Chapter 2
Metrics on Modular Spaces

Abstract In this chapter, we address the metrizability of modular spaces.

2.1 Modular Spaces

A pseudomodular \( w \) on \( X \) (cf. Fig. 1.2 on p. 5) induces an equivalence relation \( \sim \) on \( X \) as follows: given \( x, y \in X \),

\[ x \sim y \iff w^{x,y} \neq \infty \iff w_\lambda(x, y) < \infty \text{ for some } \lambda > 0, \]

where \( \lambda = \lambda(x, y) \), possibly, depends on \( x \) and \( y \). A modular space is any equivalence class with respect to \( \sim \). More explicitly, let us fix an element \( x^0 \in X \). The set

\[ X^*_w \equiv X^*_w(x^0) = \{ x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } w_\lambda(x, x^0) < \infty \} \]

is called a modular space (around \( x^0 \)), and \( x^0 \) is called the center of \( X^*_w(x^0) \) (\( x^0 \) is a representative of the equivalence class \( X^*_w(x^0) \)). Note that \( w^{x,y} \neq \infty \) for all \( x, y \in X^*_w(x^0) \).

If \( w_{+0} \) and \( w_{-0} \) are the right and left regularizations of \( w \), then (1.2.4) imply \( X^*_w(0) = X^*_w(0) = X^*_w \).

Two more modular spaces (around \( x^0 \)) can be defined making use of other equivalence relations on \( X \):

\[ X^0_w \equiv X^0_w(x^0) = \{ x \in X : w_\lambda(x, x^0) \to 0 \text{ as } \lambda \to \infty \} \]

and

\[ X^\text{fin}_w \equiv X^\text{fin}_w(x^0) = \{ x \in X : w_\lambda(x, x^0) < \infty \text{ for all } \lambda > 0 \}. \]

As above, \( X^0_w = X^0_w = X^0_w \) and \( X^\text{fin}_w = X^\text{fin}_w = X^\text{fin}_w \).

Clearly, \( X^0_w \subset X^*_w \) and \( X^\text{fin}_w \subset X^*_w \) (with proper inclusions in general). However, if \( w \) is convex, then \( X^0_w = X^*_w \) (see Proposition 1.2.3(c)); moreover, note that this property is independent of the center \( x^0 \), i.e., \( X^0_w(x^0) = X^*_w(x^0) \) for all \( x^0 \in X \).
2.2 The Basic Metric

Example 2.1.1. The inclusion relations between the three modular spaces are illustrated by the modular \( w_\lambda(x, y) = g(\lambda)d(x, y) \) on a metric space \((X, d)\) from (1.3.1):

\[
X^*_w = \begin{cases}
\{x^0\} & \text{if } g \equiv \infty, \\
X & \text{if } g \neq \infty,
\end{cases}
\]

\[
X^0_w = \begin{cases}
\{x^0\} & \text{if } \lim_{\lambda \to \infty} g(\lambda) \neq 0, \\
X & \text{if } \lim_{\lambda \to \infty} g(\lambda) = 0,
\end{cases}
\]

and

\[
X^{\text{fin}}_w = \begin{cases}
\{x^0\} & \text{if } g(\lambda) = \infty \text{ for some } \lambda > 0, \\
X & \text{if } g(\lambda) < \infty \text{ for all } \lambda > 0.
\end{cases}
\]

In particular, for modulars \( w_\lambda(x, y) = d(x, y) \) (nonconvex) and \( w_\lambda(x, y) = d(x, y)/\lambda \) (convex) from Example 1.3.2(a), we have

\[
X^0_w = \{x^0\} \subset X^*_w = X^{\text{fin}}_w = X = X^0_w = X^*_w = X^{\text{fin}}_w.
\]

In the sequel, by the modular space we mean the set \( X^*_w \) (the largest among the three) if not explicitly stated otherwise.

2.2 The Basic Metric

We begin by introducing the basic (pseudo)metric \( d^0_w \) on the modular space \( X^*_w \).

Theorem 2.2.1. Let \( w \) be a (pseudo)modular on \( X \). Set

\[
d^0_w(x, y) = \inf \{\lambda > 0 : w_\lambda(x, y) \leq \lambda\}, \quad x, y \in X \quad (\inf \emptyset = \infty).
\]

Then \( d^0_w \) is an extended (pseudo)metric on \( X \). Furthermore, if \( x, y \in X \), \( d^0_w(x, y) < \infty \) is equivalent to \( x \sim y \), and so, \( d^0_w \) is a (pseudo)metric on \( X^*_w = X^{\text{fin}}_w(x^0) \) (for any \( x^0 \in X \)).

Proof. 1. Clearly, \( d^0_w(x, y) \in [0, \infty] \), \( d^0_w(x, x) = 0 \), and \( d^0_w(x, y) = d^0_w(y, x) \) for all \( x, y \in X \). Now, suppose \( w \) is a modular on \( X \), and \( x, y \in X \) are such that \( d^0_w(x, y) = 0 \). The definition of \( d^0_w \) implies \( w_\mu(x, y) \leq \mu \) for all \( \mu > 0 \). So, for all \( \lambda > 0 \) and \( 0 < \mu < \lambda \), we have from (1.2.1): \( w_\lambda(x, y) \leq w_\mu(x, y) \leq \mu \to 0 \) as \( \mu \to +0 \). Thus \( w_\lambda(x, y) = 0 \) for all \( \lambda > 0 \), and so, by axiom (i), \( x = y \).

In order to prove the triangle inequality \( d^0_w(x, y) \leq d^0_w(x, z) + d^0_w(z, y) \) for all \( x, y, z \in X \), we assume that \( d^0_w(x, z) \) and \( d^0_w(z, y) \) are finite (otherwise, the inequality is obvious). By the definition of \( d^0_w \), given \( \lambda > d^0_w(x, z) \) and \( \mu > d^0_w(z, y) \), we find \( w_\lambda(x, z) \leq \lambda \) and \( w_\mu(z, y) \leq \mu \), and so, axiom (iii) implies

\[
w_{\lambda + \mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y) \leq \lambda + \mu.
\]

It follows that \( d^0_w(x, y) \leq \lambda + \mu \), and it remains to take into account the arbitrariness of \( \lambda \) and \( \mu \) as above.
2. If \( d_0^w(x,y) < \infty \), then, for any \( \lambda > d_0^w(x,y) \), we have \( w_\lambda(x,y) \leq \lambda < \infty \), which
means that \( x \sim y \). Conversely, suppose \( x \sim y \), i.e., \( w_\mu(x,y) < \infty \) for some \( \mu > 0 \). We set \( \lambda = \max\{\mu, w_\mu(x,y)\} \). Since \( \lambda \geq \mu \), the monotonicity (1.2.1) of \( w \) implies \( w_\lambda(x,y) \leq w_\mu(x,y) \leq \lambda \), and so, \( d_0^w(x,y) \leq \lambda < \infty \).

3. Given \( x, y \in X^*_w \), we have \( x \sim y \), and so, \( d_0^w(x,y) < \infty \). By step 1, this means that
\( d_0^w \) is a (pseudo)metric on \( X^*_w \).

The pair \((X^*_w, d_0^w)\), being a (pseudo)metric space generated by the (pseudo)modular \( w \), is called a (pseudo)metric modular space, and we will apply this terminology if we are interested in metric properties of \( X^*_w \) with respect to \( d_0^w \) (or some other metric induced by \( w \)). We call \( X^*_w \) the modular space if the main concern is its modular properties (Sects. 4.2 and 4.3), which are outside the scope of metric properties.

**Example 2.2.2.** Suppose \( w_\lambda(x,y) = g(\lambda)d(x,y) \) is the modular from (1.3.1), where
\( g : (0, \infty) \to [0, \infty] \) is a nonincreasing function, \( g \not\equiv 0 \), and \( g \not\equiv \infty \). In the examples 1–6 below, we have \( X^*_w = X \), and \( x, y \in X \) and \( \lambda_0 > 0 \) are given.

1. If \( g(\lambda) = 1/\lambda^p \) (\( p \geq 0 \)), then \( d_0^w(x,y) = (d(x,y))^{1/(p+1)} \).
2. Let \( g(\lambda) = 1 \) if \( 0 < \lambda < \lambda_0 \), and \( g(\lambda) = 0 \) if \( \lambda \geq \lambda_0 \). Then \( w \) is nonstrict and nonconvex, and \( d_0^w(x,y) = \min\{\lambda_0, d(x,y)\} \).
3. If \( g(\lambda) = 1/\lambda \) for \( 0 < \lambda < \lambda_0 \), and \( g(\lambda) = 0 \) for \( \lambda \geq \lambda_0 \), then \( w \) is nonstrict and convex, and \( d_0^w(x,y) = \min\{\lambda_0, \sqrt{d(x,y)}\} \).
4. For \( g(\lambda) = \max\{1, 1/\lambda\} \), we have: \( w \) is strict and nonconvex, and \( d_0^w \) is given by
\[
d_0^w(x,y) = \max\{d(x,y), \sqrt{d(x,y)}\}.
\]
5. If \( g(\lambda) = \infty \) for \( 0 < \lambda < \lambda_0 \), and \( g(\lambda) = 0 \) for \( \lambda \geq \lambda_0 \), then \( w \) is strict and convex, and \( d_0^w(x,y) = \lambda_0 \delta(x,y) \), where \( \delta \) is the discrete metric on \( X \).

6. Putting \( d = \delta \), for any function \( g \) as above, we have \( d_0^w(x,y) = g^0(x,y) \) with
\[
g^0 = \inf\{\lambda > 0 : g(\lambda) \leq \lambda\}.
\]

**Remark 2.2.3.**
1. If \( \rho \) is a classical modular on a real linear space \( X \) (cf. Sect. 1.3.3),
the set \( X_\rho = \{x \in X : \lim_{\alpha \to +0} \rho(\alpha x) = 0\} \) is called the modular space (with zero as its center). The modular space \( X_\rho \) is a linear subspace of \( X \), and the functional \( \cdot : X_\rho \to [0, \infty) \), given by \( |x|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq \varepsilon\} \), is an \( F \)-norm on \( X_\rho \), i.e., given \( x, y \in X_\rho \), it satisfies the conditions: (F.1) \( |x|_\rho = 0 \) iff \( x = 0 \); (F.2) \( -|x|_\rho = |x|_\rho \); (F.3) \( |x+y|_\rho \leq |x|_\rho + |y|_\rho \); and (F.4) \( c_\rho x_n - c \rho x_n \to 0 \) as \( n \to \infty \) whenever \( c_\rho c \to c \) in \( \mathbb{R} \) and \( |x_n - x|_\rho \to 0 \) as \( n \to \infty \) (where \( x_n \in X_\rho \) for \( n \in \mathbb{N} \)). The modular space \( X^*_\rho \), which is a counterpart of \( X_\rho \), does not play that significant role in our theory as \( X_\rho \) does in the classical theory of modulars (see also Remark 2.4.3(3)).

2. Under the assumptions of Proposition 1.3.5, where \( X \) is a real linear space and
\( \rho(x) = w_1(x,0) \), we also have: \( X_\rho = X^*_\rho(0) \) is a linear subspace of \( X \), and the functional \( |x|_\rho = d_0^w(x,0), x \in X^*_\rho \), is an \( F \)-norm on \( X_\rho \).
In Theorem 2.2.1 (and Example 2.2.2(6)), we have encountered the quantity
\[ g^0 = \inf \{ \lambda > 0 : g(\lambda) \leq \lambda \}, \quad (2.2.1) \]
evaluated at the nonincreasing function \( g = w^{a,y} : (0, \infty) \to [0, \infty] \), which we denoted by \( d^0_w(x,y) = (w^{a,y})^0 \). This quantity is worth a more detailed study.

Lemma 2.2.4. If \( g : (0, \infty) \to [0, \infty] \) is a nonincreasing function, then \( g^0 \in [0, \infty] \), and

(a) \( g^0 = \inf_{\lambda > 0} \max \{ \lambda, g(\lambda) \} \) (where \( \max \{ \lambda, \infty \} = \infty \) for \( \lambda > 0 \));
(b) \( g^0 < \infty \) if and only if \( g \not\equiv \infty \) (so, \( g^0 = \infty \) \( \iff g \equiv \infty \));
(c) \( g^0 \neq 0 \) if and only if \( g \not\equiv 0 \) (so, \( g^0 = 0 \) \( \iff g \equiv 0 \)).

**Proof.** 1. Let us prove inequality (\( \leq \)) in (a) and implication (\( \iff \)) in (b). We may assume \( g \not\equiv \infty \) (otherwise, (a) reads \( \inf \emptyset = \infty \) and holds trivially). For each \( \lambda > 0 \) such that \( g(\lambda) < \infty \), we set \( \lambda_1 = \max\{\lambda, g(\lambda)\} \). Then \( \lambda_1 \in (0, \infty) \), \( g(\lambda) \leq \lambda_1 \), and since \( \lambda \leq \lambda_1 \) and \( g \) is nonincreasing, \( g(\lambda_1) \leq g(\lambda) \). So, \( g(\lambda_1) \leq \lambda_1 \). It follows that \( g^0 \leq \lambda_1 = \max\{\lambda, g(\lambda)\} \). This proves (b)(\( \iff \)). Taking the infimum over all \( \lambda > 0 \) such that \( g(\lambda) < \infty \) (or all \( \lambda > 0 \)), we establish the inequality \( g^0 \leq \ldots \) in (a).

2. Let us prove inequality (\( \geq \)) in (a) and implication (\( \Rightarrow \)) in (b). Suppose \( g^0 \) is finite. Given \( \lambda_1 > g^0 \), we have \( g(\lambda_1) \leq \lambda_1 \), and so, \( g \not\equiv \infty \). This establishes (b)(\( \Rightarrow \)). Moreover (note that the monotonicity of \( g \) is not used),
\[
\inf_{\lambda > 0} \max\{\lambda, g(\lambda)\} \leq \inf_{\lambda > 0; g(\lambda) < \infty} \max\{\lambda, g(\lambda)\} = \max\{\lambda_1, g(\lambda_1)\} = \lambda_1.
\]
Passing to the limit as \( \lambda_1 \to g^0 \), we obtain the inequality \( g^0 \geq \ldots \) in (a).

3. (c)(\( \Rightarrow \)) If \( g \equiv 0 \), then \( g^0 = \inf(0, \infty) = 0 \) (equivalently, if \( g^0 \neq 0 \), then \( g \not\equiv 0 \)).
(c)(\( \iff \)) Let \( g^0 = 0 \). Then \( g(\mu) \leq \mu \) for all \( \mu > 0 \). Given \( \lambda > 0 \), for any \( 0 < \mu < \lambda \), by virtue of the monotonicity of \( g \), we get \( 0 \leq g(\lambda) \leq g(\mu) \leq \mu \). Letting \( \mu \to +0 \), we find \( g(\lambda) = 0 \) for all \( \lambda > 0 \), i.e., \( g \equiv 0 \). In other words, we have shown that \( g \not\equiv 0 \) implies \( g^0 \neq 0 \). \( \square \)

**Remark 2.2.5.** \( \) It is seen from the proof of Lemma 2.2.4(a) that
\[
g^0 = \inf \{ \max\{\lambda, g(\lambda)\} : \lambda > 0 \text{ such that } g(\lambda) < \infty \} \in [0, \infty) \text{ if } g \not\equiv \infty.
\]
Following the same lines as in the proof of Lemma 2.2.4, it may be shown that \( g^0 = \sup \{\lambda > 0 : g(\lambda) \geq \lambda\} \) (sup \( \emptyset = 0 \)) and \( g^0 = \sup_{\lambda > 0} \min\{\lambda, g(\lambda)\} \).

As a consequence of Theorem 2.2.1 and Lemma 2.2.4, we get the following

**Corollary 2.2.6.** \( d^0_w(x,y) = \inf_{\lambda > 0} \max\{\lambda, w_\lambda(x,y)\}, x, y \in X. \)

Given a nonincreasing function \( g : (0, \infty) \to [0, \infty] \), we denote by \( g_{+0} \) and \( g_{-0} \) the right and left regularizations of \( g \), defined (as in (1.2.2) and (1.2.3)) by:
$g_{+0}(\lambda) = g(\lambda + 0)$ and $g_{-0}(\lambda) = g(\lambda - 0)$ for all $\lambda > 0$. Functions $g_{+0}$ and $g_{-0}$ map $(0, \infty)$ into $[0, \infty]$ and are nonincreasing on $(0, \infty)$. Furthermore, $g_{+0}$ is continuous from the right and $g_{-0}$ is continuous from the left on $(0, \infty)$, and inequalities similar to (1.2.4) hold:

$$g(\lambda) \leq g(\lambda - 0) \leq g(\mu + 0) \leq g(\mu) \text{ in } [0, \infty] \text{ for all } 0 < \mu < \lambda. \quad (2.2.2)$$

Taking the above and (2.2.1) into account, we have

**Lemma 2.2.7.** If $g : (0, \infty) \rightarrow [0, \infty]$ is nonincreasing, then $(g_{+0})^0 = g^0 = (g_{-0})^0$.

**Proof.** Inequalities $(g_{+0})^0 \leq g^0 \leq (g_{-0})^0$ are consequences of the inclusions

$$\{\lambda > 0 : g(\lambda - 0) \leq \lambda\} \subset \{\lambda > 0 : g(\lambda) \leq \lambda\} \subset \{\lambda > 0 : g(\lambda + 0) \leq \lambda\},$$

which follow from (2.2.2). Now, we may assume that $g \neq \infty$. Then $g_{+0} \neq \infty$ and $g_{-0} \neq \infty$, which ensures that $g^0$, $(g_{+0})^0$, and $(g_{-0})^0$ are finite.

Let us show that $g^0 \leq (g_{+0})^0$. Given $\lambda > (g_{+0})^0$, choose $\mu$ such that $(g_{+0})^0 < \mu < \lambda$. By (2.2.2) and definition of $(g_{+0})^0$, we get

$$g(\lambda) \leq g(\mu + 0) = g_{+0}(\mu) \leq \mu < \lambda.$$ 

Hence $g^0 \leq \lambda$. Since $\lambda > (g_{+0})^0$ is arbitrary, we find $g^0 \leq (g_{+0})^0$.

In order to show that $(g_{-0})^0 \leq g^0$, we let $\lambda > g^0$. Then, for any $\mu > 0$ such that $g^0 < \mu < \lambda$, inequalities (2.2.2) and definition of $g^0$ imply

$$(g_{-0})(\lambda) = g(\lambda - 0) \leq g(\mu) \leq \mu < \lambda.$$ 

Therefore $(g_{-0})^0 \leq \lambda$. Letting $\lambda \rightarrow g^0$, we get $(g_{-0})^0 \leq g^0$. □

Putting, for a (pseudo)modular $w$ on $X$, $g = w_{x,y}$ in Lemma 2.2.7 and noting that $g_{\pm0} = (w_{\pm0})^{x,y}$ and $d_{w_{\pm0}}^0(x, y) = (g_{\pm0})^0$, we have

**Corollary 2.2.8.** $d_{w_{+0}}^0(x, y) = d_{w_{-0}}^0(x, y) = d_{w}^0(x, y)$ for all $x, y \in X$.

In particular, if $w$ and $w$ are (pseudo)modulars on $X$ such that $w_{+0} = W_{+0}$ or $w_{-0} = W_{-0}$, then $d_{w}^0 = d_{w}^0$ on $X \times X$.

We conclude that the right and left regularizations of a (pseudo)modular $w$ on $X$ provide no new modular spaces as compared to $X^*_w$, $X^0_w$ and $X^\text{fin}_w$ (cf. Sect. 2.1) and no new (pseudo)metrics as compared to $d_w^0$.

Yet, in Sect. 2.5, we establish the existence of continuum many (equivalent) metrics on the modular space $X^*_w$.

This section is continued by studying the basic metric $d^0_w(x, y)$ at the level of the map $g \mapsto g^0$, applied later to nonincreasing functions $g = w_{x,y}$. Our next lemma clarifies the definition of $g^0$ and Lemma 2.2.7 and, along with (2.2.1), gives a method for evaluating $g^0$ in terms of solutions of certain inequalities.
Lemma 2.2.9 (inequalities for $g^0$). Let $g : (0, \infty) \to [0, \infty]$ be a nonincreasing function with $0 < g^0 < \infty$ (i.e., $g \not\equiv 0$ and $g \not\equiv \infty$), and $\lambda > 0$. We have:

(a) $g^0 < \lambda$ if and only if $g(\lambda - 0) < \lambda$;
(b) $g^0 > \lambda$ if and only if $g(\lambda + 0) > \lambda$;
(c) $g^0 = \lambda$ if and only if $g(\lambda + 0) \leq \lambda \leq g(\lambda - 0)$.

Proof. (a)$\implies$ Suppose $g^0 < \lambda$. Given $\lambda_1$ and $\lambda_2$ such that $g^0 < \lambda_1 < \lambda_2 < \lambda$, by the monotonicity of $g$, $g(\lambda_2) \leq g(\lambda_1)$, and the definition of $g^0$ implies $g(\lambda_1) \leq \lambda_1$. Hence $g(\lambda_2) \leq \lambda_1$. Passing to the limits as $\lambda_1 \to g^0$ and $\lambda_2 \to \lambda$, we get $g(\lambda - 0) \leq g^0$, where $g^0 < \lambda$, and so, $g(\lambda - 0) < \lambda$.

(a)$\iff$ By the assumption, $g(\lambda - 0) < \lambda$, where $g(\lambda - 0) = \lim_{\mu \to \lambda - 0} g(\mu)$ and $\lambda = \lim_{\mu \to \lambda - 0} \mu$. So, there exists $\mu_0$ with $0 < \mu_0 < \lambda$ such that $g(\mu) < \mu$ for all $\mu$ with $\mu_0 \leq \mu < \lambda$. By the definition of $g^0$, we find $g^0 \leq \mu$, which implies $g^0 < \lambda$.

(b)$\implies$ Let $g^0 > \lambda$. For any $\lambda_1$ and $\lambda_2$ such that $g^0 > \lambda_2 > \lambda_1 > \lambda$, we have $g(\lambda_1) \geq g(\lambda_2) > \lambda_2$, where the last inequality follows from the definition of $g^0$: if, on the contrary, $g(\lambda_2) \leq \lambda_2$, then $g^0 \leq \lambda_2$, which contradicts the inequality $g^0 > \lambda_2$. Therefore $g(\lambda_1) > \lambda_2$. Letting $\lambda_2 \to g^0$ and $\lambda_1 \to \lambda$, we find $g(\lambda + 0) \geq g^0 > \lambda$.

(b)$\iff$ Since $\lim_{\mu \to \lambda + 0} g(\mu) = g(\lambda + 0) > \lambda = \lim_{\mu \to \lambda + 0} \mu$, there exists $\mu_0 > \lambda$ such that $g(\mu) > \mu$ for all $\mu$ with $\lambda < \mu \leq \mu_0$. It follows that $g^0 \geq \mu$ (otherwise, if $g^0 < \mu$, then the definition of $g^0$ implies $g(\mu) \leq \mu$, which is a contradiction). Since $\mu > \lambda$, we get $g^0 > \lambda$.

(c) The statement in (a) is equivalent to the following:

$$g^0 \geq \lambda \text{ if and only if } g(\lambda - 0) \geq \lambda,$$

(2.2.3)

and the one in (b) is equivalent to the assertion:

$$g^0 \leq \lambda \text{ if and only if } g(\lambda + 0) \leq \lambda.$$

(2.2.4)

From these two observations, (c) follows. $\square$

Remark 2.2.10. (a) Actually, a little bit more is shown in the proof of Lemma 2.2.9:

$g^0 < \lambda \implies g(\lambda - 0) \leq g^0 < \lambda$ in (a), and $g^0 > \lambda \implies g(\lambda + 0) \geq g^0 > \lambda$ in (b).

(b) We have $g^0 = \inf\{\lambda > 0 : g(\lambda) < \lambda\} \equiv g''$ (cf. (2.2.1) and Lemma 2.2.4).

In fact, this is clear if $g \equiv 0$ or $g \equiv \infty$, so let $0 < g^0 < \infty$. Since $\{\lambda > 0 : g(\lambda) < \lambda\} \subseteq \{\lambda > 0 : g(\lambda) \leq \lambda\}$, we get $g^0 \leq g''$. Now, given $\lambda > g^0$, inequalities (2.2.2) and Lemma 2.2.9(a) imply $g(\lambda) \leq g(\lambda - 0) < \lambda$, and so, $g'' \leq \lambda$, which yields $g'' \leq g^0$.

(c) Assuming one-sided continuity of $g$ on $(0, \infty)$, in view of (2.2.4) and (2.2.3), we get some useful particular cases of Lemma 2.2.9:

$$g^0 \leq \lambda \iff g(\lambda) \leq \lambda, \text{ provided } g \text{ is continuous from the right};$$
$$g^0 < \lambda \iff g(\lambda) < \lambda, \text{ provided } g \text{ is continuous from the left};$$
$$g^0 = \lambda \iff g(\lambda) = \lambda \text{ (i.e., } \lambda \text{ is a fixed point of } g), \text{ provided } g \text{ is continuous}.$$
To illustrate Lemma 2.2.9, consider \( g : (0, \infty) \to (0, \infty) \) defined by: 
\[
g(\lambda) = \begin{cases} 
3 & \text{if } 0 < \lambda < 1, \\
2 & \text{if } \lambda = 1, \\
0 & \text{if } \lambda > 1.
\end{cases}
\]
Clearly, \( g \) is nonincreasing and \( g^0 = \inf(1, \infty) = 1 \).

Inequalities in Lemma 2.2.9(c) are of the form:
\[
g(1 + 0) = 0 < g^0 = 1 < 3 = g(1 - 0).
\]
Although strict inequality \( g(1 - 0) = 3 > 1 = \lambda \) holds in (2.2.3), we have \( g^0 = \lambda = 1 \).
Similarly, \( g(1 + 0) = 0 < 1 = \lambda \) in (2.2.4) and \( g^0 = 1 = \lambda \).

Setting \( g = w^{x,y} \) in Lemma 2.2.9 (for \( x, y \in X_w^* \)), we obtain the following important result for modulars \( w \) on \( X \) (cf. also Remark 2.2.10(a), (c)).

**Theorem 2.2.11.** Let \( w \) be a (pseudo)modular on the set \( X \), \( X_w^* \) be the modular space, \( \lambda > 0 \), and \( x, y \in X_w^* \). Then we have:

(a) condition \( d_w^0(x, y) < \lambda \) implies \( w_{\lambda-0}(x, y) \leq d_w^0(x, y) < \lambda \), and conversely,
condition \( w_{\lambda-0}(x, y) < \lambda \) implies \( d_w^0(x, y) < \lambda \);
(b) inequality \( d_w^0(x, y) > \lambda \) implies \( w_{\lambda+0}(x, y) \geq d_w^0(x, y) > \lambda \), and conversely,
inequality \( w_{\lambda+0}(x, y) > \lambda \) implies \( d_w^0(x, y) > \lambda \);
(c) equality \( d_w^0(x, y) = \lambda \) is equivalent to \( w_{\lambda+0}(x, y) \leq \lambda \leq w_{\lambda-0}(x, y) \).

Under the continuity assumptions on \( w \), additional equivalences hold:

(d) if \( w \) is continuous from the right, then \( d_w^0(x, y) \leq \lambda \iff w_{\lambda}(x, y) \leq \lambda \);
(e) if \( w \) is continuous from the left, then \( d_w^0(x, y) < \lambda \iff w_{\lambda}(x, y) < \lambda \);
(f) if \( w \) is continuous on \((0, \infty)\), then \( d_w^0(x, y) = \lambda \iff w_{\lambda}(x, y) = \lambda \).

The conclusions of Theorem 2.2.11 are sharp (cf. Remark 2.2.10(d) and (1.3.1)).

**Example 2.2.12.** Let \( w \) be given by (1.3.2) with \( h(\lambda) = \lambda^p \) \((p > 0)\). Since \( w \) is continuous on \((0, \infty)\), by virtue of Theorem 2.2.11(f), the value \( \lambda = d_w^0(x, y) \) with \( x \neq y \) satisfies the equation \( w_\lambda(x, y) = \lambda \), that is,
\[
\lambda^{p+1} + d(x, y)\lambda - d(x, y) = 0. \tag{2.2.5}
\]
If \( p = 1 \), then solving the corresponding quadratic equation, we get
\[
d_w^0(x, y) = \frac{\sqrt{(d(x, y))^2 + 4d(x, y) - d(x, y)}}{2}. \tag{2.2.6}
\]
For \( p = 2 \), the solution \( \lambda \) of the corresponding cubic equation (2.2.5) is given by Cardano’s formula:
\[
d_w^0(x, y) = \frac{3}{2} \sqrt[3]{\frac{a}{2}} + \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{3}\right)^3} - \frac{3}{2} \sqrt{-\left(\frac{a}{2}\right)^2 + \left(\frac{a}{3}\right)^3}, \tag{2.2.7}
\]
where \( a = d(x, y) \), and the square and cube roots of positive numbers have uniquely determined positive values. The solution by radicals of the fourth-order equation (for \( p = 3 \)) can be obtained by Ferrari’s method, and is left to the interested reader.

Note that, for any function \( h \) from (1.3.2), we have \( d^0_w(x, y) < 1 \).

In fact, if \( h \) is continuous on \( (0, \infty) \), equality \( w_\lambda(x, y) = \lambda \) is of the form \( f(\lambda) = 0 \), where \( f(\lambda) = \lambda h(\lambda) - (1 - \lambda) d(x, y) \), and \( \lambda h(\lambda) \to 0 \) as \( \lambda \to +0 \). Setting \( \lambda h(\lambda) = 0 \) if \( \lambda = 0 \), we find that \( f \) is continuous on \( [0, \infty) \), \( f(0) = -d(x, y) < 0 \) (if \( x \neq y \)), and \( f(1) = h(1) > 0 \). By the Intermediate Value Theorem, \( f(\lambda) = 0 \) for some \( 0 < \lambda < 1 \), and so, \( d^0_w(x, y) = \lambda < 1 \).

In the general case, we first show that if there exists \( \mu > 0 \) such that

\[
 w_{\lambda-0}(x, y) < \mu \quad \text{for all } \lambda > 0 \text{ and } x, y \in X, \text{ then } d^0_w(x, y) < \mu \quad \text{for all } x, y \in X.
\]

Since \( w_\lambda(x, y) \leq w_{\lambda-0}(x, y) < \mu \), and this holds for \( \lambda = \mu \), we find \( d^0_w(x, y) \leq \mu \). If we assume that \( d^0_w(x, y) = \mu \) (for some \( x \neq y \)), then, by Theorem 2.2.11(b), we have \( w_\lambda(x, y) \geq w_{\lambda+0}(x, y) > \lambda \) for all \( 0 < \lambda < d^0_w(x, y) = \mu \), and so, \( w_{\lambda-0}(x, y) \) is equal to \( \lim_{\lambda \to \mu-0} w_\lambda(x, y) \geq \mu \), which contradicts the assumption. It remains to note that \( w_{\lambda-0}(x, y) < 1 = \mu \) for our modular \( w \) from (1.3.2).

One more example of a (pseudo)metric from Theorem 2.2.1 is given by the quantity \( d^0_w \) on the power set \( \mathcal{P}(X) \) of \( X \), where \( W \) is the Hausdorff pseudomodular on \( \mathcal{P}(X) \) induced by a (pseudo)modular \( w \) on \( X \). There are two ways of obtaining a distance function on \( \mathcal{P}(X) \) starting from \( w \) on \( X \), namely

\[
 w \text{ on } X \xrightarrow{\text{Theorem 2.2.1}} d^0_w \text{ on } X \xrightarrow{\text{Appendix A.1}} D_{d^0_w} \text{ on } \mathcal{P}(X)
\]

and

\[
 w \text{ on } X \xrightarrow{\text{Section 1.3.5}} W \text{ on } \mathcal{P}(X) \xrightarrow{\text{Theorem 2.2.1}} d^0_w \text{ on } \mathcal{P}(X).
\]

Fortunately, the resulting distance functions \( D_{d^0_w} \) and \( d^0_w \) coincide on \( \mathcal{P}(X) \) as the following theorem asserts.

**Theorem 2.2.13.** Let \( w \) be a (pseudo)modular on \( X \), \( D = D_{d^0_w} \) be the Hausdorff distance on \( \mathcal{P}(X) \) generated by the extended (pseudo)metric \( d^0_w \) on \( X \), and \( W \) be the Hausdorff pseudomodular on \( \mathcal{P}(X) \) induced by \( w \). Then

\[
 d^0_w(A, B) = D(A, B) \quad \text{for all } A, B \in \mathcal{P}(X).
\]

**Proof.** Since \( d^0_w(\emptyset, \emptyset) = 0 = D(\emptyset, \emptyset) \), and \( d^0_w(A, \emptyset) = \infty = D(A, \emptyset) \) for all \( A \neq \emptyset \), we may assume that \( A \neq \emptyset \) and \( B \neq \emptyset \).

\[
 (\geq) \quad \text{Suppose } d^0_w(A, B) = \inf \{ \lambda > 0 : W_\lambda(A, B) \leq \lambda \} \text{ is finite, and } \lambda > d^0_w(A, B).
\]

Applying (1.2.4) and Theorem 2.2.11(a) (cf. also Remark 2.2.10(b)), we get
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\[ W_\lambda(A, B) = \max\{E_\lambda(A, B), E_\lambda(B, A)\} < \lambda, \]

and so, \( E_\lambda(A, B) < \lambda \) and \( E_\lambda(B, A) < \lambda \). By (1.3.12), we have \( \inf_{y \in B} w_\lambda(x, y) < \lambda \) for all \( x \in A \). So, for each \( x \in A \) there exists \( y_x \in B \) (depending also on \( \lambda \)) such that \( w_\lambda(x, y_x) < \lambda \). The definition of \( d_\lambda^0 \) gives \( d_\lambda^0(x, y_x) \leq \lambda \). Since

\[ \inf_{y \in B} d_\lambda^0(x, y) \leq d_\lambda^0(x, y_x) \leq \lambda \quad \text{for all } x \in A, \]

we get \( e(A, B) = \sup_{x \in A} \inf_{y \in B} d_\lambda^0(x, y) \leq \lambda \). Similarly, \( E_\lambda(B, A) < \lambda \) implies inequality \( e(B, A) \leq \lambda \). Therefore \( D(A, B) = \max\{e(A, B), e(B, A)\} \leq \lambda \) for all \( \lambda > d_\lambda^0(A, B) \), and so, \( D(A, B) \leq d_\lambda^0(A, B) < \infty \).

\( (\leq) \) Let \( D(A, B) < \infty \), and \( \lambda > D(A, B) \) be arbitrary. Then \( \lambda > e(A, B) \) as well as \( \lambda > e(B, A) \). Inequality \( \lambda > e(A, B) = \sup_{x \in A} \inf_{y \in B} d_\lambda^0(x, y) \) implies that, given \( x \in A \), \( \lambda > \inf_{y \in B} d_\lambda^0(x, y) \). So, for every \( x \in A \) there exists \( y_x \in B \) (also depending on \( \lambda \)) such that \( \lambda > d_\lambda^0(x, y_x) \). By the definition of \( d_\lambda^0 \), we have \( w_\lambda(x, y_x) \leq \lambda \). Since

\[ \inf_{y \in B} w_\lambda(x, y) \leq w_\lambda(x, y_x) \leq \lambda \quad \text{for all } x \in A, \]

we find \( E_\lambda(A, B) = \sup_{x \in A} \inf_{y \in B} w_\lambda(x, y) \leq \lambda \). Similarly, inequality \( \lambda > e(B, A) \) implies \( E_\lambda(B, A) \leq \lambda \). Hence \( W_\lambda(A, B) = \max\{E_\lambda(A, B), E_\lambda(B, A)\} \leq \lambda \). The definition of \( d_\lambda^0 \) yields \( d_\lambda^0(A, B) \leq \lambda \) for all \( \lambda > D(A, B) \), and so, \( d_\lambda^0(A, B) \leq D(A, B) < \infty \).

\[ \square \]

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Now we treat the case when a (pseudo)modular \( w \) on \( X \) is convex: \( w \) gives rise to an additional (pseudo)metric on the modular space \( X_w^* \) to be studied below.

We make use of the following observation. As we have seen in Remark 1.2.2(d), the convexity of a (pseudo)modular \( w \) on \( X \) is equivalent to the fact that the function \( \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y) \) is a (pseudo)modular on \( X \). On the other hand, if a function \( \hat{w} \) on \((0, \infty) \times X \times X \) is initially given, then we have: \( \hat{w} \) is a (pseudo)modular on \( X \) if and only if \( \hat{w}_\lambda(x, y) = \hat{w}_\lambda(x, y)/\lambda \) is a convex (pseudo)modular on \( X \).

From Sect. 2.1, we find

\[ X_w^0 \subset X_w = X_w^* \quad \text{and} \quad X_w^{\text{fin}} = X_w^{\text{fin}} \subset X_w^* = X_w^*. \]  \hspace{1cm} (2.3.1)

By Theorem 2.2.1, \( \hat{w} \) generates a (pseudo)metric on \( X_w^* \) of the form

\[ d_{\hat{w}}^0(x, y) = \inf\{\lambda > 0 : \hat{w}_\lambda(x, y) \leq \lambda\} = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}. \]  \hspace{1cm} (2.3.2)

The last expression is given in terms of \( w \) and is denoted by \( d_w^*(x, y) \).
Properties of $d_w^*$ are gathered in the following theorem, where Theorem 2.2.1 and Corollary 2.2.6 are applied to $\hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y)$ and expressed via $w$.

**Theorem 2.3.1.** Let $w$ be a convex (pseudo)modular on $X$. Then

$$d_w^*(x, y) \equiv \inf \{ \lambda > 0 : w_\lambda(x, y) \leq 1 \} = \inf_{\lambda > 0} \max \{ \lambda, \lambda w_\lambda(x, y) \}, \quad x, y \in X,$$

is an extended (pseudo)metric on $X$ (with $d_w^*(x, y) < \infty \iff x \sim y$), whose restriction to the modular space $X^*$ is a (pseudo)metric on $X^*$.

Furthermore, $d_w^0$ and $d_w^*$ are nonlinearly equivalent in the following sense: given $x, y \in X^*$, we have

$$\min\{d_w^*(x, y), \sqrt{d_w^*(x, y)}\} \leq d_w^0(x, y) \leq \max\{d_w^*(x, y), \sqrt{d_w^*(x, y)}\},$$

(2.3.4)

or, equivalently (written in a different way),

$$d_w^0(x, y) \cdot \min\{1, d_w^0(x, y)\} \leq d_w^*(x, y) \leq d_w^0(x, y) \cdot \max\{1, d_w^0(x, y)\}.$$  

(2.3.5)

Only the second part of Theorem 2.3.1 is to be verified. For this, we need some precise inequalities for $d_w^* = d_w^0$, which are reformulated from Theorem 2.2.11 (applied to $\hat{w}$) in terms of $w$ and stated, for ease of reference, as

**Theorem 2.3.2.** Let $w$ be a convex (pseudo)modular on $X$, $\lambda > 0$, and $x, y \in X^*$. Then we have:

(a) $d_w^*(x, y) < \lambda$ implies $w_{\lambda-0}(x, y) \leq d_w^*(x, y)/\lambda < 1$, and conversely,

$$w_{\lambda-0}(x, y) < 1 \implies d_w^*(x, y) < \lambda;$$

(b) $d_w^*(x, y) > \lambda$ implies $w_{\lambda+0}(x, y) \geq d_w^*(x, y)/\lambda > 1$, and conversely,

$$w_{\lambda+0}(x, y) > 1 \implies d_w^*(x, y) > \lambda;$$

(c) $d_w^*(x, y) = \lambda$ is equivalent to $w_{\lambda+0}(x, y) \leq 1 \leq w_{\lambda-0}(x, y)$.

In addition, under the continuity assumptions on $w$, we get:

(d) $d_w^*(x, y) \leq \lambda \iff w_{\lambda}(x, y) \leq 1$, provided $w$ is continuous from the right;

(e) $d_w^*(x, y) < \lambda \iff w_\lambda(x, y) < 1$, provided $w$ is continuous from the left;

(f) $d_w^*(x, y) = \lambda \iff w_\lambda(x, y) = 1$, provided $w$ is continuous on $(0, \infty)$.

**Proof (of Theorem 2.3.1 (second part)).** In steps 1 and 2, we show that inequalities $d_w^0(x, y) < 1$ and $d_w^*(x, y) < 1$ are equivalent, and if one of them holds, then

$$d_w^*(x, y) \leq d_w^0(x, y) \leq \sqrt{d_w^*(x, y)}.$$

(2.3.6)

Since $d_w^*(x, y) < 1$ implies $d_w^*(x, y) \leq \sqrt{d_w^*(x, y)}$, inequality (2.3.6) proves (2.3.4).

1. Suppose $d_w^0(x, y) < 1$. Let us show that $d_w^*(x, y) \leq d_w^0(x, y)$ (and so, $d_w^*(x, y) < 1$).

In fact, for any number $\lambda$ such that $d_w^0(x, y) < \lambda < 1$, the definition of $d_w^0$ gives
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2. Assume that $d^*_w(x, y) < 1$. Let us prove that $d^*_w(x, y) \leq \sqrt{d^*_w(x, y)}$, which is the right-hand side inequality in (2.3.6) (and so, $d^0_w(x, y) < 1$). Since $d^*_w(x, y) \leq \sqrt{d^*_w(x, y)} < 1$, for any $\lambda$ such that $\sqrt{d^*_w(x, y)} < \lambda < 1$, inequalities (1.2.4) and, by virtue of convexity of $w$, Theorem 2.3.2(a) imply

$$w_\lambda(x, y) \leq w_{\lambda-0}(x, y) \leq \frac{d^*_w(x, y)}{\lambda} < \frac{\lambda^2}{\lambda} = \lambda.$$ 

By the definition of $d^*_w$, $d^*_w(x, y) \leq \lambda$. Letting $\lambda$ tend to $\sqrt{d^*_w(x, y)}$, we obtain the desired inequality.

As a consequence of steps 1 and 2, inequalities $d^0_w(x, y) \geq 1$ and $d^*_w(x, y) \geq 1$ are equivalent, as well. In steps 3 and 4, we show that if one of these inequalities holds, then

$$\sqrt{d^*_w(x, y)} \leq d^0_w(x, y) \leq d^*_w(x, y). \quad (2.3.7)$$

Since $d^*_w(x, y) \geq 1$ implies $d^*_w(x, y) \geq \sqrt{d^*_w(x, y)}$, (2.3.7) establishes (2.3.4).

3. Inequality $d^0_w(x, y) \geq 1$ implies $d^0_w(x, y) \leq d^*_w(x, y)$: in fact, by the definition of $d^*_w$, $w_\lambda(x, y) \leq 1$ for all $\lambda > d^*_w(x, y)$, and since $\lambda > 1$, $w_\lambda(x, y) < \lambda$. From the definition of $d^0_w$, we get $d^0_w(x, y) \leq \lambda$. The assertion follows thanks to the arbitrariness of $\lambda > d^*_w(x, y)$.

4. Suppose $d^0_w(x, y) \geq 1$, and let us show that $\sqrt{d^*_w(x, y)} \leq d^0_w(x, y)$, which is the left-hand side inequality in (2.3.7). Given $\lambda > d^0_w(x, y)$, we have $w_\lambda(x, y) \leq \lambda$, and since $\lambda > 1$, $\lambda^2 > \lambda$. The convexity of $w$ and (1.2.5) imply

$$w_{\lambda^2}(x, y) \leq \frac{\lambda}{\lambda^2} w_\lambda(x, y) \leq \frac{\lambda}{\lambda^2} \cdot \lambda = 1,$$

whence $d^*_w(x, y) \leq \lambda^2$. Letting $\lambda$ go to $d^0_w(x, y)$, we get $d^*_w(x, y) \leq (d^0_w(x, y))^2$.

\[\square\]

Remark 2.3.3. 1. If $w$ is nonconvex, the quantity $d^*_w(x, y) \in [0, \infty]$ from (2.3.3) has only two properties: $d^*_w(x, x) = 0$, and $d^*_w(x, y) = d^*_w(y, x)$. It follows from (2) in this Remark that $d^*_w(x, y) = 0 \nRightarrow x = y$, and from (4)—that the triangle inequality may not hold for $d^*_w$.

2. The convexity of $w$ is essential for inequalities (2.3.4) and (2.3.5): modular (1.3.2) is nonconvex, and $d^0_w$ is a well-defined metric on $X$ (e.g., (2.2.6) and (2.2.7)), but, since $w_\lambda(x, y) < 1$ for all $\lambda > 0$, we have $d^*_w(x, y) = 0$ for all $x, y \in X$ (and, in particular, $d^*_w$ is not a metric on $X$).

3. In the proof of Theorem 2.3.1, the implications in steps 1 and 3, which are of the form $d^0_w(x, y) < 1 \Rightarrow d^*_w(x, y) \leq d^0_w(x, y)$, and $d^*_w(x, y) \geq 1 \Rightarrow d^0_w(x, y) \leq d^*_w(x, y)$, do not rely on the convexity of $w$ and are valid for those (pseudo)modulurs $w$, for which the quantity $d^*_w(x, y)$ is well-defined. The example in (2) above is consistent with the former implication.
4. For the modular $w_{\lambda}(x, y) = d(x, y)/\lambda^p$ ($p > 0$) from Example 2.2.2(1), we have $d^0_w(x, y) = (d(x, y))^{1/(p+1)}$ and $d^*_w(x, y) = (d(x, y))^{1/p}$, where we note that $d^*_w$ is a metric on $X$ if and only if $w$ is convex, i.e., $p \geq 1$. So, for $p \geq 1$, setting $a = d(x, y)$, inequalities (2.3.6) and (2.3.7) assume the form:

$$a^p \leq a^{p+1} \leq a^1 \leq a^p$$

if $0 \leq a < 1$, and $a^p \leq a^{p+1} \leq a^1$ if $a \geq 1$.

5. Inequalities (2.3.4) are the best possible: see Example 2.3.5(1).

Remark 2.3.4. If $\rho$ is a classical convex modular on a real linear space $X$ (cf. Sect. 1.3.3 and Remark 2.2.3), then the modular space $X_{\rho}$ coincides with the set $X_{\rho}^* = \{x \in X : \rho(\alpha x) < \infty \text{ for some } \alpha > 0\}$, and the functional $\|x\|_{\rho} = \inf \{\varepsilon > 0 : \rho(x/\varepsilon) \leq 1\}$ ($x \in X_{\rho}^*$) is a norm on $X_{\rho} = X_{\rho}^*$, which is nonlinearly equivalent to the $F$-norm $\|x\|_{\rho}$ in the same sense as in Theorem 2.3.1. Moreover, under the assumptions of Proposition 1.3.5, where $X$ is a linear space and $\rho(x) = w_1(x, 0)$, we have: $X_{\rho}^* = X_w^*(0) = X_{\rho}$ is a linear subspace of $X$, and the functional $\|x\|_{\rho} = d^*_w(x, 0), x \in X_{\rho}^*$, is a norm on $X_{\rho}^*$.

2. Similar to Corollary 2.2.8, if $w$ is convex, then $d^*_{w+0} = d^*_w = d^*_w$ on $X \times X$. In fact, $(\hat{w}^\lambda)_{\lambda}(x, y) \equiv \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y)$ is also a (pseudo)modular on $X$, and $(w_{\pm0})^\lambda = (\hat{w}_{\pm0} = (w^\lambda)^{\pm0}, which can be seen as follows. Given $\lambda > 0$ and $x, y \in X, (1.2.2)$ and (1.2.3) imply

$$(w_{\pm0})^\lambda_{\lambda}(x, y) = \lambda (w_{\pm0})_{\lambda}(x, y) = \lambda \lim_{\mu \to \lambda_{\pm0}} \mu w_\mu(x, y)$$

$$= \lim_{\mu \to \lambda_{\pm0}} (w^\lambda_{\mu}(x, y) = (w^\lambda)^{\pm0}(x, y) = ((w^\lambda)^{\pm0})_{\hat{\lambda}}(x, y).$$

By virtue of (2.3.3) and (2.3.2), $d^*_w = d^*_w$, and Corollary 2.2.8 yields

$$d^*_{w_{\pm0}} = d^0_{(w_{\pm0})^\lambda} = d^0_{(w^\lambda)^{\pm0}} = d^*_{w_{\pm0}} = d^*_w.$$

Example 2.3.5. Consider the modular $w_\lambda(x, y) = \varphi(d(x, y)/\lambda)$ from (1.3.5), where the function $\varphi : [0, \infty) \to [0, \infty]$ is nondecreasing and such that $\varphi(0) = 0, \varphi \neq \infty, (X, d)$ is a metric space, $x, y \in X^*_w = X$, and $\lambda > 0$.

1. Let $\varphi(u) = u^p$ ($p > 0$). Then $w$ is strict, convex if $p \geq 1$, and nonconvex if $0 < p < 1$. For any $p > 0$, we have

$$d^0_w(x, y) = (d(x, y))^{p/(p+1)} \text{ and } d^*_w(x, y) = d(x, y).$$

To show that inequalities (2.3.4) are the best possible, we note that if $p = 1$, then $d^0_w(x, y) = \sqrt{d^*_w(x, y)}$, and if $p > 1$, then ($w$ is convex and) we find

$$d^0_w(x, y) = (d^*_w(x, y))^{p/(p+1)} \to d^*_w(x, y) \text{ as } p \to \infty.$$
2. Let \( w \) be the \((a, 0)\)-modular from (1.3.9). If \( a = \infty \), then \( w \) is nonstrict and convex, and we have: \( d_w^0(x, y) = d_w^a(x, y) = d(x, y) \). Now, if \( a > 0 \), then \( w \) is nonstrict and nonconvex, and we have: \( d_w^0(x, y) = \min\{a, d(x, y)\} \), \( d_w^a(x, y) = 0 \) if \( a \leq 1 \), and \( d_w^a(x, y) = d(x, y) \) if \( a > 1 \).

3. If \( \varphi(u) = u \) for \( 0 \leq u \leq 1 \), and \( \varphi(u) = 1 \) for \( u > 1 \), then the modular

\[
w_\lambda(x, y) = \begin{cases} 1 & \text{if } 0 < \lambda < d(x, y), \\
\frac{d(x, y)}{\lambda} & \text{if } \lambda \geq d(x, y),
\end{cases}
\]

is strict and nonconvex, and \( d_w^0(x, y) = \min\{1, \sqrt{d(x, y)}\} \).

4. Let \( \varphi(u) = 0 \) for \( 0 \leq u \leq 1 \), and \( \varphi(u) = u - 1 \) for \( u > 1 \). We have:

\[
w_\lambda(x, y) = \frac{d(x, y)}{\lambda} - 1 \quad \text{if } 0 < \lambda < d(x, y), \quad \text{and} \quad w_\lambda(x, y) = 0 \quad \text{if } \lambda \geq d(x, y),
\]

is nonstrict and convex, and (note that \( d_w^0(x, y) < d(x, y) \) if \( x \neq y \))

\[
d_w^0(x, y) = \sqrt{1 + 4d(x, y)} - 1 \quad \text{and} \quad d_w^a(x, y) = \frac{d(x, y)}{2}.
\]

5. Suppose \( \varphi(0) = 0, \varphi(u) = 1 \) if \( 0 < u < 1 \), and \( \varphi(u) = u \) if \( u \geq 1 \). Given \( \lambda > 0 \) and \( x, y \in X \), we have: \( w_\lambda(x, y) = 0 \) if \( x = y \), and if \( x \neq y \),

\[
w_\lambda(x, y) = \frac{d(x, y)}{\lambda} \quad \text{if } 0 < \lambda \leq d(x, y), \quad \text{and} \quad w_\lambda(x, y) = 1 \quad \text{if } \lambda > d(x, y).
\]

Then the modular \( w \) is strict and nonconvex, \( d_w^0(x, y) = \max\{1, \sqrt{d(x, y)}\} \) if \( x \neq y \), and \( d_w^0(x, y) = 0 \) if \( x = y \).

6. Suppose \( \varphi \) is given by: \( \varphi(u) = u \) if \( 0 \leq u \leq 1 \), \( \varphi(u) = 1 \) if \( 1 < u < 2 \), and \( \varphi(u) = u - 1 \) if \( u \geq 2 \). The corresponding modular \( w \) is strict and nonconvex, and we have: \( d_w^0(x, y) = \sqrt{d(x, y)} \) if \( d(x, y) \leq 1 \), \( d_w^0(x, y) = 1 \) if \( 1 < d(x, y) < 2 \), and \( d_w^0(x, y) = \frac{1}{2}(\sqrt{1 + 4d(x, y)} - 1) \) if \( d(x, y) \geq 2 \).

### 2.4 Modulares and Metrics on Sequence Spaces

Let \((M, d)\) be a metric space, \(X = M^\mathbb{N}\)—the set of all sequences \(x = \{x_n\}\) from \(M\), and \(x^o = \{x_n^o\} \subseteq M\)—a given sequence (the center of a modular space). In this section, we study two special modulares defined on \(X\).

1. The modular \( w \) from (1.3.10) with \( \varphi(u) = u^p (p > 0) \) and \( h(\lambda) = \lambda^q \) \((q \geq 1)\) is strict and continuous, and it is convex if \( p \geq 1 \). The modular spaces (around \(x^o\)) are given by
$$X_w^* = X_w^0 = X_w^{\text{fin}} = \left\{ x = \{x_n\} \in X : \sum_{n=1}^{\infty} (d(x_n, x_n^0))^p < \infty \right\}$$

(if \( M = \mathbb{R} \) with metric \( d(x,y) = |x-y| \) and \( x^0 = 0 \) \( \{0\}_{n=1}^{\infty} \), then \( X_w^*(0) \) is the usual space \( \ell_p \) of all real \( p \)-summable sequences).

Let \( H(\lambda) = \lambda (h(\lambda))^p = \lambda^{pq+1} \). The metric \( d_w^0 \) on \( X_w^* \) is of the form:

$$d_w^0(x,y) = H^{-1}\left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right) = \left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right)^{1/(pq+1)}$$

where \( H^{-1}(\mu) = \mu^{1/(pq+1)} \) is the inverse function of \( H \) on \([0, \infty)\).

If \( p \geq 1 \), then \( w \) is convex, and we also have metric \( d_w^* \) on \( X_w^* \) of the form:

$$d_w^*(x,y) = h^{-1}\left( \left[ \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right]^{1/p} \right) = \left( \sum_{n=1}^{\infty} (d(x_n, y_n))^p \right)^{1/pq}$$

where \( h^{-1} : [0, \infty) \to [0, \infty) \) is the inverse function of \( h \) (see Example 1.3.10, and Appendix A.1 concerning general superadditive functions \( h \)).

2. Given \( \lambda > 0 \) and \( x = \{x_n\}, y = \{y_n\} \in X = M^\mathbb{N} \), we set

$$w_\lambda(x,y) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, y_n)}{\lambda} \right)^{1/n}.$$  \hspace{1cm} (2.4.1)

**Proposition 2.4.1.** \( w = \{w_\lambda\}_{\lambda > 0} \) is a strict nonconvex continuous modular on \( X \).

**Proof.** Axioms (i), (ii), and (iii) are clear, and axiom (iii) follows from inequalities (1.3.11) with \( \varphi(u) = u^{1/n} \) and \( h(\lambda) = \lambda \).

In order to see that \( w \) is nonconvex, we show that \( X_w^0(x^0) \neq X_w^*(x^0) \) for some \( x^0 \in X \) (cf. Sect. 2.1). Choose any \( x^0 \in M \) and \( x \in M, x \neq x^0 \), and let \( x^0 = \{x_n^0\}_{n=1}^{\infty} \) and \( x = \{x_n\}_{n=1}^{\infty} \) also denote the corresponding constant sequences from \( X \). Given \( \lambda > d(x,x^0) > 0 \), we find

$$w_\lambda(x,x^0) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, x_n^0)}{\lambda} \right)^{1/n} = \lim_{n \to \infty} \left( \frac{d(x_n, x_n^0)}{\lambda} \right)^{1/n} = 1,$$

and so, \( x \in X_w^*(x^0) \setminus X_w^0(x^0) \).

Let us show that \( w_\lambda(x,y) \leq w_{\lambda+0}(x,y) \) and \( w_{\lambda-0}(x,y) \leq w_\lambda(x,y) \) for all \( \lambda > 0 \) and \( x,y \in X \), which, by virtue of inequalities (1.2.4), establish the continuity property of \( w \). For any \( n \in \mathbb{N} \) and \( \mu > \lambda \), the definition of \( w \) implies

$$\left( \frac{d(x_n, y_n)}{\mu} \right)^{1/n} \leq w_\mu(x,y),$$
and so, as $\mu \to \lambda + 0$, we get
\[
\left( \frac{d(x_n, y_n)}{\lambda} \right)^{1/n} \leq w_{\lambda+0}(x, y).
\]
Taking the supremum over all $n \in \mathbb{N}$, we obtain the first inequality above. Now, given $\lambda, \mu > 0$, we have
\[
w_\mu (x, y) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, y_n)}{\lambda} \right)^{1/n} \cdot \left( \frac{\lambda}{\mu} \right)^{1/n} \leq w_\lambda (x, y) \cdot \sup_{n \in \mathbb{N}} (\lambda/\mu)^{1/n}
\]
\[
= w_\lambda (x, y) \cdot \max \{1, \lambda/\mu\}, \quad x, y \in X.
\] (2.4.2)

It follows that if $0 < \mu < \lambda$, then $w_\mu (x, y) \leq w_\lambda (x, y) \cdot \lambda/\mu$, and so, passing to the limit as $\mu \to \lambda - 0$, we get $w_{\lambda-0}(x, y) \leq w_\lambda (x, y)$. □

Note that (2.4.2) with $y = x^0$ proves that $X_\mu^{\text{fin}}(x^0) = X_\mu^*(x^0)$, and establishes the following characterization of this modular space in terms of sequences $x = \{x_n\}$ and $x^0 = \{x^0_n\}$ themselves:
\[
x \in X_\mu^*(x^0) \quad \text{if and only if} \quad w_1(x, x^0) = \sup_{n \in \mathbb{N}} \left( d(x_n, x^0_n) \right)^{1/n} < \infty. \quad (2.4.3)
\]

The modular space $X_\mu^0(x^0)$ is characterized in the following way.

**Proposition 2.4.2.** Given $x \in X$, $x \in X_\mu^0(x^0)$ if and only if $\lim_{n \to \infty} \left( d(x_n, x^0_n) \right)^{1/n} = 0$.

**Proof.** Suppose $x \in X_\mu^0(x^0)$. Then $w_\lambda (x, x^0) \to 0$ as $\lambda \to \infty$, and so, for each $\varepsilon > 0$ there exists $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that
\[
w_{\lambda_0}(x, x^0) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, x^0_n)}{\lambda_0} \right)^{1/n} \leq \varepsilon. \quad (2.4.4)
\]
This inequality is equivalent to
\[
\left( d(x_n, x^0_n) \right)^{1/n} \leq (\lambda_0)^{1/n} \cdot \varepsilon \quad \text{for all} \quad n \in \mathbb{N}. \quad (2.4.5)
\]
Passing to the limit superior as $n \to \infty$, we get
\[
\limsup_{n \to \infty} \left( d(x_n, x^0_n) \right)^{1/n} \leq \varepsilon.
\]
Due to the arbitrariness of $\varepsilon > 0$, $(d(x_n, x^0_n))^{1/n} \to 0$ as $n \to \infty$. 
Now, assume that \((d(x_n, x_n^o))^{1/n} \to 0\) as \(n \to \infty\). Given \(\epsilon > 0\), there exists a number \(n_0 = n_0(\epsilon) \in \mathbb{N}\) such that \((d(x_n, x_n^o))^{1/n} < \epsilon\) for all \(n > n_0\). Setting

\[
\lambda_1(\epsilon) = \max\{1, 1/\epsilon^{n_0}\} \cdot \max_{1 \leq n \leq n_0} d(x_n, x_n^o)
\]

and noting that

\[
d(x_n, x_n^o) = \frac{d(x_n, x_n^o)}{\epsilon^n} \leq \lambda_1(\epsilon) \cdot \epsilon^n \text{ for all } 1 \leq n \leq n_0,
\]

we obtain (2.4.5) with \(\lambda_0 = \lambda_0(\epsilon) = \max\{1, \lambda_1(\epsilon)\}\). It follows that inequality (2.4.4) holds, whence, by virtue of (1.2.1), \(w_\lambda(x, x^o) \leq w_{\lambda_0}(x, x^o) \leq \epsilon\) for all \(\lambda \geq \lambda_0\). This means that \(w_\lambda(x, x^o) \to 0\) as \(\lambda \to \infty\), i.e., \(x \in X_w^*(x^o)\).

The metric \(d_w^0\) on the modular space \(X_w^0(x^o)\) is given by

\[
d_w^0(x, y) = \sup_{n \in \mathbb{N}} \left( d(x_n, y_n) \right)^{(1/(n+1)}), \quad x, y \in X_w^*(x^o).
\]

(2.4.6)

Recalling that \(w\) is nonconvex, we note that \(d_w^*(x, y) = \sup_{n \in \mathbb{N}} d(x_n, y_n)\) is only an extended metric on \(X_w^*(x^o)\) and \(X\) (however, \(d_w^*(x, y)\) is a metric on the set of all bounded sequences in \(M\); see Remark 2.4.3 below).

Writing \(x = \{x_n\} \in c(x^o)\) if \(\lim_{n \to \infty} d(x_n, x_n^o) = 0\), and \(x = \{x_n\} \in \ell_\infty(x^o)\) if \(\sup_{n \in \mathbb{N}} d(x_n, x_n^o) < \infty\), we have the following (proper) inclusion relations:

\[
X_w^0(x^o) \subset c(x^o) \subset \ell_\infty(x^o) \subset X_w^{\infty}(x^o) = X_w^*(x^o).
\]

(2.4.7)

(Here \(c(x^o)\) is the set of all sequences in \(M\), which are metrically equivalent to \(x^o = \{x_n\}\), and \(\ell_\infty(x^o)\) is the set of all sequences in \(M\), which are bounded relative to \(x^o\).) The first inclusion is a consequence of Proposition 2.4.2, and the third one is established as follows: if \(b = \sup_{n \in \mathbb{N}} d(x_n, x_n^o) < \infty\), then, for all \(\lambda > 0\), we have:

\[
w_\lambda(x, x^o) = \sup_{n \in \mathbb{N}} \left( \frac{d(x_n, x^o)}{\lambda} \right)^{1/n} \leq \sup_{n \in \mathbb{N}} \left( \frac{b}{\lambda} \right)^{1/n} = \max\{1, b/\lambda\} < \infty.
\]

Remark 2.4.3. 1. If \(x^o = \{x_n^o\}\) is a convergent sequence in \(M\), then every sequence \(x = \{x_n\} \in c(x^o)\) is also convergent in \(M\) (to the limit of \(x^o\)), and if \(x^o\) is bounded in \(M\) (i.e., \(\sup_{n, m \in \mathbb{N}} d(x^o, x^o_m) < \infty\)), then every \(x \in \ell_\infty(x^o)\) is also bounded in \(M\).

2. In the particular case when \(M = \mathbb{R}\) with metric \(d(x, y) = |x - y|\) and \(x^o = 0\) is the zero sequence, we have: \(c_0 = c(0)\) is the set of all real sequences convergent to zero, and \(\ell_\infty = \ell_\infty(0)\) is the set of all bounded real sequences. The following examples are illustrative (see (2.4.7)): (a) \(\{1/n\} \in c_0 \setminus X_w^0(0)\); (b) \(\{2^n\} \in X_w^*(0) \setminus \ell_\infty\); (c) \(\{2^{-n^2}\} \in X_w^0(0)\); (d) \(\{2^{n^2}\} \not\in X_w^*(0)\); (e) if \(x = \{n\}\), then \(x \in X_w^*(0)\), \(d_w^0(x, 0) = \sup_{n \in \mathbb{N}} n^{1/(n+1)} < \infty\), while \(d_w^*(x, 0) = \sup_{n \in \mathbb{N}} n = \infty\).
3. The classical $F$-norm $|x|_\rho = d_w^0(x, 0) = \sup_{n \in \mathbb{N}} |x_n|^{1/(n+1)}$, corresponding to
\[ \rho(x) = w_1(x, 0) \] with $w$ from (2.4.1) and $M = \mathbb{R}$, is well-defined for $x = \{x_n\}$
from $X_\rho = X_w^0(0) \subset c_0$ and satisfies conditions (F.1)–(F.4) from Remark 2.2.3.
However, on the larger modular space $X_\rho^* = X_w^*(0)$ (see Remark 2.3.4(1)), the
functional $| \cdot |_\rho$ does not satisfy the continuity condition (F.4): for instance, if
$x = (2^{n+1})_{n=1}^\infty$ and $\alpha_n = 1/k$, then $x \in X_\rho^* \setminus X_\rho$ and $\alpha_k \to 0$ as $k \to \infty$, but
\[ |\alpha_k x|_\rho = \sup_{n \in \mathbb{N}} (\alpha_k \cdot 2^{n+1})^{1/(n+1)} = 2 \sup_{n \in \mathbb{N}} \left( \frac{1}{k} \right)^{1/(n+1)} = 2 \quad \text{for all} \quad k \in \mathbb{N}. \]

2.5 Intermediate Metrics

In Theorem 2.2.1 and Corollary 2.2.6, we have seen two expressions for metric $d_w^0$
on $X_w^*$ (see also Theorem 2.3.1 if $w$ is convex). In this section, we define and study
infinitely many metrics on the modular space $X_w^*$.

**Theorem 2.5.1.** Let $w$ be a (pseudo)modular on the set $X$. Given $0 \leq \theta \leq 1$ and
$x, y \in X$, setting
\[ d_w^0(x, y) = \inf_{\lambda > 0} \left[ (1 - \theta) \max\{\lambda, w_\lambda(x, y)\} + \theta (\lambda + w_\lambda(x, y)) \right], \quad (2.5.1) \]
we have: $d_w^0$ is an extended (pseudo)metric on $X$, and a (pseudo)metric on the
modular space $X_w^* = X_w^*(x^0)$ for any $x^0 \in X$, and the following (sharp) inequalities
hold:
\[ d_w^0(x, y) \leq (1-\theta)d_w^0(x, y) + \theta d_w^1(x, y) \leq d_w^0(x, y) \leq d_w^1(x, y) \leq 2d_w^0(x, y). \quad (2.5.2) \]

**Proof.** Clearly, $0 \leq d_w^0(x, y) \leq \infty$ for all $x, y \in X$ and $0 \leq \theta \leq 1$.

1. First, we prove our theorem for $\theta = 0$ and $\theta = 1$ simultaneously (for $d_w^0$, this
is the second proof). Given $u, v \in [0, \infty]$, we denote by $u \oplus v$ either $\max\{u, v\}$
or $u + v$ (and $u \oplus v = \infty$ if $u = \infty$ or $v = \infty$). Then $d_w^0(x, y)$ and $d_w^1(x, y)$ are
expressed by the formula:
\[ d_w^0(x, y) = \inf_{\lambda > 0} \lambda \oplus w_\lambda(x, y), \quad x, y \in X. \quad (2.5.3) \]

1a. If $x, y \in X_w^*$, then $d_w^0(x, y) < \infty$. In fact, since $x \sim y$, there exists $\lambda_0 > 0$
such that $w_{\lambda_0}(x, y) < \infty$, and so, the set $\{\lambda \oplus w_\lambda(x, y) : \lambda > 0\} \setminus \{\infty\}$
is nonempty and bounded from below by 0 (i.e., is contained in $[0, \infty]$).

1b. Given $x \in X$, we have, by (i), $\lambda \oplus w_\lambda(x, x) = \lambda \oplus 0 = \lambda$ for all $\lambda > 0$, and
so, $d_w^0(x, x) = \inf_{\lambda > 0} \lambda = 0$. Now, suppose $w$ is a modular. Let $x, y \in X$, and
$d_w^0(x, y) = 0$. If we show that $w_\lambda(x, y) = 0$ for all $\lambda > 0$, then axiom (i) will
imply $x = y$. On the contrary, assume that $w_{\lambda_0}(x, y) \neq 0$ for some $\lambda_0 > 0$. Given $\lambda > 0$, we have two cases: if $\lambda \geq \lambda_0$, then

$$\lambda \oplus w_\lambda(x, y) \geq \lambda \oplus 0 = \lambda \geq \lambda_0,$$

and if $\lambda < \lambda_0$, then, by the monotonicity (1.2.1) of $w$, we find

$$\lambda \oplus w_\lambda(x, y) \geq 0 \oplus w_\lambda(x, y) = w_\lambda(x, y) \geq w_{\lambda_0}(x, y).$$

Hence $\lambda \oplus w_\lambda(x, y) \geq \min\{\lambda_0, w_{\lambda_0}(x, y)\} = \lambda_1$ for all $\lambda > 0$. By the definition of $d^\oplus_w$, we get $d^\oplus_w(x, y) \geq \lambda_1 > 0$, which contradicts the assumption.

lc. Axiom (ii) for $w$ implies the symmetry property of $d^\oplus_w$.

ld. Let us establish the triangle inequality $d^\oplus_w(x, y) \leq d^\oplus_w(x, z) + d^\oplus_w(z, y)$ for all $x, y, z \in X$. The inequality is clear if at least one summand on the right is infinite. So, we assume that both of them are finite. By (2.5.3), given $\varepsilon > 0$, there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda \oplus w_\lambda(x, z) \leq d^\oplus_w(x, z) + \varepsilon \quad \text{and} \quad \mu \oplus w_\mu(z, y) \leq d^\oplus_w(z, y) + \varepsilon.$$

Since $\oplus$ is max or $+$, (2.5.3) and axiom (iii) imply

$$d^\oplus_w(x, y) \leq (\lambda + \mu) \oplus w_{\lambda+\mu}(x, y) \leq (\lambda + \mu) \oplus (w_\lambda(x, z) + w_\mu(z, y)) \leq \bigl(\lambda \oplus w_\lambda(x, z)\bigr) + \bigl(\mu \oplus w_\mu(z, y)\bigr) \leq d^\oplus_w(x, z) + \varepsilon + d^\oplus_w(z, y) + \varepsilon.$$

(2.5.4)

It remains to take into account the arbitrariness of $\varepsilon > 0$.

2. That $d^\ominus_w$ is well-defined, nondegenerate (when $w$ is a modular), and symmetric can be proved along the same lines as in steps 1a–1c. Let us show that $d^\ominus_w$ satisfies the triangle inequality. Suppose $d^\ominus_w(x, z)$ and $d^\ominus_w(z, y)$ are finite. Given $\varepsilon > 0$, by virtue of (2.5.1), there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$(1 - \theta) \max\{\lambda, w_\lambda(x, z)\} + \theta (\lambda + w_\lambda(x, z)) \leq d^\ominus_w(x, z) + \varepsilon,$$

$$(1 - \theta) \max\{\mu, w_\mu(z, y)\} + \theta (\mu + w_\mu(z, y)) \leq d^\ominus_w(z, y) + \varepsilon.$$

Taking into account (2.5.1), axiom (iii) and the last inequality in (2.5.4), we get:

$$d^\ominus_w(x, y) \leq (1 - \theta) \max\{\lambda + \mu, w_{\lambda+\mu}(x, y)\} + \theta (\lambda + \mu + w_{\lambda+\mu}(x, y)) \leq \max\{\lambda, w_\lambda(x, z)\} + \max\{\mu, w_\mu(z, y)\} + \theta (\lambda + w_\lambda(x, z) + w_\mu(z, y)) \leq (1 - \theta) \bigl[\max\{\lambda, w_\lambda(x, z)\} + \max\{\mu, w_\mu(z, y)\}\bigr] + \theta \bigl[\lambda + w_\lambda(x, z) + \mu + w_\mu(z, y)\bigr]$$

+ $\theta (\lambda + w_\lambda(x, z) + w_\mu(z, y))$.
2.5 Intermediate Metrics

\[ \begin{aligned}
&= \left[(1 - \theta) \max\{\lambda, w_\lambda(x, z)\} + \theta(\lambda + w_\lambda(x, z))\right] \\
&\quad + \left[(1 - \theta) \max\{\mu, w_\mu(z, y)\} + \theta(\mu + w_\mu(z, y))\right] \\
&\leq d^0_w(x, z) + \varepsilon + d^0_w(z, y) + \varepsilon.
\end{aligned} \]

By the arbitrariness of \( \varepsilon > 0 \), the triangle inequality for \( d^0_w \) follows.

3. The inequalities \( \max\{u, v\} \leq u + v \leq 2 \max\{u, v\} \) for \( u, v \geq 0 \) imply

\[ d^0_w(x, y) \leq d^1_w(x, y) \leq 2d^0_w(x, y) \quad \text{for all} \quad x, y \in X. \tag{2.5.5} \]

This proves also the first and fourth inequalities in (2.5.2). Since, for any \( \lambda > 0 \),

\[ d^0_w(x, y) \leq \max\{\lambda, w_\lambda(x, y)\} \quad \text{and} \quad d^1_w(x, y) \leq \lambda + w_\lambda(x, y), \]

we find

\[
(1 - \theta)d^0_w(x, y) + \theta d^1_w(x, y) \leq (1 - \theta) \max\{\lambda, w_\lambda(x, y)\} + \theta(\lambda + w_\lambda(x, y)) \\
\leq \lambda + w_\lambda(x, y),
\]

which establishes the second and third inequalities in (2.5.2).

The sharpness of inequalities (2.5.2) is elaborated in Examples 2.5.5 and 2.5.6.

Remark 2.5.2. Not only intermediate (pseudo)metrics \( d^\theta_w \) between \( d^0_w \) and \( d^1_w \) can be introduced as in (2.5.1): given \( \alpha, \beta \geq 0 \) with \( \alpha + \beta \neq 0 \), we set

\[ d^{\alpha, \beta}_w(x, y) = \inf_{\lambda > 0} \left[ \alpha \max\{\lambda, w_\lambda(x, y)\} + \beta(\lambda + w_\lambda(x, y))\right], \quad x, y \in X. \]

In this case, we have \( d^{\alpha, \beta}_w(x, y) = (\alpha + \beta)d^\theta_w(x, y) \) with \( \theta = \beta / (\alpha + \beta) \).

Remark 2.5.3. Different binary operations \( \oplus \) on \([0, \infty)\) can be used in formula (2.5.3) to define \( d^{\oplus}_w(x, y) \), but then only the generalized triangle inequality holds:

\[ d^{\oplus}_w(x, y) \leq C(d^{\oplus}_w(x, z) + d^{\oplus}_w(z, y)) \quad \text{with} \quad C > 1. \tag{2.5.6} \]

This can be seen as follows. Suppose \( \varphi : [0, \infty) \to [0, \infty) \) is a continuous function such that \( \varphi(0) = 0 \), \( \varphi(u) > 0 \) for \( u > 0 \) and, for some constant \( C > 1 \),

\[ \varphi\left(\frac{u + v}{C}\right) \leq \varphi(u) + \varphi(v) \leq \varphi(u + v) \quad \text{for all} \quad u, v \geq 0. \tag{2.5.7} \]
(Here the right-hand side inequality is the superadditivity property of \( \varphi \), which is satisfied, e.g., by any convex function \( \varphi \); see Appendix A.1). Denoting by \( \varphi^{-1} \) the inverse function of \( \varphi \) and setting

\[
  u \oplus v = \varphi^{-1}(\varphi(u) + \varphi(v)) \quad \text{for all } u, v \geq 0,
\]

we find, from (2.5.7), that

\[
  u \oplus v \leq u + v \leq C(u \oplus v).
\]  

(2.5.9)

For instance, if \( \varphi(u) = u^\rho \) with \( \rho > 1 \), then \( u \oplus v = (u^\rho + v^\rho)^{1/\rho} \), and inequalities (2.5.9) hold with sharp constant \( C = 2^{1-(1/\rho)} \), and if \( \varphi(u) = e^u - 1 \), then \( u \oplus v = \log(e^u + e^v - 1) \), and (2.5.9) hold with sharp constant \( C = 2 \). Now, in order to obtain (2.5.6), we take into account (2.5.3) and (2.5.9), and find that the right-hand side in (2.5.4) is less than or equal to

\[
  \left( \lambda + \mu \right) + \left( w_\lambda(x, z) + w_\mu(z, y) \right) = \left( \lambda + w_\lambda(x, z) \right) + \left( \mu + w_\mu(z, y) \right)
\]

\[
  \leq C \left[ \left( \lambda + w_\lambda(x, z) \right) + \left( \mu + w_\mu(z, y) \right) \right]
\]

\[
  \leq C \left[ d^\Theta_w(x, z) + \varepsilon + d^\Theta_w(z, y) + \varepsilon \right], \quad \varepsilon > 0.
\]

The generalized triangle inequality (2.5.6) can also be obtained if, instead of \( d^\Theta_w(x, y) \) from (2.5.3), we consider the quantity

\[
  d^\Theta_w(x, y) = \inf_{\lambda > 0} \left( \max \left\{ \lambda, w_\lambda(x, y) \right\} \right) \oplus \left( \lambda + w_\lambda(x, y) \right)
\]

with the operation \( \oplus \) on \([0, \infty)\) of the form (2.5.8).

As in Corollary 2.2.8, the right \( w_{+0} \) and left \( w_{-0} \) regularizations of \( w \) do not produce new metrics of the form (2.5.1) in the following sense.

**Proposition 2.5.4.** \( d^\Theta_{w_{+0}}(x, y) = d^\Theta_{w_{-0}}(x, y) = d^\Theta_w(x, y) \) for all \( 0 \leq \theta \leq 1 \) and \( x, y \in X \).

**Proof.** For instance, let us verify this for \( \theta = 1 \). By virtue of (1.2.4), we have

\[
  \lambda + w_{\lambda+0}(x, y) \leq \lambda + w_\lambda(x, y) \leq \lambda + w_{\lambda-0}(x, y) \quad \text{for all } \lambda > 0,
\]

whence \( d^1_{w_{+0}}(x, y) \leq d^1_w(x, y) \leq d^1_{w_{-0}}(x, y) \).

Let us show that \( d^1_{w_{+0}}(x, y) \geq d^1_w(x, y) \). Suppose \( d^1_{w_{+0}}(x, y) < \infty \), and \( u > d^1_{w_{+0}}(x, y) \). Let \( u > u_1 > d^1_{w_{+0}}(x, y) \). By (2.5.1) with \( \theta = 1 \), there exists \( \lambda_1 > 0 \) such that

\[
  \lim_{\lambda \to \lambda_1+0} \left( \lambda + w_\lambda(x, y) \right) = \lambda_1 + w_{\lambda_1+0}(x, y) \leq u_1 < u.
\]
It follows that \( \lambda_2 + w_{\lambda_2}(x, y) < u \) for some \( \lambda_2 > \lambda_1 \), which implies

\[
d_w^1(x, y) = \inf_{\lambda > 0} (\lambda + w_\lambda(x, y)) \leq \lambda_2 + w_{\lambda_2}(x, y) < u,
\]

and it remains to pass to the limit as \( u \to d_{w+0}^1(x, y) \).

Now, we show that \( d_w^1(x, y) \geq d_{w-0}^1(x, y) \). Let \( d_w^1(x, y) < \infty \), and \( u > d_w^1(x, y) \). Choose \( u_1 \) such that \( u > u_1 > d_w^1(x, y) \). By (2.5.1) with \( \theta = 1 \), there exists \( \mu_1 > 0 \) such that \( \mu_1 + w_{\mu_1}(x, y) \leq u_1 < u \). It follows from (1.2.4) that

\[
w_{\lambda - 0}(x, y) \leq w_{\mu_1}(x, y) < u - \mu_1 \quad \text{for all} \quad \lambda > \mu_1,
\]

and so,

\[
d_{w-0}^1(x, y) \leq \lambda_1 + w_{\lambda - 0}(x, y) < \lambda_1 + u - \mu_1.
\]

Passing to the limit as \( \lambda_1 \to \mu_1 + 0 \), we get \( d_{w-0}^1(x, y) \leq u \), and it remains to take into account the arbitrariness of \( u \) as above. \( \square \)

**Example 2.5.5 (metric \( d_w^1 \)).**

1. Let \( w_{\lambda}(x, y) = \lambda^{-p}d(x, y) \) be of the form (1.3.1) with \( p > 0 \). By Example 2.2.2(1), \( d_w^0(x, y) = (d(x, y))^{1/(p+1)} \).

   Let us calculate \( d_w^1(x, y) = \inf_{\lambda > 0} f(\lambda) \), where \( f(\lambda) = \lambda + \lambda^{-p}d(x, y) \) (and \( x \neq y \)). The derivative \( f'(\lambda) = 1 - p\lambda^{-p-1}d(x, y) \) vanishes at \( \lambda_0 = (pd(x, y))^{1/(p+1)} \), \( f'(\lambda) < 0 \) if \( 0 < \lambda < \lambda_0 \), and \( f'(\lambda) > 0 \) if \( \lambda > \lambda_0 \), and so, \( f \) attains the global minimum on \((0, \infty)\) at the point \( \lambda_0 \), which is equal to

\[
d_w^1(x, y) = f(\lambda_0) = y(p) \cdot (d(x, y))^{1/(p+1)} \quad \text{for all} \quad x, y \in X,
\]

where

\[
y(p) = (p + 1)p^{-p/(p+1)}, \quad p > 0.
\]

Note that \( 1 < y(p) \leq 2 \), \( y(p) = 2 \) if and only if \( p = 1 \), and \( y(1/p) = y(p) \).

The inequalities for \( y(p) \) can be established directly by taking the logarithm and investigating the resulting function for extrema, or they follow from (2.5.5). In particular, if \( p = 1 \), the expressions for \( d_w^0 \) and \( d_w^1 \) are of the form:

\[
d_w^0(x, y) = \sqrt{d(x, y)} \quad \text{and} \quad d_w^1(x, y) = 2\sqrt{d(x, y)}, \quad x, y \in X.
\]

2. Formulas for \( d_w^0 \) and \( d_w^1 \) above are valid in a somewhat more general case when a (pseudo)modular \( w \) on \( X \) is \( p \)-homogeneous with \( p > 0 \) in the sense that

\[
w_{\lambda}(x, y) = \lambda^{-p}w_1(x, y) \quad \text{for all} \quad \lambda > 0 \text{ and } x, y \in X.
\]
In this case, we have

\[ d_w^0(x, y) = (w_1(x, y))^{1/(p+1)} \quad \text{and} \quad d_w^1(x, y) = \gamma(p) \cdot (w_1(x, y))^{1/(p+1)}. \]

(2.5.10)

One more example of a \( p \)-homogeneous modular \( w \) on a metric space \((X, d)\) is given by \( w_3(x, y) = (d(x, y)/\lambda)^p = \lambda^{-p}w_1(x, y)\) (see Example 2.3.5(1)).

3. Given a metric space \((X, d)\) and a convex function \( \varphi : [0, \infty) \to [0, \infty) \) vanishing at zero only, we set (cf. (1.3.5))

\[ w_\lambda(x, y) = \lambda \varphi\left(\frac{d(x, y)}{\lambda}\right), \quad \lambda > 0, \quad x, y \in X. \]

Then \( w \) is a strict modular on \( X \) (cf. (1.3.8)), and since \( \varphi \) is increasing, continuous, and admits the continuous inverse \( \varphi^{-1} \), we find

\[ d_w^0(x, y) = \inf \{ \lambda > 0 : \varphi(d(x, y)/\lambda) \leq 1 \} = d(x, y)/\varphi^{-1}(1). \]

In particular, if \( \varphi(u) = u^p \) with \( p > 1 \), we have \( d_w^0(x, y) = d(x, y) \), and taking into account that

\[ w_\lambda(x, y) = \lambda \left(\frac{d(x, y)}{\lambda}\right)^p = \lambda^{-(p-1)}(d(x, y))^p = \lambda^{-(p-1)}w_1(x, y), \]

we conclude from (2.5.10) (replacing \( p \) there by \( p - 1 \)) that

\[ d_w^1(x, y) = \gamma(p - 1) \cdot (w_1(x, y))^{1/p} = p(p - 1)^{(1-p)/p} \cdot d(x, y). \]

4. Setting \( w_\lambda(x, y) = e^{-\lambda}d(x, y) \) and following the same reasoning as in Example 2.5.5(1), we get

\[ d_w^0(x, y) = \begin{cases} 
  d(x, y) & \text{if } d(x, y) \leq 1, \\
  1 + \log d(x, y) & \text{if } d(x, y) > 1, 
\end{cases} \quad x, y \in X. \]

Example 2.5.6 (metric \( d_w^0 \)). In order to be able to calculate the value \( d_w^0(x, y) \) from (2.5.1) explicitly for all \( 0 \leq \theta \leq 1 \), here once again we consider the modular \( w_\lambda(x, y) = \lambda^{-\theta}d(x, y) \) of the form (1.3.1) with \( p > 0 \). Since the cases \( \theta = 0 \) and \( \theta = 1 \) were considered in Example 2.5.5(1), we are left with the case when \( 0 < \theta < 1 \) (in calculations below, we assume that \( x \neq y \)).

To begin with, we note that \( d_w^0(x, y) = \inf_{\lambda > 0} f(\theta, \lambda) \), where the function \( f(\theta, \lambda) \) under the infimum sign in (2.5.1) is expressed as

\[ f(\theta, \lambda) = \begin{cases} 
  f_1(\lambda) \equiv w_\lambda(x, y) + \theta \lambda & \text{if } \lambda \leq w_\lambda(x, y), \\
  f_2(\lambda) \equiv \lambda + \theta w_\lambda(x, y) & \text{if } \lambda > w_\lambda(x, y), 
\end{cases} \]
with \( f_1(\lambda) = \lambda^{-\rho} d(x, y) + \theta \lambda \) and \( f_2(\lambda) = \lambda + \theta \lambda^{-\rho} d(x, y) \), and the inequality 
\( \lambda \leq w_2(x, y) = \lambda^{-\rho} d(x, y) \) is equivalent to \( \lambda \leq \lambda_0 = d_0(x, y) = (d(x, y))^{1/(p+1)} \).
Hence

\[
d^\theta_w(x, y) = \min_{\theta < \lambda \leq \lambda_0} \{ \inf_{\lambda < \lambda_0} f_1(\lambda), \inf_{\lambda > \lambda_0} f_2(\lambda) \}, \tag{2.5.11}
\]

where we note that \( f_1(\lambda_0) = f_2(\lambda_0) = \lambda_0(1 + \theta) \).

The derivative \( f'_1(\lambda) = -\lambda^{-\rho - 1}pd(x, y) + \theta \) is equal to zero only at the point 
\( \lambda_1 = \lambda_0(p/\theta)^{1/(p+1)} \), \( f'_1(\lambda) < 0 \) if \( 0 < \lambda < \lambda_1 \), and \( f'_1(\lambda) > 0 \) if \( \lambda > \lambda_1 \), and so, the global minimum of \( f_1 \) on \((0, \infty)\) is attained at \( \lambda_1 \) and is equal to 
\[
f_1(\lambda_1) = \lambda_0 \gamma(p) \lambda^{\rho/(p+1)}.
\]

Similarly, the derivative \( f'_2(\lambda) = 1 - \lambda^{-\rho - 1} \theta pd(x, y) \) is equal to zero at the point 
\( \lambda_2 = \lambda_0(\theta p)^{1/(p+1)} \), \( f'_2(\lambda) < 0 \) if \( 0 < \lambda < \lambda_2 \), and \( f'_2(\lambda) > 0 \) for \( \lambda > \lambda_2 \), and so, \( f_2 \) attains the global minimum on \((0, \infty)\) at \( \lambda_2 \), where it has the value 
\[
f_2(\lambda_2) = \lambda_0 \gamma(p) \theta^{1/(p+1)}.
\]

Given \( p > 0 \) and \( 0 \leq \theta \leq 1 \), we have four cases: (I) \( p \geq 1 \) and \( \theta \leq 1/p \); (II) \( p > 1 \) and \( 1/p < \theta \); (III) \( p < 1 \) and \( \theta \leq p \); and (IV) \( p < 1 \) and \( p < \theta \).

**Cases (I), (III).** We have \( p \geq 1 \geq \theta \) in case (I), and \( p > \theta \) in case (III), and so, \( \lambda_0 \leq \lambda_1 \). Since \( f_1 \) decreases on \((0, \lambda_1)\), the value \( \inf_{\lambda \leq \lambda_0} f_1(\lambda) \) is equal to 
\( f_1(\lambda_0) = \lambda_0(1 + \theta) \). Also, we have \( \theta p \leq 1 \) in case (I), and \( \theta p < 1 \) in case (III), and so, \( \lambda_2 \leq \lambda_0 \). Since \( f_2 \) increases on \([\lambda_2, \infty)\), the value \( \inf_{\lambda > \lambda_0} f_2(\lambda) \) is equal to 
\( f_2(\lambda_0) = \lambda_0(1 + \theta) \). By virtue of (2.5.11), 
\( d^\theta_w(x, y) = \lambda_0(1 + \theta) \).

**Case (II).** As in case (I), since \( p > 1 \geq \theta \), \( \inf_{\lambda \leq \lambda_0} f_1(\lambda) = \lambda_0(1 + \theta) \).
Furthermore, \( \theta p > 1 \) implies \( \lambda_0 < \lambda_2 \), where \( \lambda_2 \) is the point of minimum of \( f_2 \) on \([\lambda_0, \infty)\), and so, 
\[
\inf_{\lambda > \lambda_0} f_2(\lambda) = f_2(\lambda_2) = f_2(\lambda_0) = \lambda_0(1 + \theta) = \inf_{\lambda \leq \lambda_0} f_1(\lambda).
\]

It follows from (2.5.11) that 
\( d^\theta_w(x, y) = f_2(\lambda_2) = \lambda_0 \gamma(p) \theta^{1/(p+1)} \).

**Case (IV).** Inequality \( p < \theta \) implies \( \lambda_1 < \lambda_0 \), and since \( \lambda_1 \) is the point of minimum of \( f_1 \) on \((0, \lambda_0)\), we find 
\[
\inf_{\lambda \leq \lambda_0} f_1(\lambda) = f_1(\lambda_1) < f_1(\lambda_0) = \lambda_0(1 + \theta).
\]

As in case (III), since \( \theta p < 1 \), \( \inf_{\lambda > \lambda_0} f_2(\lambda) = \lambda_0(1 + \theta) \). By (2.5.11), we conclude that 
\( d^\theta_w(x, y) = f_1(\lambda_1) = \lambda_0 \gamma(p) \theta^{p/(p+1)} \).
In this way, we have shown that

\[
d_w^\theta(x, y) = (d(x, y))^{1/(p+1)} \cdot \begin{cases} 
1 + \theta & \text{if } 0 \leq \theta \leq 1/p \leq 1 \\
0 & \text{if } 0 \leq \theta \leq p < 1, \\
\gamma(p)\theta^{1/(p+1)} & \text{if } 0 < 1/p < \theta \leq 1, \\
\gamma(p)\theta^{\theta/(p+1)} & \text{if } 0 < p < \theta \leq 1. 
\end{cases}
\] (2.5.12)

A few comments on this formula are in order. If \( \theta = 0 \) or \( \theta = 1 \), then it gives back the values \( d_w^0(x, y) \) and \( d_w^1(x, y) \) from Example 2.5.5(1). If \( p > 1 \) and \( \theta = 1/p \) in the third line of (2.5.12), then \( \gamma(p)\theta^{1/(p+1)} = 1 + \theta \) (as in the first line). Similarly, if \( p < 1 \) and \( \theta = p \) in the fourth line of (2.5.12), then \( \gamma(p)\theta^{\theta/(p+1)} = 1 + \theta \).

Note that, for any \( p > 0 \) and \( 0 \leq \theta \leq 1 \), we have (cf. (2.5.2))

\[
(1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) = (1 - \theta + \theta\gamma(p)) \cdot (d(x, y))^{1/(p+1)}.
\]

For \( p \neq 1 \), we have \( 1 < \gamma(p) < 2 \), so if (a) \( p > 1 \) and \( 0 < \theta < 1 \), or (b) \( p < 1 \) and \( 0 < \theta \leq p \), then \( 1 - \theta + \theta\gamma(p) < 1 + \theta \), and so,

\[
(1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) < d_w^0(x, y), \quad x \neq y.
\]

Now, if \( p = 1 \), then \( \gamma(p) = 2 \) and \( 1 - \theta + \theta\gamma(p) = 1 + \theta \), which imply

\[
d_w^0(x, y) = (1 + \theta)\sqrt{d(x, y)} = (1 - \theta)d_w^0(x, y) + \theta d_w^1(x, y) \quad \text{for all } 0 \leq \theta \leq 1.
\]

For a convex (pseudo)modular \( w \) on \( X \), \( \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y) \) is a (pseudo)modular on \( X \), so setting \( d_w^\theta* = d_{\hat{w}}^\theta \) and applying Theorem 2.5.1, we get

**Theorem 2.5.7.** If \( w \) is a convex (pseudo)modular on \( X \) and \( 0 \leq \theta \leq 1 \), then

\[
d_w^\theta*(x, y) = \inf_{\lambda > 0} \left( (1 - \theta) \max \{\lambda, \lambda w_\lambda(x, y)\} + \theta(\lambda + \lambda w_\lambda(x, y)) \right), \quad x, y \in X,
\]

is an extended (pseudo)metric on \( X \) and a (pseudo)metric on \( X_w^* \), and

\[
d_w^\theta*(x, y) \leq (1 - \theta)d_w^\theta*(x, y) + \theta d_w^1*(x, y) \leq d_w^\theta*(x, y) \leq d_w^1*(x, y) \leq 2d_w^\theta*(x, y),
\]

where (see (2.3.3)) \( d_w^\theta*(x, y) = d_w^\theta*(x, y) \).

**Remark 2.5.8.** Given \( 0 \leq \theta \leq 1 \), \( d_w^\theta(x, y) < 1 \) implies \( d_w^\theta*(x, y) \leq d_w^\theta(x, y) \). In fact, for any \( r \) such that \( d_w^\theta(x, y) < r < 1 \) there exists \( \lambda = \lambda(r) > 0 \) such that

\[
(1 - \theta) \max \{\lambda, w_\lambda(x, y)\} + \theta(\lambda + w_\lambda(x, y)) \leq r < 1.
\]
It follows that \( \lambda = (1 - \theta)\lambda + \theta \lambda < 1 \),
\[
\max\{\lambda, \lambda w_\lambda (x, y)\} \leq \max\{\lambda, w_\lambda (x, y)\} \quad \text{and} \quad \lambda + \lambda w_\lambda (x, y) \leq \lambda + w_\lambda (x, y),
\]
and so,
\[
d_\lambda^0 (x, y) \leq (1 - \theta) \max\{\lambda, \lambda w_\lambda (x, y)\} + \theta(\lambda + \lambda w_\lambda (x, y)) \leq r.
\]
It remains to pass to the limit as \( r \to d_\lambda^0 (x, y) \).

Example 2.5.9. Let \( p \geq 1 \) and \( w_\lambda (x, y) = (d(x, y)/\lambda)^p \) be the \( p \)-homogeneous modular from Example 2.3.5(1). Then, by Example 2.5.5(1), (2),
\[
d_\lambda^1 (x, y) = \gamma(p) \cdot (d(x, y))^{p/(p+1)} \quad \text{and} \quad d_\lambda^{1*} (x, y) = \begin{cases} d(x, y) & \text{if } p = 1, \\ \gamma(p-1)d(x, y) & \text{if } p > 1. \end{cases}
\]

2.6 Bibliographical Notes and Comments

Sections 2.1 and 2.2. Modular spaces \( X_\lambda^* \) and \( X_\lambda^0 \) were introduced in Chistyakov [22] and studied in [24, 25, 28]. The space \( X_\lambda^0 \) is a counterpart of the classical modular space \( X_\rho \) defined in Musielak and Orlicz [77]; see Remark 2.2.3(1), in which the main results of [77] are briefly described. As condition (p.4) from Sect.1.3.3 is crucial for defining the \( F \)-norm \( |x|_\rho \) on \( X_\rho \), axiom (iii) in Definition 1.2.1 is a proper tool to define the (pseudo)metric \( d_\lambda^0 (x, y) \) on the space \( X_\lambda^* \), which is larger than \( X_\lambda^0 \).

The properties of \( d_\lambda^0 (x, y) \) are based on the properties of quantity \( g^0 \) from (2.2.1) (recall that \( d_\lambda^0 (x, y) = (w^{x,y})^0 \)). This allows us to obtain an alternative expression for the (pseudo)metric \( d_\lambda^0 (x, y) \) in Corollary 2.2.6.

Modular space \( X_\lambda^{\text{fin}} \) is (natural and) new. Its role will be more clear below (see Theorem 3.3.8): some ‘duality’ holds between the modular spaces.

Corollary 2.2.8 was first established in Chistyakov [28].

Lemma 2.2.9 and Theorem 2.2.11 are sharp refinements of Theorem 2.10 from Chistyakov [24]. Counterparts of Theorem 2.2.11(d), (e) for classical modulars are presented in Maligranda [68, Theorem 1.4].

Theorem 2.2.13 is new.

Section 2.3. In the convex case, the results of the classical modular theory are presented in Remark 2.3.4(1). They were established by Nakano [81, Sect. 81], Musielak and Orlicz [78], and Orlicz [90] (for \( s \)-convex modulars with \( 0 < s \leq 1 \)). For Orlicz modulars (i.e., integral modulars of the form \( \rho(x) = \int_{\Omega} \varphi(|x(t)|)d\mu \)), the norm \( \|x\|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq 1\} \) on \( X_\rho^* \) was considered by Morse and Transue [73] and Luxemburg [66]. Note that the norm \( \|x\|_\rho \) is the Minkowski functional \( p_A (x) = \inf\{\varepsilon > 0 : x/\varepsilon \in A\} \) of the convex set \( A = \{x: \rho(x) \leq 1\} \).
Furthermore, Musielak and Orlicz [78] proved inequalities of the form (2.3.6) and (2.3.7) for classical convex modulars $\rho$, and Orlicz [90] established the representation $\|x\|_\rho = \inf_{t>0} \sup \{t^{-1}, \rho(tx)t^{-1}\}$ (cf. the second equality in (2.3.3)).

The (pseudo)metric $d_w^*(x,y)$ on $X^*_w$ was introduced in Chistyakov [22]. It is seen from the expressions for $d_w^*(x,y)$ and $\|x\|_\rho$ that $d_w^*(x,y)$ is a counterpart of the norm $\|x\|_\rho$. Interestingly, the idea of definition of $d_w^*(x,y) = (\hat{\omega}_{x,y})^0$ has no relation with the idea of Minkowski’s functional of a convex set, and relies on $g^0$ from (2.2.1), however, by virtue of the ‘embedding’ (1.3.3), for convex modulars on linear spaces, we get $\|x\|_\rho = d_w^*(x,0)$ (see Remark 2.3.4(1)).

Section 2.4. The first modular stands for illustrative purposes—its idea is to generalize, in a straightforward way, the well-known space $\ell_p$ of $p$-summable sequences. The second modular (2.4.1), mentioned in [24, Example 3.2], is more interesting and studied in detail (see also Example 4.2.7(2)). Note that modular (2.4.1) can be obtained, via (1.3.3), from the classical modular $\|x\|_\rho = \sup_{n\in\mathbb{N}} \sqrt{|x_n|}$ for $x = \{x_n\} \in \mathbb{R}^\mathbb{N}$, see Rolewicz [95, Example 1.2.3].

Section 2.5. The whole material of Sect. 2.5 is new. Connections with the classical modular theory are as follows. Metric $d^\theta_w(x,y)$ from (2.5.1) for $\theta = 1$ is a counterpart of the $F$-norm $|x|_\rho^1 = \inf_{t>0}(1 + t\rho(tx))/t$, $x \in X_\rho$, from Koshi and Shimogaki [53], where inequality $|x|_\rho \leq |x|_\rho^1 \leq 2|x|_\rho$ of the form (2.5.5) was also established; here $|x|_\rho = \inf \{\varepsilon > 0 : \rho(x/\varepsilon) \leq \varepsilon\}$ is the Musielak-Orlicz $F$-norm.

The idea to define the operation $\oplus$ in (2.5.8) is taken from Musielak [74] and Musielak and Peetre [79] (see also Musielak [75, Sect. 3]).

The classical variant of Example 2.5.5 was elaborated in Maligranda [68, p. 4].

Metric $d^\theta_w^*(x,y)$ from Theorem 2.5.7 for $\theta = 1$ is a counterpart of the Amemiya norm $\|x\|_\rho^1 = \inf_{t>0}(1 + \rho(tx))/t$, $x \in X^*_\rho = X_\rho$ (see Nakano [81, Sect. 81], Hudzik and Maligranda [48], Maligranda [68, p. 6], Musielak [75, Theorem 1.10]).

For more information about the modular theory on linear spaces and Orlicz spaces we refer to Adams [1], Kozlowski [55], Krasnosel’skiǐ and Rutickiǐ [56], Lindenstrauss and Tzafriri [65], Luxemburg [66], Maligranda [68], Musielak [75], Nakano [80, 81], Orlicz [89], Rao and Ren [92, 93], Rolewicz [95].