty\)), then every \(x \in \ell_\infty(x^o)\) is also bounded in \(M\).

2. In the particular case when \(M = \mathbb{R}\) with metric \(d(x, y) = |x - y|\) and \(x^o = 0\) is the zero sequence, we have: \(c_0 = c(0)\) is the set of all real sequences convergent to zero, and \(\ell_\infty = \ell_\infty(0)\) is the set of all bounded real sequences. The following examples are illustrative (see (2.4.7)): (a) \(\{1/n\} \in c_0 \setminus X^0_w(0)\); (b) \(\{2^n\} \in X^*_w(0) \setminus \ell_\infty\); (c) \(\{2^{-n^2}\} \in X^0_w(0)\); (d) \(\{2^{n^2}\} \not\in X^*_w(0)\); (e) if \(x = \{n\}\), then \(x \in X^*_w(0)\), \(d^0_w(x, 0) = \sup_{n \in \mathbb{N}} n^{1/(n+1)} < \infty\), while \(d^*_w(x, 0) = \sup_{n \in \mathbb{N}} n^{1/(n+1)} = \infty\).
3. The classical $F$-norm $|x|_\rho = d_w^0(x, 0) = \sup_{n \in \mathbb{N}} |x_n|^{1/(n+1)}$, corresponding to $\rho(x) = w_1(x, 0)$ with $w$ from (2.4.1) and $M = \mathbb{R}$, is well-defined for $x = \{x_n\}$ from $X_\rho = X_w^0(0) \subset c_0$ and satisfies conditions (F.1)–(F.4) from Remark 2.2.3. However, on the larger modular space $X_\rho^* = X_w^*(0)$ (see Remark 2.3.4(1)), the functional $| \cdot |_\rho$ does not satisfy the continuity condition (F.4): for instance, if $x = (2^{n+1})_{n=1}^\infty$ and $\alpha_k = 1/k$, then $x \in X_\rho^* \setminus X_\rho$ and $\alpha_k \to 0$ as $k \to \infty$, but

$$|\alpha_k x|_\rho = \sup_{n \in \mathbb{N}} (\alpha_k \cdot 2^{n+1})^{1/(n+1)} = 2 \sup_{n \in \mathbb{N}} \left(\frac{1}{k}\right)^{1/(n+1)} = 2 \text{ for all } k \in \mathbb{N}.$$ 

### 2.5 Intermediate Metrics

In Theorem 2.2.1 and Corollary 2.2.6, we have seen two expressions for metric $d_w^0$ on $X_w^*$ (see also Theorem 2.3.1 if $w$ is convex). In this section, we define and study infinitely many metrics on the modular space $X_w^*$.

**Theorem 2.5.1.** Let $w$ be a (pseudo)modular on the set $X$. Given $0 \leq \theta \leq 1$ and $x, y \in X$, setting

$$d_w^\theta(x, y) = \inf_{\lambda > 0} \left[(1 - \theta) \max\{\lambda, w_\lambda(x, y)\} + \theta(\lambda + w_\lambda(x, y))\right].$$

we have: $d_w^\theta$ is an extended (pseudo)metric on $X$, and a (pseudo)metric on the modular space $X_w^* = X_w^*(x^*)$ for any $x^* \in X$, and the following (sharp) inequalities hold:

$$d_w^\theta(x, y) \leq (1 - \theta)d_w^\theta(x, y) + \theta d_w(x, y) \leq d_w^\theta(x, y) \leq d_w^1(x, y) \leq 2d_w^0(x, y).$$

**Proof.** Clearly, $0 \leq d_w^\theta(x, y) \leq \infty$ for all $x, y \in X$ and $0 \leq \theta \leq 1$.

1. First, we prove our theorem for $\theta = 0$ and $\theta = 1$ simultaneously (for $d_w^0$, this is the second proof). Given $u, v \in [0, \infty]$, we denote by $u \oplus v$ either $\max\{u, v\}$ or $u + v$ (and $u \oplus v = \infty$ if $u = \infty$ or $v = \infty$). Then $d_w^0(x, y)$ and $d_w^1(x, y)$ are expressed by the formula:

$$d_w^\ominus(x, y) = \inf_{\lambda > 0} \lambda \ominus w_\lambda(x, y), \quad x, y \in X.$$ 

1a. If $x, y \in X_w^*$, then $d_w^\ominus(x, y) < \infty$. In fact, since $x \sim y$, there exists $\lambda_0 > 0$ such that $w_{\lambda_0}(x, y) < \infty$, and so, the set $\{\lambda \ominus w_\lambda(x, y) : \lambda > 0\} \setminus \{\infty\}$ is nonempty and bounded from below by 0 (i.e., is contained in $[0, \infty)$).

1b. Given $x \in X$, we have, by (i'), $\lambda \ominus w_\lambda(x, x) = \lambda \ominus 0 = \lambda$ for all $\lambda > 0$, and so, $d_w^\ominus(x, x) = \inf_{\lambda > 0} \lambda = 0$. Now, suppose $w$ is a modular. Let $x, y \in X$, and $d_w^\ominus(x, y) = 0$.

If we show that $w_\lambda(x, y) = 0$ for all $\lambda > 0$, then axiom (i) will
imply \( x = y \). On the contrary, assume that \( w_{\lambda_0}(x, y) \neq 0 \) for some \( \lambda_0 > 0 \). Given \( \lambda > 0 \), we have two cases: if \( \lambda \geq \lambda_0 \), then

\[
\lambda \oplus w_\lambda(x, y) \geq \lambda \oplus 0 = \lambda \geq \lambda_0,
\]

and if \( \lambda < \lambda_0 \), then, by the monotonicity (1.2.1) of \( w \), we find

\[
\lambda \oplus w_\lambda(x, y) \geq 0 \oplus w_\lambda(x, y) = w_\lambda(x, y) \geq w_{\lambda_0}(x, y).
\]

Hence \( \lambda \oplus w_\lambda(x, y) \geq \min\{\lambda_0, w_{\lambda_0}(x, y)\} = \lambda_1 \) for all \( \lambda > 0 \). By the definition of \( d^\oplus_w \), we get \( d^\oplus_w(x, y) \geq \lambda_1 > 0 \), which contradicts the assumption.

1c. Axiom (ii) for \( w \) implies the symmetry property of \( d^\oplus_w \).

1d. Let us establish the triangle inequality \( d^\oplus_w(x, y) \leq d^\oplus_w(x, z) + d^\oplus_w(z, y) \) for all \( x, y, z \in X \). The inequality is clear if at least one summand on the right is infinite. So, we assume that both of them are finite. By (2.5.3), given \( \varepsilon > 0 \), there exist \( \lambda = \lambda(\varepsilon) > 0 \) and \( \mu = \mu(\varepsilon) > 0 \) such that

\[
\lambda \oplus w_\lambda(x, z) \leq d^\oplus_w(x, z) + \varepsilon \quad \text{and} \quad \mu \oplus w_\mu(z, y) \leq d^\oplus_w(z, y) + \varepsilon.
\]

Since \( \oplus \) is max or +, (2.5.3) and axiom (iii) imply

\[
d^\oplus_w(x, y) \leq (\lambda + \mu) \oplus w_{\lambda + \mu}(x, y) \leq (\lambda + \mu) \oplus (w_\lambda(x, z) + w_\mu(z, y)) \leq (\lambda \oplus w_\lambda(x, z)) + (\mu \oplus w_\mu(z, y)) \leq d^\oplus_w(x, z) + \varepsilon + d^\oplus_w(z, y) + \varepsilon.
\]

It remains to take into account the arbitrariness of \( \varepsilon > 0 \).

2. That \( d^\theta_w \) is well-defined, nondegenerate (when \( w \) is a modular), and symmetric can be proved along the same lines as in steps 1a–1c. Let us show that \( d^\theta_w \) satisfies the triangle inequality. Suppose \( d^\theta_w(x, z) \) and \( d^\theta_w(z, y) \) are finite. Given \( \varepsilon > 0 \), by virtue of (2.5.1), there exist \( \lambda = \lambda(\varepsilon) > 0 \) and \( \mu = \mu(\varepsilon) > 0 \) such that

\[
(1 - \theta) \max\{\lambda, w_\lambda(x, z)\} + \theta(\lambda + w_\lambda(x, z)) \leq d^\theta_w(x, z) + \varepsilon,
\]

\[
(1 - \theta) \max\{\mu, w_\mu(z, y)\} + \theta(\mu + w_\mu(z, y)) \leq d^\theta_w(z, y) + \varepsilon.
\]

Taking into account (2.5.1), axiom (iii) and the last inequality in (2.5.4), we get:

\[
d^\theta_w(x, y) \leq (1 - \theta) \max\{\lambda + \mu, w_{\lambda + \mu}(x, y)\} + \theta(\lambda + \mu + w_{\lambda + \mu}(x, y)) \leq (1 - \theta) \left[ \max\{\lambda, w_\lambda(x, z)\} + \max\{\mu, w_\mu(z, y)\} \right] + \theta \left[ \lambda + w_\lambda(x, z) + \mu + w_\mu(z, y) \right] + \varepsilon.
\]
By the arbitrariness of $\varepsilon > 0$, the triangle inequality for $d_w^0$ follows.

3. The inequalities $\max\{u, v\} \leq u + v \leq 2\max\{u, v\}$ for $u, v \geq 0$ imply

\[
d_w^0(x, y) \leq d_w^1(x, y) \leq 2d_w^0(x, y) \quad \text{for all} \quad x, y \in X. \tag{2.5.5}
\]

This proves also the first and fourth inequalities in (2.5.2). Since, for any $\lambda > 0$,

\[
d_w^0(x, y) \leq \max\{\lambda, w_\lambda(x, y)\} \quad \text{and} \quad d_w^1(x, y) \leq \lambda + w_\lambda(x, y),
\]

we find

\[
(1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) \leq (1 - \theta)\max\{\lambda, w_\lambda(x, y)\} + \theta(\lambda + w_\lambda(x, y))
\]

\[
\leq \lambda + w_\lambda(x, y),
\]

which establishes the second and third inequalities in (2.5.2). \hfill \Box

The sharpness of inequalities (2.5.2) is elaborated in Examples 2.5.5 and 2.5.6.

Remark 2.5.2. Not only intermediate (pseudo)metrics $d_w^0$ between $d_0^w$ and $d_1^w$ can be introduced as in (2.5.1): given $\alpha, \beta \geq 0$ with $\alpha + \beta \neq 0$, we set

\[
d_w^{\alpha,\beta}(x, y) = \inf_{\lambda > 0} \left[ \alpha \max\{\lambda, w_\lambda(x, y)\} + \beta(\lambda + w_\lambda(x, y)) \right], \quad x, y \in X.
\]

In this case, we have $d_w^{\alpha,\beta}(x, y) = (\alpha + \beta)d_w^0(x, y)$ with $\theta = \beta / (\alpha + \beta)$.

Remark 2.5.3. Different binary operations $\oplus$ on $[0, \infty)$ can be used in formula (2.5.3) to define $d_w^{\oplus}(x, y)$, but then only the generalized triangle inequality holds:

\[
d_w^{\oplus}(x, y) \leq C(d_w^{\oplus}(x, z) + d_w^{\oplus}(z, y)) \quad \text{with} \quad C > 1. \tag{2.5.6}
\]

This can be seen as follows. Suppose $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and, for some constant $C > 1$,

\[
\varphi\left(\frac{u + v}{C}\right) \leq \varphi(u) + \varphi(v) \leq \varphi(u + v) \quad \text{for all} \quad u, v \geq 0. \tag{2.5.7}
\]
(Here the right-hand side inequality is the superadditivity property of $\varphi$, which is satisfied, e.g., by any convex function $\varphi$; see Appendix A.1). Denoting by $\varphi^{-1}$ the inverse function of $\varphi$ and setting

$$u \oplus v = \varphi^{-1}(\varphi(u) + \varphi(v)) \quad \text{for all} \quad u, v \geq 0,$$

we find, from (2.5.7), that

$$u \oplus v \leq u + v \leq C(u \oplus v). \quad (2.5.9)$$

For instance, if $\varphi(u) = u^\rho$ with $\rho > 1$, then $u \oplus v = (u^\rho + v^\rho)^{1/\rho}$, and inequalities (2.5.9) hold with sharp constant $C = 2^{1/(1/\rho)}$, and if $\varphi(u) = e^u - 1$, then $u \oplus v = \log(e^u + e^v - 1)$, and (2.5.9) hold with sharp constant $C = 2$. Now, in order to obtain (2.5.6), we take into account (2.5.3) and (2.5.9), and find that the right-hand side in (2.5.4) is less than or equal to

$$(\lambda + \mu) + (w_\lambda(x, z) + w_\mu(z, y)) = (\lambda + w_\lambda(x, z)) + (\mu + w_\mu(z, y))$$

$$\leq C[(\lambda + w_\lambda(x, z)) + (\mu + w_\mu(z, y))]$$

$$\leq C[d^\Theta_w(x, z) + \varepsilon + d^\Theta_w(z, y) + \varepsilon], \quad \varepsilon > 0.$$

The generalized triangle inequality (2.5.6) can also be obtained if, instead of $d^\Theta_w(x, y)$ from (2.5.3), we consider the quantity

$$d^\Theta_w(x, y) = \inf_{\lambda > 0} \left(\max\{\lambda, w_\lambda(x, y)\}\right) \oplus (\lambda + w_\lambda(x, y))$$

with the operation $\oplus$ on $[0, \infty)$ of the form (2.5.8).

As in Corollary 2.2.8, the right $w_{+0}$ and left $w_{-0}$ regularizations of $w$ do not produce new metrics of the form (2.5.1) in the following sense.

**Proposition 2.5.4.** $d^\Theta_{w_{+0}}(x, y) = d^\Theta_{w_{-0}}(x, y) = d^\Theta_w(x, y)$ for all $0 \leq \theta \leq 1$ and $x, y \in X$.

**Proof.** For instance, let us verify this for $\theta = 1$. By virtue of (1.2.4), we have

$$\lambda + w_{\lambda,0}(x, y) \leq \lambda + w_\lambda(x, y) \leq \lambda + w_{\lambda,0}(x, y) \quad \text{for all} \quad \lambda > 0,$$

whence $d^1_{w_{+0}}(x, y) \leq d^1_w(x, y) \leq d^1_{w_{-0}}(x, y)$.

Let us show that $d^1_{w_{+0}}(x, y) \geq d^1_w(x, y)$. Suppose $d^1_{w_{+0}}(x, y) < \infty$, and $u > d^1_{w_{+0}}(x, y)$. Let $u > u_1 > d^1_{w_{+0}}(x, y)$. By (2.5.1) with $\theta = 1$, there exists $\lambda_1 > 0$ such that

$$\lim_{\lambda \to \lambda_1+0} \left(\lambda + w_\lambda(x, y)\right) = \lambda_1 + w_{\lambda_1,0}(x, y) \leq u_1 < u.$$
It follows that $\lambda_2 + w_{\lambda_2}(x, y) < u$ for some $\lambda_2 > \lambda_1$, which implies

$$d_w^1(x, y) = \inf_{\lambda > 0} (\lambda + w_{\lambda}(x, y)) \leq \lambda_2 + w_{\lambda_2}(x, y) < u,$$

and it remains to pass to the limit as $u \to d_{w^+}(x, y)$.

Now, we show that $d_w^1(x, y) \geq d_{w^-}(x, y)$. Let $d_w^1(x, y) < \infty$, and $u > d_w^1(x, y)$. Choose $u_1$ such that $u > u_1 > d_w^1(x, y)$. By (2.5.1) with $\theta = 1$, there exists $\mu_1 > 0$ such that $\mu_1 + w_{\mu_1}(x, y) \leq u_1 < u$. It follows from (1.2.4) that

$$w_{\lambda_1 - 0}(x, y) \leq w_{\mu_1}(x, y) < u - \mu_1 \quad \text{for all} \quad \lambda_1 > \mu_1,$$

and so,

$$d_{w^-}(x, y) \leq \lambda_1 + w_{\lambda_1 - 0}(x, y) < \lambda_1 + u - \mu_1.$$

Passing to the limit as $\lambda_1 \to \mu_1 + 0$, we get $d_{w^+}(x, y) \leq u$, and it remains to take into account the arbitrariness of $u$ as above.

\[\Box\]

**Example 2.5.5 (metric $d_w^1$).**

1. Let $w_\lambda(x, y) = \lambda^{-p}d(x, y)$ be of the form (1.3.1) with $p > 0$. By Example 2.2.2(1), $d_w^0(x, y) = (d(x, y))^{1/(p+1)}$.

Let us calculate $d_w^1(x, y) = \inf_{\lambda > 0} f(\lambda)$, where $f(\lambda) = \lambda + \lambda^{-p}d(x, y)$ (and $x \neq y$). The derivative $f'(\lambda) = 1 - p\lambda^{-p-1}d(x, y)$ vanishes at $\lambda_0 = (pd(x, y))^{1/(p+1)}$, $f'(\lambda) < 0$ if $0 < \lambda < \lambda_0$, and $f'(\lambda) > 0$ if $\lambda > \lambda_0$, and so, $f$ attains the global minimum on $(0, \infty)$ at the point $\lambda_0$, which is equal to

$$d_w^1(x, y) = f(\lambda_0) = \gamma(p) \cdot (d(x, y))^{1/(p+1)} \quad \text{for all} \quad x, y \in X,$$

where

$$\gamma(p) = (p + 1)p^{-p/(p+1)}, \quad p > 0.$$

Note that $1 < \gamma(p) \leq 2$, $\gamma(p) = 2$ if and only if $p = 1$, and $\gamma(1/p) = \gamma(p)$. The inequalities for $\gamma(p)$ can be established directly by taking the logarithm and investigating the resulting function for extrema, or they follow from (2.5.5). In particular, if $p = 1$, the expressions for $d_w^0$ and $d_w^1$ are of the form:

$$d_w^0(x, y) = \sqrt{d(x, y)} \quad \text{and} \quad d_w^1(x, y) = 2\sqrt{d(x, y)}, \quad x, y \in X.$$

2. Formulas for $d_w^0$ and $d_w^1$ above are valid in a somewhat more general case when a (pseudo)modular $w$ on $X$ is $p$-homogeneous with $p > 0$ in the sense that

$$w_\lambda(x, y) = \lambda^{-p}w_1(x, y) \quad \text{for all} \quad \lambda > 0 \text{ and } x, y \in X.$$
In this case, we have
\[ d_w^0(x, y) = (w_1(x, y))^{1/(p+1)} \quad \text{and} \quad d_w^1(x, y) = \gamma(p) \cdot (w_1(x, y))^{1/(p+1)}. \]
(2.5.10)

One more example of a \( p \)-homogeneous modular \( w \) on a metric space \((X, d)\) is given by \( w_\lambda(x, y) = (d(x, y)/\lambda)^p = \lambda^{-p} w_1(x, y) \) (see Example 2.3.5(1)).

3. Given a metric space \((X, d)\) and a convex function \( \varphi : [0, \infty) \to [0, \infty) \) vanishing at zero only, we set (cf. (1.3.5))
\[ w_\lambda(x, y) = \lambda \varphi\left( \frac{d(x, y)}{\lambda} \right), \quad \lambda > 0, \quad x, y \in X. \]

Then \( w \) is a strict modular on \( X \) (cf. (1.3.8)), and since \( \varphi \) is increasing, continuous, and admits the continuous inverse \( \varphi^{-1} \), we find
\[ d_w^0(x, y) = \inf \{ \lambda > 0 : \varphi(d(x, y)/\lambda) \leq 1 \} = d(x, y)/\varphi^{-1}(1). \]

In particular, if \( \varphi(u) = u^p \) with \( p > 1 \), we have \( d_w^0(x, y) = d(x, y) \), and taking into account that
\[ w_\lambda(x, y) = \lambda \left( \frac{d(x, y)}{\lambda} \right)^p = \lambda^{-(p-1)} (d(x, y))^p = \lambda^{-(p-1)} w_1(x, y), \]
we conclude from (2.5.10) (replacing \( p \) there by \( p - 1 \)) that
\[ d_w^1(x, y) = \gamma(p - 1) \cdot (w_1(x, y))^{1/p} = p(p - 1)^{(1-p)/p} \cdot d(x, y). \]

4. Setting \( w_\lambda(x, y) = e^{-\lambda} d(x, y) \) and following the same reasoning as in Example 2.5.5(1), we get
\[ d_w^1(x, y) = \begin{cases} d(x, y) & \text{if } d(x, y) \leq 1, \\ 1 + \log d(x, y) & \text{if } d(x, y) > 1, \end{cases} \quad x, y \in X. \]

Example 2.5.6 (metric \( d_w^0 \)). In order to be able to calculate the value \( d_w^0(x, y) \) from (2.5.1) explicitly for all \( 0 \leq \theta \leq 1 \), here once again we consider the modular \( w_\lambda(x, y) = \lambda^{-\theta} d(x, y) \) of the form (1.3.1) with \( p > 0 \). Since the cases \( \theta = 0 \) and \( \theta = 1 \) were considered in Example 2.5.5(1), we are left with the case when \( 0 < \theta < 1 \) (in calculations below, we assume that \( x \neq y \)).

To begin with, we note that \( d_w^0(x, y) = \inf_{\lambda>0} f(\theta, \lambda) \), where the function \( f(\theta, \lambda) \) under the infimum sign in (2.5.1) is expressed as
\[ f(\theta, \lambda) = \begin{cases} f_1(\lambda) = w_\lambda(x, y) + \theta \lambda & \text{if } \lambda \leq w_\lambda(x, y), \\ f_2(\lambda) = \lambda + \theta w_\lambda(x, y) & \text{if } \lambda > w_\lambda(x, y), \end{cases} \]
with \( f_1(\lambda) = \lambda^{-p}d(x, y) + \theta \lambda \) and \( f_2(\lambda) = \lambda + \theta \lambda^{-p}d(x, y) \), and the inequality 
\( \lambda \leq w_2(x, y) = \lambda^{-p}d(x, y) \) is equivalent to \( \lambda \leq \lambda_0 \equiv d_w^0(x, y) = (d(x, y))^{1/(p+1)} \). Hence
\[
d_w^\theta(x, y) = \min \left\{ \inf_{\theta \lambda \leq \lambda_0} f_1(\lambda), \inf_{\lambda > \lambda_0} f_2(\lambda) \right\}, \quad (2.5.11)
\]
where we note that \( f_1(\lambda_0) = f_2(\lambda_0) = \lambda_0(1 + \theta) \).

The derivative \( f_1'(\lambda) = -\lambda^{-p-1}pd(x, y) + \theta \) is equal to zero only at the point 
\( \lambda_1 = \lambda_0(p/\theta)^{1/(p+1)} \), \( f_1'(\lambda) < 0 \) if \( 0 < \lambda < \lambda_1 \), and \( f_1'(\lambda) > 0 \) if \( \lambda > \lambda_1 \), and so, the global minimum of \( f_1 \) on \((0, \infty)\) is attained at \( \lambda_1 \) and is equal to
\[
f_1(\lambda_1) = \lambda_0\gamma(p)\theta^{1/(p+1)}.
\]

Similarly, the derivative \( f_2'(\lambda) = 1 - \lambda^{-p-1}\theta pd(x, y) \) is equal to zero at the point 
\( \lambda_2 = \lambda_0(\theta p)^{1/(p+1)} \), \( f_2'(\lambda) < 0 \) if \( 0 < \lambda < \lambda_2 \), and \( f_2'(\lambda) > 0 \) for \( \lambda > \lambda_2 \), and so, \( f_2 \) attains the global minimum on \((0, \infty)\) at \( \lambda_2 \), where it has the value
\[
f_2(\lambda_2) = \lambda_0\gamma(p)\theta^{1/(p+1)}.
\]

Given \( p > 0 \) and \( 0 \leq \theta \leq 1 \), we have four cases: (I) \( p \geq 1 \) and \( \theta \leq 1/p \); (II) \( p > 1 \) and \( 1/p < \theta \); (III) \( p < 1 \) and \( \theta \leq p \); and (IV) \( p < 1 \) and \( p < \theta \).

**Cases (I), (III).** We have \( p \geq 1 \geq \theta \) in case (I), and \( p \geq \theta \) in case (III), and so, \( \lambda_0 \leq \lambda_1 \). Since \( f_1 \) decreases on \((0, \lambda_1]\), the value \( \inf_{\lambda \leq \lambda_0} f_1(\lambda) \) is equal to \( f_1(\lambda_0) = \lambda_0(1 + \theta) \). Also, we have \( \theta p \leq 1 \) in case (I), and \( \theta p < 1 \) in case (III), and so, \( \lambda_2 \leq \lambda_0 \). Since \( f_2 \) increases on \([\lambda_2, \infty)\), the value \( \inf_{\lambda > \lambda_0} f_2(\lambda) \) is equal to \( f_2(\lambda_0) = \lambda_0(1 + \theta) \). By virtue of \((2.5.11)\), \( d_w^\theta(x, y) = \lambda_0(1 + \theta) \).

**Case (II).** As in case (I), since \( p > 1 \geq \theta \), \( \inf_{\lambda \leq \lambda_0} f_1(\lambda) = \lambda_0(1 + \theta) \). Furthermore, \( \theta p > 1 \) implies \( \lambda_0 < \lambda_2 \), where \( \lambda_2 \) is the point of minimum of \( f_2 \) on \([\lambda_0, \infty)\), and so,
\[
\inf_{\lambda > \lambda_0} f_2(\lambda) = f_2(\lambda_2) < f_2(\lambda_0) = \lambda_0(1 + \theta) = \inf_{\lambda \leq \lambda_0} f_2(\lambda).
\]

It follows from \((2.5.11)\) that \( d_w^\theta(x, y) = f_2(\lambda_2) = \lambda_0\gamma(p)\theta^{1/(p+1)} \).

**Case (IV).** Inequality \( p < \theta \) implies \( \lambda_1 < \lambda_0 \), and since \( \lambda_1 \) is the point of minimum of \( f_1 \) on \((0, \lambda_0]\), we find
\[
\inf_{\lambda \leq \lambda_0} f_1(\lambda) = f_1(\lambda_1) < f_1(\lambda_0) = \lambda_0(1 + \theta).
\]

As in case (III), since \( \theta p < 1 \), \( \inf_{\lambda > \lambda_0} f_2(\lambda) = \lambda_0(1 + \theta) \). By \((2.5.11)\), we conclude that \( d_w^\theta(x, y) = f_1(\lambda_1) = \lambda_0\gamma(p)\theta^{1/(p+1)} \).
In this way, we have shown that

\[
    d_w^0(x, y) = (d(x, y))^{1/(p+1)} \cdot \begin{cases} 
        1 + \theta & \text{if } 0 \leq \theta \leq 1/p \leq 1 \\
        0 \leq \theta \leq p < 1, \\
        \gamma(p)\theta^{1/(p+1)} & \text{if } 0 < 1/p < \theta \leq 1, \\
        \gamma(p)\theta^{p/(p+1)} & \text{if } 0 < p < \theta \leq 1. 
    \end{cases}
\] (2.5.12)

A few comments on this formula are in order. If \( \theta = 0 \) or \( \theta = 1 \), then it gives back the values \( d_w^0(x, y) \) and \( d_w^1(x, y) \) from Example 2.5.5(1). If \( p > 1 \) and \( \theta = 1/p \) in the third line of (2.5.12), then \( \gamma(p)\theta^{1/(p+1)} = 1 + \theta \) (as in the first line). Similarly, if \( p < 1 \) and \( \theta = p \) in the fourth line of (2.5.12), then \( \gamma(p)\theta^{p/(p+1)} = 1 + \theta \).

Note that, for any \( p > 0 \) and \( 0 \leq \theta \leq 1 \), we have (cf. (2.5.2))

\[
    (1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) = (1 - \theta + \theta\gamma(p)) \cdot (d(x, y))^{1/(p+1)}.
\]

For \( p \neq 1 \), we have \( 1 < \gamma(p) < 2 \), so if (a) \( p > 1 \) and \( 0 < \theta < 1 \), or (b) \( p < 1 \) and \( 0 < \theta \leq p \), then \( 1 - \theta + \theta\gamma(p) < 1 + \theta \), and so,

\[
    (1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) < d_w^0(x, y), \quad x \neq y.
\]

Now, if \( p = 1 \), then \( \gamma(p) = 2 \) and \( 1 - \theta + \theta\gamma(p) = 1 + \theta \), which imply

\[
    d_w^0(x, y) = (1 + \theta)\sqrt{d(x, y)} = (1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) \quad \text{for all } 0 \leq \theta \leq 1.
\]

For a convex (pseudo)modular \( w \) on \( X \), \( \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y) \) is a (pseudo)modular on \( X \), so setting \( d_w^\theta = d_{\hat{w}}^\theta \) and applying Theorem 2.5.1, we get

**Theorem 2.5.7.** If \( w \) is a convex (pseudo)modular on \( X \) and \( 0 \leq \theta \leq 1 \), then

\[
    d_w^\theta(x, y) = \inf_{\lambda > 0} \left[ (1 - \theta)\max\{\lambda, \lambda w_\lambda(x, y)\} + \theta(\lambda + \lambda w_\lambda(x, y)) \right], \quad x, y \in X,
\]

is an extended (pseudo)metric on \( X \) and a (pseudo)metric on \( X_w^* \), and

\[
    d_w^\theta(x, y) \leq (1 - \theta)d_w^\theta(x, y) + \theta d_w^1(x, y) \leq d_w^\theta(x, y) \leq d_w^1(x, y) \leq 2d_w^\theta(x, y),
\]

where (see (2.3.3)) \( d_w^\theta(x, y) = d_{\hat{w}}^\theta(x, y) \).

**Remark 2.5.8.** Given \( 0 \leq \theta \leq 1 \), \( d_w^0(x, y) < 1 \) implies \( d_w^\theta(x, y) \leq d_w^0(x, y) \). In fact, for any \( r \) such that \( d_w^0(x, y) < r < 1 \) there exists \( \lambda = \lambda(r) > 0 \) such that

\[
    (1 - \theta)\max\{\lambda, w_\lambda(x, y)\} + \theta(\lambda + w_\lambda(x, y)) \leq r < 1.
\]
It follows that $\lambda = (1 - \theta)\lambda + \theta\lambda < 1$, 

$$\max\{\lambda, \lambda w_\lambda(x, y)\} \leq \max\{\lambda, w_\lambda(x, y)\} \quad \text{and} \quad \lambda + \lambda w_\lambda(x, y) \leq \lambda + w_\lambda(x, y),$$

and so,

$$d^{\theta\ast}_w(x, y) \leq (1 - \theta) \max\{\lambda, \lambda w_\lambda(x, y)\} + \theta(\lambda + \lambda w_\lambda(x, y)) \leq r.$$

It remains to pass to the limit as $r \to d^0_w(x, y)$.

**Example 2.5.9.** Let $p \geq 1$ and $w_\lambda(x, y) = (d(x, y)/\lambda)^p$ be the $p$-homogeneous modular from Example 2.3.5(1). Then, by Example 2.5.5(1), (2),

$$d^1_w(x, y) = \gamma(p) \cdot (d(x, y))^{p/(p+1)} \quad \text{and} \quad d^{1\ast}_w(x, y) = \begin{cases} d(x, y) & \text{if } p = 1, \\ \gamma(p - 1)d(x, y) & \text{if } p > 1. \end{cases}$$

2.6 Bibliographical Notes and Comments

Sections 2.1 and 2.2. Modular spaces $X^*_w$ and $X^0_w$ were introduced in Chistyakov [22] and studied in [24, 25, 28]. The space $X^0_w$ is a counterpart of the classical modular space $X_\rho$ defined in Musielak and Orlicz [77]; see Remark 2.2.3(1), in which the main results of [77] are briefly described. As condition ($\rho.4$) from Sect.1.3.3 is crucial for defining the $F$-norm $|x|_\rho$ on $X_\rho$, axiom (iii) in Definition 1.2.1 is a proper tool to define the (pseudo)metric $d^0_w(x, y)$ on the space $X^*_w$, which is larger than $X^0_w$.

The properties of $d^0_w(x, y)$ are based on the properties of quantity $g^0$ from (2.2.1) (recall that $d^0_w(x, y) = (w^{x,y})^0$). This allows us to obtain an alternative expression for the (pseudo)metric $d^0_w(x, y)$ in Corollary 2.2.6.

Modular space $X^\text{lin}_w$ is (natural and) new. Its role will be more clear below (see Theorem 3.3.8): some ‘duality’ holds between the modular spaces.

Corollary 2.2.8 was first established in Chistyakov [28].

Lemma 2.2.9 and Theorem 2.2.11 are sharp refinements of Theorem 2.10 from Chistyakov [24]. Counterparts of Theorem 2.2.11(d), (e) for classical modulars are presented in Maligranda [68, Theorem 1.4].

Theorem 2.2.13 is new.

**Section 2.3.** In the convex case, the results of the classical modular theory are presented in Remark 2.3.4(1). They were established by Nakano [81, Sect. 81], Musielak and Orlicz [78], and Orlicz [90] (for $s$-convex modulars with $0 < s \leq 1$). For Orlicz modulars (i.e., integral modulars of the form $\rho(x) = \int_G \varphi(|x(t)|)d\mu(t)$, the norm $\|x\|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq 1\}$ on $X^*_\rho$ was considered by Morse and Transue [73] and Luxemburg [66]. Note that the norm $\|x\|_\rho$ is the Minkowski functional $p_A(x) = \inf\{\varepsilon > 0 : x/\varepsilon \in A\}$ of the convex set $A = \{x : \rho(x) \leq 1\}$.
Furthermore, Musielak and Orlicz [78] proved inequalities of the form (2.3.6) and (2.3.7) for classical convex modulars σ, and Orlicz [90] established the representation \([x]_\sigma = \inf_{t>0} \sup\{t^{-1}, \rho(tx)t^{-1}\}\) (cf. the second equality in (2.3.3)).

The (pseudo)metric \(d_w^*(x,y)\) on \(X^*_w\) was introduced in Chistyakov [22]. It is seen from the expressions for \(d_w^*(x,y)\) and \([x]_\rho\) that \(d_w^*(x,y)\) is a counterpart of the norm \([x]_\rho\). Interestingly, the idea of definition of \(d_w^*(x,y) = (\tilde{w}(x,y))^0\) has no relation with the idea of Minkowski’s functional of a convex set, and relies on \(g^0\) from (2.2.1), however, by virtue of the ‘embedding’ (1.3.3), for convex modulars \(\sigma\) on linear spaces, we get \([x]_\rho = d_w^*(x,0)\) (see Remark 2.3.4(1)).

Section 2.4. The first modular stands for illustrative purposes—its idea is to generalize, in a straightforward way, the well-known space \(\ell_p\) of \(p\)-summable sequences. The second modular (2.4.1), mentioned in [24, Example 3.2], is more interesting and studied in detail (see also Example 4.2.7(2)). Note that modular (2.4.1) can be obtained, via (1.3.3), from the classical modular \([x]_\rho = \sup_{n\in\mathbb{N}} \sqrt{|x_n|}\) for \(x = \{x_n\} \in \mathbb{R}^\mathbb{N}\), see Rolewicz [95, Example 1.2.3].

Section 2.5. The whole material of Sect. 2.5 is new. Connections with the classical modular theory are as follows. Metric \(d_w^0(x,y)\) from (2.5.1) for \(\theta = 1\) is a counterpart of the \(F\)-norm \([x]_\rho^0 = \inf_{t>0}(1 + t\rho(tx))/t, x \in X_\rho,\) from Koshi and Shimogaki [53], where inequality \([x]_\rho \leq [x]_\rho^1 \leq 2[x]_\rho\) of the form (2.5.5) was also established; here \([x]_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq \varepsilon\}\) is the Musielak-Orlicz \(F\)-norm.

The idea to define the operation \(\oplus\) in (2.5.8) is taken from Musielak [74] and Musielak and Peetre [79] (see also Musielak [75, Sect. 3]).

The classical variant of Example 2.5.5 was elaborated in Maligranda [68, p. 4].

Metric \(d_w^0(x,y)\) from Theorem 2.5.7 for \(\theta = 1\) is a counterpart of the Amemiya norm \([x]_\rho^1 = \inf_{t>0}(1 + \rho(tx))/t, x \in X_\rho^* = X_\rho,\) (see Nakano [81, Sect. 81], Hudzik and Maligranda [48], Maligranda [68, p. 6], Musielak [75, Theorem 1.10]).

For more information about the modular theory on linear spaces and Orlicz spaces we refer to Adams [1], Kozlowski [55], Krasnosel’skiĭ and Rutickiĭ [56], Lindenstrauss and Tzafriri [65], Luxemburg [66], Maligranda [68], Musielak [75], Nakano [80, 81], Orlicz [89], Rao and Ren [92, 93], Rolewicz [95].
Metric Modular Spaces
Chistyakov, V.V.
2015, XIII, 137 p. 2 illus. in color., Softcover
ISBN: 978-3-319-25281-0