# Chapter 2
# Heavy-Particle Spacetime Symmetries and Building Blocks

What is heavy particle effective theory? Imagine a charged particle $\phi$ with mass $M$ in its rest frame, interacting with other particles of energy $\Lambda \ll M$ through exchange of photons. The momentum transfer in such collisions are of order $\Lambda$, allowing the heavy particle to be off-shell by only that amount. Hence, to a good approximation, the heavy particle remains at rest and appears to the light degrees of freedom as a static source of charge. In this situation, we may expand the momentum of $\phi$ about its large component as

$$p^\mu = Mv^\mu + k^\mu \,, \tag{2.1}$$

where $v^\mu$ is the velocity of $\phi$ (e.g., $v^\mu = (1,0,0,0)$ in the rest frame), and the residual momentum $k^\mu$ is $\mathcal{O}(\Lambda)$ and accounts for the dynamics of $\phi$ off-shell. Hence, the "hard" energy scale $M$ (equivalently, the momentum scale $Mv^\mu$) is irrelevant to the system, and we may integrate it out by passing from a fully relativistic description of $\phi$ to a heavy particle description where only the "soft" momentum mode, $k^\mu$, remains.

At leading-order in $1/M$, the theory describing $\phi$ is

$$\mathcal{L} = \bar{\phi}_v iv \cdot D\phi_v + \dots \,, \tag{2.2}$$

where $D^\mu$ is the covariant derivative including the gauge interaction. The heavy particle field $\phi_v$, derived from the full relativistic field $\phi$, is labelled by its velocity $v$ (and has mass dimension $3/2$). The ellipsis in Eq. (2.2) denotes interactions higher order in the heavy mass expansion, which may be systematically included to meet the demands of precision, depending on the size of $\Lambda/M$. For example, $\phi$'s interactions with photons, through $\mathcal{O}(1/M)$, are given by

$$\mathcal{L} = \bar{\phi}_v \left\{ iv \cdot D + c_1 \frac{D_\perp^2}{2M} + c_F \frac{\sigma^\perp_{\mu\nu} F^{\mu\nu}}{4M} + c_F' \frac{\tilde{\sigma}^\perp_{\mu\nu} F^{\mu\nu}}{4M} + \dots \right\} \phi_v \,, \tag{2.3}$$



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where $\tilde{T}^{\mu\nu} = T_{\alpha\beta}\epsilon^{\alpha\beta\mu\nu}$ and $a_\perp^\mu = a^\mu - v^\mu v \cdot a$. The higher-order kinetic, magnetic dipole and electric dipole terms are parameterized by Wilson coefficients $c_1$, $c_F$ and $c'_F$, respectively. From a low-energy perspective, these higher-dimension operators represent new physics, albeit not new particles but new interactions. The higher dimension operators depend on additional microscopic properties of the particle $\phi$ such as its spin, its mass and its electromagnetic moments.

The purpose of such a framework, as with any useful effective theory, is for controlled calculations of physical observables. Heavy particle effective theory systematically captures the dominant interactions at each order in $1/M$, and at lower orders, in particular, the few operators appearing are remarkably simple. These simplifications are tied to new symmetries arising in the heavy particle limit, and allow for tractable analysis of otherwise complicated processes. Precise calculations, including the resummation of large logarithms $\sim \log\frac{M}{\Lambda}$, are also made possible by the separation of scales $\Lambda$ and $M$ through the effective theory. Heavy particle methods find a wide range of applications in particle, nuclear and atomic physics [21, 40, 70, 89].

So far we have sketched heavy particle effective theory as a framework for describing a massive particle of energy $M$ interacting with degrees of freedom having much smaller energy $\Lambda \ll M$, by systematic expansion in $\Lambda/M$. To fully answer "what is heavy particle effective theory?" we must precisely state how such a theory is constructed. A closely related question is "what are the spacetime symmetries of the Lagrangian in Eq. (2.3)?" From the intuitive picture of a heavy particle in its rest frame, rotational invariance is manifest, but what about boost invariance? With the appearance of the vector $v^\mu$ in the Lagrangian, it is unclear how to implement Lorentz transformations.

When the underlying UV theory for the heavy particle is known, we may derive the effective theory Lagrangian by introducing a field redefinition in the full theory. For example, in terms of an arbitrary (spacetime independent) time-like unit vector $v^\mu$, the decomposition of a quark field $Q(x)$ of mass $M$,

$$Q(x) = e^{-iMv\cdot x}\left[h_v(x) + H_v(x)\right] , \tag{2.4}$$

with $\not{v} h_v = h_v$ and $\not{v} H_v = -H_v$, defines an effective heavy quark field $h_v(x)$, and after integrating out the antiparticle field $H_v(x)$, we arrive at the effective Lagrangian for a heavy quark,

$$\bar{Q}(i\not{D} - M)Q \quad\to\quad \bar{h}_v iv\cdot D h_v + \mathcal{O}(1/M). \tag{2.5}$$

Invariance of observables under small changes of $v$, so-called "reparameterization invariance", enforces certain constraints on the coefficients of the effective Lagrangian [84]. These constraints are consistent with the requirements of Lorentz invariance, e.g. as imposed by matching effective theory $S$ matrix elements to Lorentz-invariant full theory $S$ matrix elements. However, this construction raises several questions. Is reparameterization invariance a sufficient condition for Lorentz invariance? How do we derive a reparameterization transformation law without first constructing the underlying theory and explicitly integrating out degrees of freedom? For applications involving a composite particle such as the proton, or hypothetical new particles such as dark matter whose underlying theory is unknown, we cannot in an obvious way introduce $v$ as a parameter inside of a field redefinition. What is the significance of $v$ in such cases? What is the general method for constructing a Lorentz invariant heavy particle effective field theory?

In this chapter we present the formalism of induced representations of the Lorentz group, Wigner's "little group" construction [103], for application to field transformation laws. The parameter $v$ enters as an arbitrary reference vector in the little group construction. The relationship between Lorentz invariance and reparameterization invariance is stated precisely, and a class of allowable reparameterization transformations is obtained. We find that a standard ansatz for implementing reparameterization invariance breaks down starting at order $1/M^4$. We explain this subtlety and its resolution. A large literature exists on topics relating to reparameterization invariance, especially as applied to heavy quark Lagrangians [16, 44, 75, 80, 83, 84, 86, 98]. We aim to present a conceptually clear statement of the constraints imposed by Lorentz invariance on heavy particle effective theories.

Recent investigations using heavy particle effective theory demand high orders in the $1/M$ expansion (see e.g. [17, 58, 62]). To avoid a proliferation of undetermined constants, and to enable efficient computations, it is important to recognize that many Wilson coefficients are linked by Lorentz invariance to coefficients appearing at lower orders. At a practical level, we derive the explicit field transformation laws that can be consistently used to build Lorentz invariant Lagrangians, providing a complete solution for Wilson coefficient constraints, to arbitrary order in $1/M$.

While a top-down derivation of heavy particle effective theory, e.g., starting from QCD to derive heavy quark effective theory (HQET) through the field redefinition in (2.4), must map onto the framework presented here, our construction does not rely on knowing the underlying ultraviolet completion, or on explicitly integrating out degrees of freedom. A bottom-up construction is key for applications of heavy particle effective theory to dark matter. Let us summarize the main points for general construction of heavy particle effective theories:

1. A heavy-particle field $h_v$ is identified with a representation of the little group for massive particles, determining its field transformation laws under rotations and boosts. It carries a label $v$ associated with the time-like unit vector $v^\mu$ that defines the little group (cf. Sects. 2.2 and 2.3).
2. The little group for massive particles is isomorphic to $SO(3)$, and therefore has field representations carrying spin $s = 0, 1/2, 1, \dots$. A heavy particle field of arbitrary spin may be represented in covariant notation using a Dirac spinor-vector with appropriate constraints. For example, a spin-$1/2$ heavy-particle field has $2(1/2) + 1$ degrees of freedom and can be written as a Dirac spinor, $h_v$, obeying $\not{v} h_v = h_v$ as a projection constraint (cf. Sect. 2.4).
3. For a self-conjugate heavy particle we impose an additional parity equivalent to a modified *CPT* transformation (cf. Sect. 2.4).

Having determined the building blocks and their transformation laws under symmetries, interactions with heavy-particle fields can be constructed in the usual way: write down the most general set of gauge- and Lorentz-invariant operators in terms of heavy fields $h_v$ , $h'_v$ , . . . , the time-like unit vector $v^\mu$, and other relativistic degrees of freedom (e.g., gauge fields, SM matter fields) up to a given order in the $1/M$ power counting. For self-conjugate fields the additional parity is imposed.

This rest of this chapter is organized as follows. In Sect. 2.1 we briefly review the construction of Lorentz invariant field theories based on finite dimensional representations of the Lorentz group. In Sect. 2.2 we introduce the formalism of induced representations and investigate the necessary conditions for a Lorentz invariant $S$ matrix. Section 2.3 establishes the connection between Lorentz invariance and reparameterization invariance. A subtlety in the identification of allowable reparameterization transformations is explained, and a correct solution to the invariance equation (2.52) is found for applications to $1/M^4$ heavy fermion Lagrangians. Section 2.4 presents formalism for arbitrary-spin heavy particles and for heavy particles derived from self-conjugate relativistic fields. In Sects. 2.5–2.8 we apply the formalism for construction of the parity and time-reversal invariant effective Lagrangian for a heavy fermion interacting with an Abelian gauge field, i.e., NRQED, through order $1/M^4$. The implementation of Lorentz invariance in the effective theory becomes nontrivial at this order, and we obtain a complete solution for Wilson coefficient constraints using both variational and invariant operator methods (cf. Sect. 2.6). We also present results of one-photon matching which verify the coefficient constraints (cf. Sect. 2.7), and Lagrangians describing pure photon and four-fermion interactions (cf. Sect. 2.8). We close the chapter with a discussion in Sect. 2.9.

## 2.1 Finite Dimensional Representations of the Lorentz Algebra

The standard method for constructing Lorentz invariant Lagrangians postulates the field transformation law

$$\phi_a(x) \to M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x) \,, \tag{2.6}$$

where $M(\Lambda)$ is a finite dimensional (coordinate-independent and, in general, non-unitary) representation of the Lorentz group. In infinitesimal form, including also spacetime translations $\phi(x) \to \phi(x-a)$, we have

$$\delta\phi = i(a_0 h - \boldsymbol{a}\cdot\boldsymbol{p} - \boldsymbol{\theta}\cdot\boldsymbol{j} + \boldsymbol{\eta}\cdot\boldsymbol{k})\phi \,, \tag{2.7}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are infinitesimal rotation and boost parameters, and the generators of the Poincaré group acting on fields are[1]

$$h = i\partial_t\,, \tag{2.8a}$$

$$\boldsymbol{p} = -i\boldsymbol{\partial}\,, \tag{2.8b}$$

$$\boldsymbol{j} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{\Sigma}\,, \tag{2.8c}$$

$$\boldsymbol{k} = \boldsymbol{r}h - t\boldsymbol{p} \pm i\boldsymbol{\Sigma}\,, \tag{2.8d}$$

with $\Sigma^i$ the $(2s+1)$-dimensional matrix generators of the spin-$s$ representation of rotations (e.g. for spin-$1/2$ Weyl fermions, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}/2$ with $\sigma^i$ the Pauli matrices). Using (2.6) it is straightforward to construct Lorentz invariant actions, and correspondingly to prove Lorentz invariance of the $S$ matrix. Let us briefly review this procedure.[2]

Recall the Poincaré algebra for generators of time translations $H$, space translations $P^i$, rotations $J^i$, and boosts $K^i$:

$$[H, P^i] = 0\,, \tag{2.9a}$$

$$[H, J^i] = 0\,, \tag{2.9b}$$

$$[P^i, P^j] = 0\,, \tag{2.9c}$$

$$[J^i, P^j] = i\epsilon^{ijk}P^k\,, \tag{2.9d}$$

$$[J^i, J^j] = i\epsilon^{ijk}J^k\,, \tag{2.9e}$$

$$[J^i, K^j] = i\epsilon^{ijk}K^k\,, \tag{2.9f}$$

$$[H, K^i] = -iP^i\,. \tag{2.9g}$$

$$[P^i, K^j] = -iH\delta^{ij}\,, \tag{2.9h}$$

$$[K^i, K^j] = -i\epsilon^{ijk}J^k\,. \tag{2.9i}$$

Having built a Lagrangian that is invariant under (2.7), we may construct the corresponding conserved charges. Using (2.8), we find in canonical quantization that these charges obey the commutation relations (2.9).

Lorentz invariance of the $S$ matrix demands that the free-particle charges, denoted by $H_0$, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{K}_0$, commute with the scattering operator, $S = \lim_{T\to\infty}\Omega(T)^\dagger\Omega(-T)$, where $\Omega(T) = e^{iHT}e^{-iH_0T}$. We assume that momentum

[1] In this chapter we use bold letters for Euclidean three-vectors, e.g. $\boldsymbol{\partial} = (\boldsymbol{\partial}^i) = (\boldsymbol{\partial}_i) = (\partial_x, \partial_y, \partial_z)$.

[2] For a pedagogical discussion, see [102].

and angular momentum operators for the interacting theory are unchanged from the free theory and furthermore demand translational and rotational invariance of the interaction

$$\boldsymbol{P} = \boldsymbol{P}_0\,, \quad \boldsymbol{J} = \boldsymbol{J}_0\,, \quad [H - H_0, \boldsymbol{P}_0] = [H - H_0, \boldsymbol{J}_0] = 0\,. \tag{2.10}$$

Then $[\boldsymbol{P}_0, S] = [\boldsymbol{J}_0, S] = 0$, and by the definition of $S$ also $[H_0, S] = 0$. Finally, if one can show (2.9g) and that an asymptotic smoothness condition for $\Delta\boldsymbol{K} = \boldsymbol{K} - \boldsymbol{K}_0$ is obeyed, it follows that

$$\begin{aligned}[\boldsymbol{K}_0, S] &= \lim_{T\to\infty}[\boldsymbol{K}_0, \Omega(T)^\dagger\Omega(-T)] \\ &= \lim_{T\to\infty}\Big\{ -[e^{iH_0T}\Delta\boldsymbol{K}e^{-iH_0T}]\Omega(T)^\dagger\Omega(-T) \\ &\quad + \Omega(T)^\dagger\Omega(-T)[e^{-iH_0T}\Delta\boldsymbol{K}e^{iH_0T}]\Big\} = 0\,,\end{aligned} \tag{2.11}$$

completing the proof of the Lorentz invariance of the $S$-matrix. For later application, we note that of the commutation relations involving $\boldsymbol{K}$, it is only necessary to show the relation (2.9g); relations (2.9f), (2.9h) and (2.9i) are not required to complete the proof.[3]

## 2.2 Effective Field Theory and the Little Group

The field transformation law (2.6), based on finite dimensional representations of the Lorentz group, is not suitable for heavy particle effective field theories. For example, the associated irreducible representations of the Lorentz group are chiral, in conflict with the low-energy limit of a parity conserving theory such as QED or QCD. Let us consider instead the class of infinite dimensional induced representations. We first review their appearance in transformations of physical states, and then apply them as transformations acting on fields.

### *2.2.1 Little Group Formalism*

Consider Lorentz transformations acting on the Hilbert space of physical states for a spin-$s$ particle of mass $M$. These transformations are implemented by an induced representation [103]. In terms of a fixed timelike reference vector $v^\mu$

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[3]In fact, these relations *do* follow from the observation that having proven Lorentz invariance of the $S$ matrix, it can be shown that $H$, $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ are related to their free counterparts by the similarity transformation $\Omega(\pm\infty)$ [102].

(we assume $v^2 = 1$), define the associated "little group" as the subgroup of Lorentz transformations leaving $v$ invariant, $\Lambda v = v$. The little group for massive particles is isomorphic to $SO(3)$, the group of rotations. Let $L(p)$ denote a standard Lorentz transformation taking $Mv$ to $p$, yielding a (momentum-dependent) mapping of the Lorentz group into the little group,

$$\Lambda \to W(\Lambda, p) = L(\Lambda p)^{-1}\Lambda L(p)\,. \tag{2.12}$$

We may define physical states to transform schematically as

$$|p,m\rangle \to U(\Lambda,p)|p,m\rangle = \sum_{m'=-s}^{s} D_{m'\,m}[W(\Lambda,p)]|\Lambda p, m'\rangle\,, \tag{2.13}$$

where $p^0 = \sqrt{M^2 + \boldsymbol{p}^2}$, and $D(W)$ is a spin-$s$ representation matrix for rotations. A representation for the little group thus induces a representation for the full Lorentz group.

A convenient choice for the standard Lorentz transformation is $L(p) = \Lambda(p/M, v)$, where $\Lambda(w,v)$ denotes the generalized rotation in the plane of the unit vectors $v$ and $w$ such that $\Lambda(w,v)v = w$. This matrix is given by $\Lambda(w,v) = \exp[-i\theta \mathcal{J}_{\alpha\beta} w^\alpha v^\beta]$, with the Lorentz generators $\mathcal{J}_{\alpha\beta}$ defined in Eq. (2.65) and the angle $\theta$ chosen appropriately [84]. In the vector and spinor representations we have, respectively

$$\begin{aligned}\Lambda(w,v)^\mu{}_\nu &= g^\mu{}_\nu - \frac{1}{1+v\cdot w}\left(w^\mu w_\nu + v^\mu v_\nu\right)\\ &\quad + w^\mu v_\nu - v^\mu w_\nu + \frac{v\cdot w}{1+v\cdot w}\left(w^\mu v_\nu + v^\mu w_\nu\right)\,,\end{aligned} \tag{2.14a}$$

$$\Lambda_{\frac{1}{2}}(w,v) = \frac{1+\not{w}\not{v}}{\sqrt{2(1+v\cdot w)}}\,. \tag{2.14b}$$

It is straightforward to verify that for elements of the little group, i.e. "rotations" with $\mathcal{R}v = v$, this choice of $L(p)$ implies

$$W(\mathcal{R},p) = \mathcal{R}\,, \tag{2.15}$$

a property that greatly simplifies the construction of invariant Lagrangians, cf. Sects. 2.2.3, 2.3.1 and 2.3.2 below. Other choices of $L(p)$ do not share this property. For example, suppose that we introduce a spacelike vector $s^\mu$ with $s^2 = -1$. Then we may define $L'(p) = R(p)B(p)$, with $B(p)$ a boost taking $Mv^\mu$ to $MB(p)^\mu{}_\nu v^\nu = (v\cdot p)v^\mu + \sqrt{(v\cdot p)^2 - M^2}s^\mu$, and $R(p)$ a rotation taking $MB(p)^\mu{}_\nu v^\nu$ to $p^\mu$. Such an $L'(p)$ provides a simple interpretation of $U[L(p)]|Mv,m\rangle$ in terms of helicity eigenstates (note that the spacelike vector is required to define a direction for helicity decomposition), but this consideration is secondary to the simplicity of (2.15) for our present purposes.

The remaining independent Lorentz generators represent "boosts" that shift $v$. They can be chosen as $\mathcal{B}(q) = \Lambda(v - q/M, v)$ with $(v - q/M)^2 = 1$. The appearance of the $1/M$ factor in $v - q/M$ will be explained in Sect. 2.2.3 below. For an infinitesimal momentum $q$, which obeys $v \cdot q = \mathcal{O}(q^2)$, these boosts are given by

$$\mathcal{B}(q)^\mu{}_\nu = g^\mu{}_\nu + \frac{v^\mu q_\nu - q^\mu v_\nu}{M} + \mathcal{O}(q^2)\,, \tag{2.16a}$$

$$\mathcal{B}_{\frac{1}{2}}(q) = 1 - \frac{\not{q}\not{v}}{2M} + \mathcal{O}(q^2)\,. \tag{2.16b}$$

For the transformation (2.13), we find

$$W(\mathcal{B}(q), p) = 1 - \frac{i}{2}\left[\frac{1}{M(M + v \cdot p)}(q^\alpha p_\perp^\beta - p_\perp^\alpha q^\beta)\right]\mathcal{J}_{\alpha\beta} + \mathcal{O}(q^2), \tag{2.17}$$

where for any four-vector $k$ we define $k_\perp^\mu \equiv k^\mu - (v \cdot k)v^\mu$.

### 2.2.2 Field Transformation Law and Lorentz Invariance

In place of (2.6) let us postulate the transformation law for free massive fields,

$$\phi_a(x) \to D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x)\,. \tag{2.18}$$

For notational simplicity consider the special choice $v = (1, 0, 0, 0)$. Equation (2.18) together with Eq. (2.17) corresponds to replacing the boost generator (2.8d) by[4]

$$\boldsymbol{k} = \boldsymbol{r}h - t\boldsymbol{p} \pm i\frac{\boldsymbol{\Sigma} \times \boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\,. \tag{2.19}$$

The generators (2.8a)–(2.8c) together with (2.19) will satisfy the Poincaré algebra when acting on fields satisfying

$$i\partial_t\phi = \pm\sqrt{M^2 - \boldsymbol{\partial}^2}\,\phi\,. \tag{2.20}$$

It follows that the conserved charges derived from a free field Lagrangian invariant under (2.18) will satisfy (2.9).

[4]For spin-$1/2$ particles, (2.19) may also be obtained by performing a Foldy-Wouthuysen transformation on Eq. (2.8d) [45].

In contrast to (2.6), transformation (2.18) acts on the field coordinates, spoiling gauge invariance. To include gauge interactions, we promote the partial derivatives in (2.18) to covariant derivatives $D_\mu = \partial_\mu - igA^A_\mu t^A \equiv \partial_\mu - igA_\mu$,

$$\phi_a(x) \to D[W(\Lambda, iD)]_{ab}\phi_b(\Lambda^{-1}x)\,, \tag{2.21}$$

and correspondingly the infinitesimal generators become

$$h = i\partial_t\,, \tag{2.22a}$$

$$\boldsymbol{p} = -i\boldsymbol{\partial}\,, \tag{2.22b}$$

$$\boldsymbol{j} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{\Sigma}\,, \tag{2.22c}$$

$$\boldsymbol{k} = \boldsymbol{r}h - t\boldsymbol{p} \pm i\frac{\boldsymbol{\Sigma} \times \boldsymbol{D}}{M + \sqrt{M^2 - \boldsymbol{D}^2}} + \mathcal{O}(g)\,. \tag{2.22d}$$

In the expansion of $\boldsymbol{D}/(M + \sqrt{M^2 - \boldsymbol{D}^2})$ we assume a choice of ordering for the covariant derivatives. The $\mathcal{O}(g)$ terms in $\boldsymbol{k}$ denote field strength-dependent corrections that vanish for the non-interacting theory (i.e. $g \to 0$). Such $\mathcal{O}(g)$ terms can be introduced so that the resulting invariant Lagrangian is in "canonical form", i.e. where the only time derivative acting on $\phi$ appears in the leading term,

$$\mathcal{L} = \bar{\phi}(iD_t + \dots)\phi\,. \tag{2.23}$$

The existence of suitable field strength-dependent terms, ensuring a boost generator $\boldsymbol{k}$ which yields a non-zero invariant Lagrangian, is implied by the all-orders construction in Sect. 2.3 and Appendix A. The explicit form of these corrections is not required for the following argument.

Although the field-dependent generators (2.22) do not obey simple commutation relations, we may nevertheless show that the $S$ matrix derived from the resulting invariant action is Lorentz invariant (and hence that the conserved charges in the interacting theory satisfy the Poincare algebra). To see this, we assume as before the relations (2.10). Relation (2.9g) is satisfied if the explicit time dependence of the conserved charge $\boldsymbol{K}$ satisfies $\partial\boldsymbol{K}/\partial t = -\boldsymbol{P}$, so that

$$0 = \frac{d}{dt}\boldsymbol{K} = \frac{\partial}{\partial t}\boldsymbol{K} + i[H, \boldsymbol{K}] = -\boldsymbol{P} + i[H, \boldsymbol{K}]\,. \tag{2.24}$$

The fact that $\partial\boldsymbol{K}/\partial t = -\boldsymbol{P}$ follows from the assumed form of the infinitesimal generators (2.22). For the boost $\phi \to (1 + i\boldsymbol{\eta}\cdot\boldsymbol{k})\phi$, we find the conserved charge[5]

[5]The first ellipsis in (2.25) includes possible contributions from a surface term in $\delta\mathcal{L}$, which do not affect the term with explicit $t$ dependence in (2.25).

$$\boldsymbol{K} = \sum_\phi i \int d^3x \frac{\delta\mathcal{L}}{\delta\dot{\phi}} \boldsymbol{k}\,\phi + \ldots = \sum_\phi i \int d^3x \frac{\delta\mathcal{L}}{\delta\dot{\phi}} \left[-t\boldsymbol{p}\right]\phi + \ldots = -t\boldsymbol{P} + \ldots \,. \tag{2.25}$$

Here the important point is that the remaining terms have no *explicit* time dependence, so that (2.24) follows.

Let us close this section with two comments. First, the choice $v = (1,0,0,0)$ is not essential to the argument. The generators for arbitrary $v$ can be obtained by a coordinate change using a boost which takes $(1,0,0,0)$ to $v$. While the resulting explicit expressions for rotation and boost generators become more complicated, the demonstration of Lorentz invariance is not essentially changed. Second, having specified an ordering for covariant derivatives appearing in the boost generator $\boldsymbol{k}$, additional field strength-dependent corrections are determined at each order in $1/M$ by enforcing that the resulting invariant Lagrangian is in canonical form. We illustrate this with an explicit example in the following subsection. The existence of such a generator is implied by the analysis of Sect. 2.3 and Appendix A.

### 2.2.3 $1/M$ Expansion and Lagrangian Constraints

To enable the $1/M$ expansion we extract the rest mass by the field redefinition,

$$\phi(x) = e^{-iMt}\phi'(x)\,. \tag{2.26}$$

In phenomenological applications it is also convenient to work with non-relativistic field normalization

$$\phi'(x) = \left(\frac{M^2}{M^2 - \boldsymbol{D}^2}\right)^{\frac{1}{4}} \phi''(x)\,. \tag{2.27}$$

We enforce invariance under (2.22a), (2.22b) and (2.22c) by ensuring translational invariance (no explicit factors of $x^\mu$) and rotational invariance. For the boost transformation (2.22d) we use $\boldsymbol{\eta} = -\boldsymbol{q}/M$ in (2.7) to preserve the power counting $D_t = \mathcal{O}(1/M)$ in (2.29). This explains the appearance of $1/M$ in (2.16). The resulting $1/M$ expansion becomes[6]

$$\phi'' \to e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left\{1 + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2} + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}\boldsymbol{D}^2}{4M^4} - \frac{\boldsymbol{\Sigma}\times\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2}\left[1 + \frac{\boldsymbol{D}^2}{4M^2}\right] + \mathcal{O}(g, 1/M^5)\right\}\phi''\,. \tag{2.28}$$

[6]For notational clarity we leave the coordinate change $x \to x' = \mathcal{B}^{-1}x$ implicit and suppress primes on coordinates and derivatives in (2.28) and (2.29).

Gauge fields are assumed to transform as usual, in the vector representation of the Lorentz group. Combined with derivatives acting on the transformed coordinate in (2.28), we have

$$D_t \to D_t + \frac{1}{M}\boldsymbol{q}\cdot\boldsymbol{D}\,, \quad \boldsymbol{D} \to \boldsymbol{D} + \frac{1}{M}\boldsymbol{q}D_t\,. \tag{2.29}$$

To illustrate the constraints, consider the canonical form of the Abelian gauged heavy spin-1/2 fermion effective Lagrangian (i.e., NRQED) through $\mathcal{O}(1/M^3)$. Identifying $\phi'' = \psi$ as a two-component spinor and setting $g = -e$ we obtain [77, 86]

$$\begin{aligned}\mathcal{L} = \psi^\dagger\Big\{ & iD_t + c_2\frac{\boldsymbol{D}^2}{2M} + c_4\frac{\boldsymbol{D}^4}{8M^3} + c_F e\frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D e\frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + ic_S e\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2}\\ & + c_{W1}e\frac{\{\boldsymbol{D}^2,\boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2}e\frac{D^i\boldsymbol{\sigma}\cdot\boldsymbol{B}D^i}{4M^3} + c_{p'p}e\frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D}+\boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3}\\ & + ic_M e\frac{\{\boldsymbol{D}^i,[\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} + c_{A1}e^2\frac{\boldsymbol{B}^2-\boldsymbol{E}^2}{8M^3} - c_{A2}e^2\frac{\boldsymbol{E}^2}{16M^3} + \mathcal{O}(1/M^4)\Big\}\psi\,.\end{aligned} \tag{2.30}$$

Here we have defined $E^i = (-i/e)[D_t, D^i]$, $\epsilon^{ijk}B^k \equiv (i/e)[D^i, D^j]$. Under (2.28), a straightforward computation yields

$$\delta\mathcal{L} = \frac{1}{M}\delta\mathcal{L}_1 + \frac{1}{M^2}\delta\mathcal{L}_2 + \frac{1}{M^3}\delta\mathcal{L}_3 + \ldots\,, \tag{2.31}$$

where using $\boldsymbol{\Sigma} = \boldsymbol{\sigma}/2$ in (2.28),

$$\delta\mathcal{L}_1 = \psi^\dagger\left[(1-c_2)i\boldsymbol{q}\cdot\boldsymbol{D}\right]\psi\,, \tag{2.32a}$$

$$\delta\mathcal{L}_2 = \psi^\dagger\left[-\frac{1}{2}(1-c_2)\{\boldsymbol{q}\cdot\boldsymbol{D},D_t\} + \frac{e}{4}(1-2c_F+c_S)\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{E}\right]\psi\,, \tag{2.32b}$$

$$\begin{aligned}\delta\mathcal{L}_3 = \psi^\dagger\Big[ & \frac{e}{8}c_D[D_t,\boldsymbol{q}\cdot\boldsymbol{E}] + \frac{e}{8}(c_F-c_D+2c_M)\,\boldsymbol{q}\cdot[\boldsymbol{\partial}\times\boldsymbol{B}] + \frac{i}{4}(c_2-c_4)\{\boldsymbol{q}\cdot\boldsymbol{D},\boldsymbol{D}^2\}\\ & + \frac{ie}{8}c_S\{D_t,\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{E}\} + \frac{ie}{8}(c_2+2c_F-c_S-2c_{W1}+2c_{W2})\{\boldsymbol{q}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{B}\}\\ & + \frac{ie}{8}(-c_2+c_F-c_{p'p})\{\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{q}\cdot\boldsymbol{B}\}\\ & + \frac{ie}{8}(-c_F+c_S-c_{p'p})\,\boldsymbol{q}\cdot\boldsymbol{\sigma}(\boldsymbol{D}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{D})\Big]\psi\,.\end{aligned} \tag{2.32c}$$

From $\delta\mathcal{L}_1$ and $\delta\mathcal{L}_2$, we find

$$c_2 = 1\,, \quad c_S = 2c_F - 1\,. \tag{2.33}$$

The variation $\delta\mathcal{L}_3$ is equivalent to zero upon a field strength-dependent modification of the boost transformation (2.28),

$$\psi(x) \to e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \bigg\{ 1 + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2} - \frac{\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{D}}{4M^2} + \frac{ic_D}{8M^3} e\boldsymbol{q}\cdot\boldsymbol{E} + \frac{c_S}{8M^3} e\boldsymbol{q}\cdot\boldsymbol{\sigma}\times\boldsymbol{E} + \mathcal{O}\left(\frac{1}{M^4}\right) \bigg\} \psi(\mathcal{B}^{-1}x)\,, \tag{2.34}$$

and upon enforcing the constraints [58, 86]

$$c_4 = 1\,, \quad 2c_M = c_D - c_F\,, \quad c_{W2} = c_{W1} - 1\,, \quad c_{p'p} = c_F - 1\,. \tag{2.35}$$

The computation of the complete Lagrangian at $\mathcal{O}(1/M^4)$ is presented in Sect. 2.6.

## 2.3 Reparametrization Invariance and Invariant Operators

While in practice it may be convenient to enforce Lorentz invariance only after expanding the Lagrangian in a series of rotationally-invariant, but not Lorentz invariant, operators, it is interesting to consider formalism that permits an explicitly Lorentz-invariant construction. This formalism also addresses the question of existence of a suitable boost generator, extending (2.34) to arbitrary order in $1/M$.

This section begins by introducing covariant notation that can either be used in place of the $v = (1,0,0,0)$ formalism above, or used to construct manifestly invariant operators. The relation between Lorentz invariance and reparameterization invariance is then demonstrated, and a general discussion of the invariant operator method is presented. In particular, we derive the necessary invariance equation (2.52) and present the solution to order $1/M^3$. A systematic, all-orders solution of the invariance equation is given in Appendix A.

### *2.3.1 Covariant Notation*

The formalism of Sect. 2.4 allows us to straightforwardly extend the discussion to a general reference vector $v$ and to arbitrary spin. Consider a term in the Lagrangian of the schematic form

$$\bar{\phi}_v \bigg\{ \cdots v^\mu \cdots D^\mu \cdots \gamma^\mu \cdots \bigg\} \phi_v\,, \tag{2.36}$$

where indices are contracted with $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. Invariance under generalized rotations of such a term in the action follows using the field transformation (2.15),

$$\phi_v(x) \to \mathcal{R}\phi_v(x')\,, \tag{2.37}$$

where $x' \equiv \mathcal{R}^{-1}x$. The transformation of the derivative and the gauge field are as usual,

$$\partial^\mu \to \partial^\mu = \mathcal{R}^\mu{}_\nu \partial'^\nu\,, \quad A^\mu \to \mathcal{R}^\mu{}_\nu A^\nu(x')\,. \tag{2.38}$$

If the Lagrangian is already constructed such that all vector and spinor indices are contracted in (2.36), we can easily see that the Lagrangian is invariant under generalized rotations using the identities

$$v^\mu = \mathcal{R}^\mu{}_\nu v^\nu\,, \qquad \gamma^\mu = \mathcal{R}_{\frac{1}{2}}\left(\mathcal{R}^\mu{}_\nu \gamma^\nu\right)\mathcal{R}_{\frac{1}{2}}^{-1}\,. \tag{2.39}$$

According to (2.17), the infinitesimal boosts are implemented by

$$\phi_v(x) \to W(\mathcal{B}, iD)\phi_v(x')\,, \tag{2.40}$$

where $x' \equiv \mathcal{B}^{-1}x$, together with the transformation of the derivative and gauge field,

$$\partial^\mu \to \partial^\mu = \mathcal{B}^\mu{}_\nu \partial'^\nu\,, \quad A^\mu(x) \to \mathcal{B}^\mu{}_\nu A^\nu(x')\,. \tag{2.41}$$

We may proceed as in Sect. 2.2.3 above to construct invariant combinations of Lagrangian interactions of the form (2.36), order by order in $1/M$.

As an explicit example, let us focus presently on the phenomenologically important one-heavy particle sector of a spin-$1/2$ theory. To enable the $1/M$ expansion and convert to non-relativistic normalization, we introduce the field redefinition as in (2.26) and (2.27),

$$\psi_v(x) = e^{-iMv\cdot x}N(v, iD)\psi'_v(x)\,, \quad N(v, iD) = \left(\frac{M^2}{M^2 + D_\perp^2}\right)^{\frac{1}{4}}\,. \tag{2.42}$$

The boost transformation (2.40) becomes

$$\psi'_v \to e^{iq\cdot x}\tilde{W}_{\frac{1}{2}}(\mathcal{B}, iD + Mv)\psi'_v\,, \tag{2.43}$$

where

$$\tilde{W}(\mathcal{B}, iD + Mv) = N(v + q/M, iD - q)^{-1}W(\mathcal{B}, iD + Mv)N(v, iD)\,. \tag{2.44}$$

The $1/M$ expansion of this transformation is the extension to arbitrary $v$, for spin-$1/2$, of the previous (2.28):

$$\psi'_v \to e^{iq\cdot x}\left\{1 + \frac{iq\cdot D_\perp}{2M^2} - \frac{iq\cdot D_\perp D_\perp^2}{4M^4} + \frac{1}{4M^2}\sigma_{\alpha\beta}q^\alpha D_\perp^\beta\left[1 - \frac{D_\perp^2}{4M^2}\right] + \mathcal{O}(g, 1/M^5)\right\}\psi'_v\,. \tag{2.45}$$

Similarly, we find the extension to arbitrary $v$ of the transformations (2.29)

$$v \cdot D \to v \cdot D + \frac{1}{M} q \cdot D_\perp \,, \quad D_\perp^\mu \to D_\perp^\mu - \frac{1}{M} q^\mu (v \cdot D). \tag{2.46}$$

Using these transformations one can build an invariant Lagrangian, which (in the Abelian case) is equivalent to the extension of the Lagrangian (2.30) to arbitrary $v$ with the same constraints (2.33) and (2.35).

### *2.3.2 Reparametrization Invariance*

We can reformulate the transformation law for generalized boosts by using the identities,

$$v^\mu = \mathcal{B}^\mu{}_\nu (\mathcal{B}^{-1})^\nu_\rho v^\rho \equiv \mathcal{B}^\mu_\nu w^\nu \,, \quad \gamma^\mu = \mathcal{B}_{\frac{1}{2}} \left( \mathcal{B}^\mu{}_\nu \gamma^\nu \right) \mathcal{B}_{\frac{1}{2}}^{-1} \,. \tag{2.47}$$

In place of (2.40) and (2.41) the transformation of any operator of the form (2.36) is identical to the transformation obtained by the substitutions

$$v \to w = v + q/M \,, \quad \phi_v \to \phi_w \equiv \mathcal{B}^{-1} W(\mathcal{B}, iD_\mu) \phi_v \,, \tag{2.48}$$

with no transformation of the coordinate and gauge field. The rules (2.48), with suitable choice for $W$, may be identified with the rules obtained by enforcing "reparameterization invariance" [84]. However, we emphasize that from the present perspective, we are not changing the reference vector $v$, but simply noticing the equivalence of (2.40) and (2.41) on the one hand, and (2.48) on the other hand, when acting on operators of the form (2.36).

### *2.3.3 Invariant Operator Method*

It is not obvious that a non-zero Lagrangian, invariant under (2.40) and (2.41) to arbitrary order, will exist. For example, in (2.31) invariance relies on the possibility to enforce $\delta \mathcal{L}_n = 0$ by modifying the boost generator as in (2.34) and enforcing relations as in (2.33) and (2.35). It is not evident that this procedure can be extended to arbitrary order. We present here a method of constructing operators that are manifestly invariant under a particular choice of boost generator, to arbitrary order in $1/M$. The details of the construction are given in Appendix A.

The embedding of the little group into constrained representations of the full Lorentz group (cf. Sect. 2.4) provides a framework for constructing explicitly invariant operators. Suppose that we find an operator $\Gamma(v, iD)$ such that

$$\Gamma(\Lambda^{-1} v, iD) \Lambda^{-1} W(\Lambda, iD) = \Gamma(v, iD) \,, \tag{2.49}$$

when acting on fields $\phi_v$ obeying the appropriate constraints, as given in Sect. 2.4 (e.g. $\not{v}\phi_v = \phi_v$ for spin-$1/2$). It follows from the rules (2.48) that the combination

$$\Phi_v \equiv \Gamma(v, iD)\phi_v \tag{2.50}$$

is invariant under the reparameterization implementation (2.48) of generalized boosts. Provided that invariance under generalized rotations (2.37)–(2.39) is maintained, we may build operators that are explicitly invariant. For example, in the spin-$1/2$ case

$$\bar{\Psi}_v i\not{D}\,\Psi_v\,, \quad \bar{\Psi}_v\Psi_v\,, \quad \bar{\Psi}_v i\sigma^{\mu\nu}[D_\mu, D_\nu]\Psi_v\,, \tag{2.51}$$

are invariant. Note that because of Eq. (2.15) the only constraints on $\Gamma(v, iD)$ from Eq. (2.49) come from boosts $\Lambda = \mathcal{B}$.

Applying field redefinitions as in (2.42), the condition (2.49) for $\Gamma$ becomes

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}\tilde{W}(\mathcal{B}, iD + Mv) = \Gamma(v, iD)\,. \tag{2.52}$$

We will refer to (2.52) as the "invariance equation". Provided that such a $\Gamma(v, iD)$ can be found, the field

$$\Phi'_v(x) \equiv \Gamma(v, iD)\phi'_v(x) \tag{2.53}$$

obeys a simple transformation law under the reparameterization implementation of generalized boosts (2.48),

$$\Phi'_v \to \Phi'_w \equiv e^{iq\cdot x}\Phi'_v\,. \tag{2.54}$$

Noting that $e^{-iq\cdot x}(iD^\mu + Mw^\mu)e^{iq\cdot x} = iD^\mu + Mv^\mu$, invariant operators may thus be built from contractions of polynomials of $\gamma^\mu$ and $v^\mu + iD^\mu/M$, between $\bar{\Phi}'_v$ and $\Phi'_v$. For example in the spin-$1/2$ case,

$$\bar{\Psi}'_v(i\not{D} + M\not{v})\Psi'_v\,, \quad \bar{\Psi}'_v\Psi'_v\,, \quad \bar{\Psi}'_v i\sigma^{\mu\nu}[D_\mu, D_\nu]\Psi'_v\,, \tag{2.55}$$

are invariant.

### 2.3.4 Solution for $\Gamma(v, iD)$

The key element of the invariant operator construction is a solution of the invariance equation (2.52). Without loss of generality, let us set $N(v, iD) = 1$; the solution for general $N$ can then be obtained by $\Gamma(v, iD) \to \Gamma(v, iD)N(v, iD)^{-1}$. The method presented can be easily extended to arbitrary spin, but for illustration we focus on the one-heavy particle sector of a spin-$1/2$ theory.

In order to obtain a solution in closed form for the free theory, and to make contact with previous work, it is convenient to take the free theory limit for $W_{\frac{1}{2}}(\mathcal{B}, i\partial + Mv)$ of the form [84]

$$W_{\frac{1}{2}}(\mathcal{B}, i\partial + Mv) = \mathcal{B}_{\frac{1}{2}}\,\Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\rm free}, v + q/M)^{-1}\Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\rm free}, v) \quad (2.56)$$

$$= 1 + \frac{1}{4M^2}\sigma_\perp^{\mu\nu} q_\mu \partial_\nu \left[1 - \frac{iv\cdot\partial}{M} + \frac{1}{M^2}\left((iv\cdot\partial)^2 - \frac{1}{4}(i\partial_\perp)^2\right)\right] + \mathcal{O}(1/M^5)\,,$$

where $\Lambda_{\frac{1}{2}}(u, v)$ was defined in (2.14), $\mathcal{V}^\mu_{\rm free} \equiv v^\mu + i\partial^\mu/M$ and $\hat{\mathcal{V}}^\mu_{\rm free} \equiv \mathcal{V}^\mu_{\rm free}/|\mathcal{V}_{\rm free}|$. We have also used that $/v\psi_v = \psi_v$. Inspection of (2.52) shows that an all-orders solution can be written for $\Gamma$ in the non-interacting theory,

$$\Gamma(v, i\partial) = \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\rm free}, v) = 1 + \frac{i\partial\!\!\!/_\perp}{2M} + \frac{1}{M^2}\left[-\frac{1}{8}(i\partial_\perp)^2 - \frac{1}{2} i\partial\!\!\!/_\perp iv\cdot\partial\right]$$
$$+ \frac{1}{M^3}\left[\frac{1}{4}(i\partial_\perp)^2 iv\cdot\partial + \frac{i\partial\!\!\!/_\perp}{2}\left(-\frac{3}{8}(i\partial_\perp)^2 + (iv\cdot\partial)^2\right)\right] + \mathcal{O}(1/M^4)\,. \quad (2.57)$$

In the interacting theory it turns out that one cannot simply replace $\partial$ by $D$ in (2.57) to obtain a solution for $\Gamma(v, iD)$. It is instead necessary to add specific field strength dependent terms, first to $W$ (as in (2.58) and (A.2a) below) in order to satisfy consistency conditions, and then to $\Gamma$ in order to solve the invariance equation (2.52). The computations of Appendix A show that a solution for $\Gamma(v, iD)$ will exist if we specify

$$W_{\frac{1}{2}}(\mathcal{B}, iD + Mv) = 1 + \frac{1}{4M^2}\sigma^\perp_{\mu\nu} q^\mu D^\nu_\perp \left(1 - \frac{iv\cdot D}{M}\right) + \mathcal{O}(1/M^4)\,, \quad (2.58)$$

with (2.58) reducing to (2.56) at $g = 0$. Let us proceed through $\mathcal{O}(1/M^3)$, writing

$$\Gamma = 1 + \frac{1}{M}\Gamma^{(1)} + \frac{1}{M^2}\Gamma^{(2)} + \frac{1}{M^3}\Gamma^{(3)} + \dots\,, \quad (2.59)$$

and deriving a solution to the invariance equation (2.52) order by order in $1/M$. In Appendix A we present a systematic construction that extends the solution to arbitrary order.

Modulo terms that vanish when acting on $\psi_v$ with $v\!\!\!/\psi_v = \psi_v$, we find

$$\Gamma^{(1)} = \frac{1}{2} iD\!\!\!\!/_\perp\,. \quad (2.60a)$$

$$\Gamma^{(2)} = -\frac{1}{8}(iD_\perp)^2 - \frac{1}{2} iD\!\!\!\!/_\perp iv\cdot D + gA\sigma^{\mu\nu}G_{\mu\nu} + gB\gamma^\mu v^\nu G_{\mu\nu}\,. \quad (2.60b)$$

$$\Gamma^{(3)} = \frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left[-\frac{3}{8}(iD_\perp)^2 + (iv\cdot D)^2\right]$$
$$- \frac{g}{8}G_{\mu\nu}v^\mu D_\perp^\nu - \frac{g}{16}\sigma_\perp^{\mu\nu}G_{\mu\nu}i\not{D}_\perp\,, \qquad (2.60c)$$

where we define $[iD_\mu, iD_\nu] = igG_{\mu\nu}$. Starting at order $1/M^2$ the solution is not unique. However, since we will consider arbitrary factors of $\mathcal{V}^\mu \equiv v^\mu + iD^\mu/M$ when constructing invariant operators, we can set $A = B = 0$ by considering instead of $\Gamma$, the operator $\Gamma'$ given by

$$\Gamma(v, iD) = (1 - iA\sigma_{\mu\nu}[\mathcal{V}^\mu, \mathcal{V}^\nu] - iB\gamma_\mu\mathcal{V}_\nu[\mathcal{V}^\mu, \mathcal{V}^\nu] + \dots)\,\Gamma'(v, iD)\,. \qquad (2.61)$$

Similarly, we have absorbed additional $1/M^3$ terms in (2.60c). The remaining terms in (2.60) have free derivatives $D_\mu$ acting to the right, and cannot be removed as in (2.61).

A complete basis of bilinears required through order $1/M^3$ is

$$\mathcal{L} = \bar{\Psi}_v\Big\{M(\not{\mathcal{V}} - 1) - a_F g\frac{\sigma^{\mu\nu}G_{\mu\nu}}{4M} + ia_D g\frac{\{\mathcal{V}_\mu, [M\mathcal{V}_\nu, G^{\mu\nu}]\}}{16M^2}$$
$$- a_{W1}g\frac{[M\mathcal{V}^\alpha, [M\mathcal{V}_\alpha, \sigma^{\mu\nu}G_{\mu\nu}]]}{16M^3}$$
$$+ a_{A1}g^2\frac{G_{\mu\nu}G^{\mu\nu}}{16M^3} + a_{A2}g^2\frac{\mathcal{V}_\alpha G^{\mu\alpha}G_{\mu\beta}\mathcal{V}^\beta}{16M^3}\Big\}\Psi_v\,. \qquad (2.62)$$

Performing field redefinitions to arrive at canonical form, we recover the result (2.30) with constraints (2.33) and (2.35). The computation at $\mathcal{O}(1/M^4)$ is presented in Sect. 2.6. We may perform a similar computation for heavy vector particles (or particles of arbitrary spin), and/or enforce constraints appropriate to self-conjugate fields (cf. Sect. 2.4).

The passage from (2.57) to (2.60) is not as simple as previously envisaged [84, 86], and careful attention must be paid to the interplay of Lorentz and gauge symmetry. The computations in Appendix A show that an arbitrary "covariantization" of (2.56) does *not* solve the invariance equation (2.52). The covariant little group element $W(\mathcal{B}, iD + Mv)$ must satisfy consistency conditions for a solution to exist, and specific field strength dependent terms, such as those appearing in (2.60c), are necessary in order that $\Gamma(v, iD)$ satisfy the resulting invariance equation (2.52). These considerations have previously been overlooked [84, 86]. For example, a naive covariantization of Eq. (2.57),

$$\Gamma^{\text{naive}}(v, iD) = 1 + \frac{i\not{D}_\perp}{2M} + \frac{1}{M^2}\left[-\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\not{D}_\perp iv\cdot D\right] \qquad (2.63)$$
$$+ \frac{1}{M^3}\left[\frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left(-\frac{3}{8}(iD_\perp)^2 + (iv\cdot D)^2\right)\right]$$
$$+ \mathcal{O}(1/M^4)\,,$$

is not a solution to the invariance equation. The necessity for such additional field strength dependent terms can also be seen from the fact that the right hand side of (2.63) would imply a transformation $\psi_v \to \psi_w = \Gamma^{\rm naive}(w, iD)^{-1} e^{iq\cdot x}\Gamma^{\rm naive}(v, iD)\psi_v$ that takes $\psi_v$ outside of the assumed representation space, with $\not{v}\psi_v = \psi_v$. In the heavy fermion Lagrangian, the effects of these field-strength dependent terms appear first at order $\mathcal{O}(1/M^4)$, where omission of the final term in (2.60c) would lead to incorrect $1/M^4$ Lagrangian coefficient relations (details are presented in later sections of this chapter).[7]

Before closing this section, let us summarize the value of the invariant operator method. Appendix A shows that we can find a suitable covariantization of $W(\mathcal{B}, i\partial + Mv)$ that allows solution of the invariance equation for $\Gamma(v, iD)$ to any order in $1/M$. Hence this method proves the existence of a covariantized boost operator and a non-zero, Lorentz invariant Lagrangian to arbitrary order. We may proceed in either of two ways to construct invariant Lagrangians. Firstly, we may proceed as in (2.62), where we construct manifestly invariant interactions through some fixed order in $1/M$; to achieve canonical form we must then perform field redefinitions. Alternatively, we may proceed as in (2.30) (or its generalization to arbitrary $v$), armed with the knowledge that a suitable boost generator as in (2.34) can be reconstructed order by order.

## 2.4 Higher-Spin and Self-conjugate Fields

Although the explicit results have so far focused on spin-$1/2$ fields transforming under an Abelian (i.e. complex) gauge group, the formalism extends straightforwardly to fields of arbitrary spin or to self-conjugate fields. Below we describe the formalism for embedding arbitrary spin representations within products of Dirac spinor and Lorentz vector representations of the Lorentz group. For a related discussion see e.g. [43]. We also describe constraints imposed on the effective theory deriving from self-conjugate fields, and the relation of such constraints to discrete symmetries $C$, $P$ and $T$.

### *2.4.1 Higher Spin Representations*

Irreducible higher spin representations can be built using products of the Dirac spinor and vector representations

$$\psi_v \to \Lambda_{\frac{1}{2}}\psi_v\,, \quad Z_v^\alpha \to \Lambda^\alpha{}_\beta Z_v^\beta\,, \tag{2.64}$$

[7] When building invariant fermion bilinears, the leading terms involve $iv\cdot D$ multiplying $1/M$ corrections appearing in $\Gamma(v, iD)$. Since such terms are eliminated in going to canonical form, nontrivial effects of the $1/M^3$ corrections to $\Gamma(v, iD)$ appear first at order $1/M^4$.

where $\Lambda = D(W)$ is a little group element as in Sect. 2.2.1, i.e., $\Lambda v = v$. The corresponding generators for these two representations are given by

$$\mathcal{J}_{\frac{1}{2}}^{\alpha\beta} = \frac{1}{2}\sigma^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta], \qquad (\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(g^\alpha{}_\mu g^\beta{}_\nu - g^\beta{}_\mu g^\alpha{}_\nu). \tag{2.65}$$

We enforce a maximal set of constraints to isolate the appropriate irreducible representation.

**Integer spin:** For integer spin $s = n$, consider the totally symmetric and traceless tensor $Z_v^{\mu_1\cdots\mu_n}$, which has $(n+1)^2$ degrees of freedom. Imposing

$$v_{\mu_1} Z_v^{\mu_1\cdots\mu_n} = 0 \tag{2.66}$$

yields $n^2$ additional constraints, leaving us with $2n+1 = 2s+1$ degrees of freedom as desired. Under Lorentz transformations this field transforms as

$$Z_v^{\mu_1\cdots\mu_n} \to \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} Z_v^{\nu_1\cdots\nu_n}\,. \tag{2.67}$$

Using $\Lambda^T g\Lambda = g$ and $\Lambda v = v$, it is easy to see that symmetry, tracelessness and the constraint (2.66) are preserved by this transformation.

**Half-integer spin:** For half-integer spin $s = n + 1/2$, consider the spinor-tensor $\psi_v^{\mu_1,\mu_2,\dots,\mu_n}$, which is totally symmetric in the indices $\mu_1 \dots \mu_n$ and therefore has $2(n+1)(n+2)(n+3)/3$ degrees of freedom. We impose the constraints[8]

$$\slashed{v}\psi_v^{\mu_1\cdots\mu_n} = \psi_v^{\mu_1\cdots\mu_n}\,, \quad \gamma_{\mu_1}\psi_v^{\mu_1\cdots\mu_n} = 0. \tag{2.68}$$

The second constraint yields $n(n+1)(n+5)/3$ equations, while the first projects a four-component spinor onto a two-dimensional subspace, reducing the degrees of freedom by $1/2$. In total $2(n+1) = 2s+1$ degrees of freedom remain. Under Lorentz transformations this field transforms as

$$\psi_v^{\mu_1\cdots\mu_n} \to \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} \Lambda_{\frac{1}{2}} \psi_v^{\nu_1\cdots\nu_n}\,. \tag{2.69}$$

This is symmetric in $\mu_1 \dots \mu_n$. That Eq. (2.68) are preserved follows immediately from $\Lambda v = v$ and $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$.

The construction of heavy particle Lagrangians for higher-spin fields proceeds in a similar way to the spin-1/2 case described above, with extension of the spin matrices $(1, \sigma^i)$ to the appropriate set. It is possible to choose spin matrices of the appropriate dimension to construct the Lagrangian for a heavy particle of arbitrary spin valid to any order in the $1/M$ expansion.

[8] Note that the second constraint implies $g_{\mu\nu}\psi_v^{\mu\nu\mu_3\cdots\mu_n} = 0$ and, furthermore, is equivalent to imposing $v_{\mu_1}\psi_v^{\mu_1\cdots\mu_n} = 0$ and $\epsilon_{\nu\alpha\beta\mu_1} v^\nu \sigma^{\alpha\beta}\psi_v^{\mu_1\cdots\mu_n} = 0$.

### 2.4.2 Self-conjugate Parity and CPT

The self-conjugacy of $SU(2)$ implies that for any field $\phi(x)$ transforming as in (2.8) or (2.19) with the plus sign, the field

$$\phi^c(x) = S\phi^*(x)\,, \tag{2.70}$$

transforms as in (2.8) or (2.19) with the minus sign. Here $S$ is the $(2s+1)\times(2s+1)$ similarity transformation for the spin-$s$ representation of $SU(2)$, such that $(-\Sigma^i)^* = S\Sigma^i S^{-1}$. In covariant language, this translates to the simultaneous transformations

$$\phi_v(x) \to \phi_v^c(x)\,, \quad v^\mu \to -v^\mu\,. \tag{2.71}$$

In terms of the irreducible representations constructed in Sect. 2.4.1, the field transformation in (2.71) reads[9]

$$Z_v^{\mu_1\ldots\mu_s} \to (Z_v^c)^{\mu_1\ldots\mu_s} = (Z_v^{\mu_1\ldots\mu_s})^*\,, \quad \psi_v^{\mu_1\ldots\mu_s} \to (\psi_v^c)^{\mu_1\ldots\mu_s} = \mathcal{C}(\psi_v^{\mu_1\ldots\mu_s})^*\,, \tag{2.72}$$

for integer spin and half-integer spin fields, respectively. The charge conjugation matrix $\mathcal{C}$ acts on the spinor index of $\psi_v$. It is symmetric and unitary, and obeys $\mathcal{C}^\dagger\gamma^\mu\mathcal{C} = -\gamma^{\mu *}$. The parity (2.71) arises if the effective theory is describing a full theory of a self-conjugate field (necessarily transforming in a real representation of a gauge group). For example, the effective theory field for a real scalar $\varphi = \varphi^*$ can be obtained via

$$\varphi(x) = e^{-iMv\cdot x}\varphi_v(x)/\sqrt{M} = e^{iMv\cdot x}\varphi_v^*(x)/\sqrt{M} = \varphi^*(x)\,. \tag{2.73}$$

Similarly, the effective theory for a Majorana fermion represented by a Dirac spinor $\psi_M = \psi_M^c$ can be obtained via

$$\psi_M = \sqrt{2}e^{-iMv\cdot x}(h_v + H_v) = \sqrt{2}e^{iMv\cdot x}(h_v^c + H_v^c) = \psi_M^c\,, \tag{2.74}$$

where $\not{v}h_v = h_v$ and $\not{v}H_v = -H_v$.

It follows from (2.71) that the allowed operators $\bar{\phi}_v\mathcal{O}(v)\phi_v$ in the Lagrangian representing a self-conjugate field can be chosen such that

$$\mathcal{O}(v) = \mathcal{C}\mathcal{O}(-v)^*\mathcal{C}^\dagger. \tag{2.75}$$

Since we are often interested in constructing the Lagrangian in canonical form, i.e., without higher $iv\cdot D$ derivatives acting on $\phi_v$, it is important to ask whether this condition is preserved by the requisite field redefinitions. By a similar reasoning

[9] We here choose a basis such that $S = 1$ for vectors.

to above, operators of the form $\bar{\phi}_v[iv\cdot DX(v)+X^\dagger(v)iv\cdot D]\phi_v$ appearing in the Lagrangian must be such that $X(v)=\mathcal{C}X(-v)^*\mathcal{C}^\dagger$. Hence field redefinitions of the form $\phi_v\to[1-X(v)]\,\phi_v$ achieve canonical form of the Lagrangian while preserving (2.75).

The self-conjugate parity for heavy fields described above is equivalent to imposing a modified, heavy-particle version of $CPT$, where $P$ and $T$ act on a Dirac spinor $\chi_v$ in the usual way, but $C$ acts as the identity:

$$C:\chi(t,\boldsymbol{x})\to\chi(t,\boldsymbol{x})\,,\quad P:\chi(t,\boldsymbol{x})\to\gamma_0\chi(t,-\boldsymbol{x})\,,\quad T:\chi(t,\boldsymbol{x})\to\gamma_1\gamma_3\chi(-t,\boldsymbol{x})\,. \tag{2.76}$$

In this formulation, the reference vector $v^\mu$ is unchanged while the field transformations under discrete symmetries $C,P,T$ are implemented. Hence, it may be imposed in the case where the heavy particle effective theory is written in the rest frame where the reference vector has been fixed to $v^\mu=(1,\mathbf{0})$.

## 2.5 NRQED Example: Lagrangian

Let us demonstrate the application of our formalism in the case of Nonrelativistic QED (NRQED) (i.e., the parity and time-reversal symmetric theory of a heavy spin-$1/2$ particle coupled to an Abelian gauge field) at $\mathcal{O}(1/M^4)$. We will also consider examples with multiple heavy particle fields, and other relativistic degrees of freedom beyond Abelian gauge fields.

NRQED is an effective field theory [21] describing the interactions of nonrelativistic fermions with electromagnetic fields. NRQED interactions at order $1/M^4$ have become relevant for describing radiative corrections to proton structure contributions in hydrogenic bound state spectroscopy [58, 93]. The NRQED Lagrangian, properly constrained by Lorentz invariance, trivializes the derivation of low-energy theorems of Compton scattering [95] and automatically incorporates the intricate singularity structure of scattering amplitudes [10, 99]. It can be used to rigorously compute radiative corrections to low-energy lepton-nucleon scattering, and it also provides a model-independent framework within which to analyze static properties of nucleons, such as polarizabilities and generalized electromagnetic moments [85].

Let us illustrate the formalism for constructing heavy particle Lagrangians, deriving a complete basis of operators and coefficient constraints through order $1/M^4$ for the effective theory of nonrelativistic nucleons and leptons interacting with photons.[10] An important formal issue first arises at order $1/M^4$: as discussed in the previous sections, a "reparameterization invariance" ansatz for enforcing relativistic

[10]For definiteness we will often refer to the heavy fermion $\psi$ as the "nucleon", and to a second fermion $\chi$ in Sect. 2.8 as the "lepton".

invariance breaks down at this order. We derive the correct implementation of Lorentz invariance constraints and the resulting Wilson coefficient relations (i.e., nonrenormalization theorems) through order $1/M^4$.

Let us begin by constructing the NRQED Lagrangian in the one-fermion sector through order $1/M^4$. Consider the Lagrangian for a heavy fermion coupled to an Abelian gauge field. We enforce hermiticity and invariance under parity, time-reversal and rotational symmetries. We also perform field redefinitions to eliminate time derivatives acting on the fermion field (apart from the leading term); we refer to this choice as the "canonical form" of the heavy particle Lagrangian. We thus find in the one-fermion sector,

$$\begin{aligned}\mathcal{L} = \psi^\dagger \Big\{ & iD_t + c_2 \frac{\boldsymbol{D}^2}{2M} + c_4 \frac{\boldsymbol{D}^4}{8M^3} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} \\ & + i c_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D})}{8M^2} \\ & + c_{W1} g \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2} g \frac{D^i \boldsymbol{\sigma}\cdot\boldsymbol{B} D^i}{4M^3} + c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} \\ & + i c_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} + c_{A1} g^2 \frac{\boldsymbol{B}^2 - \boldsymbol{E}^2}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\ & + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} \\ & + i c_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} + i c_{X5} g \frac{D^i \boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D}) D^i}{M^4} \\ & + i c_{X6} g \frac{\epsilon^{ijk}\sigma^i D^j [\boldsymbol{\partial}\cdot\boldsymbol{E}] D^k}{M^4} \\ & + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4} + c_{X9} g^2 \frac{[\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{M^4} \\ & + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} \\ & + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}\times[\partial_t \boldsymbol{E} - \boldsymbol{\partial}\times\boldsymbol{B}]}{M^4} + \mathcal{O}(1/M^5) \Big\} \psi \,. \end{aligned} \tag{2.77}$$

We have defined $D_t = \partial/\partial t + igZA^0$, $D^i = \partial/\partial x^i - igZA^i$, where $-gZ = -e$, $+e$ or $0$ for an electron, proton or neutron, respectively. The operators up to $1/M^3$ were previously listed in [21, 77, 86]. We use the summation convention $X^iY^i \equiv \sum_{i=1}^3 X^iY^i$, and define $[X,Y] \equiv XY - YX$, $\{X,Y\} \equiv XY + YX$ to denote commutators and anticommutators as usual. Square brackets around quantities imply that derivatives act only within the bracket. Electric and magnetic fields are defined as usual by $\boldsymbol{E} = -[\partial_t \boldsymbol{A}] - [\boldsymbol{\partial} A^0]$ and $\boldsymbol{B} = [\boldsymbol{\partial}\times\boldsymbol{A}]$. By the definition of $\boldsymbol{E}$ and $\boldsymbol{B}$, $[\boldsymbol{\partial}\cdot\boldsymbol{B}] = 0$ and $[\partial_t \boldsymbol{B} + \boldsymbol{\partial}\times\boldsymbol{E}] = 0$.

The most general term in (2.77) is obtained by constructing all possible rotationally invariant, hermitian combinations of $iD_t$, $D^i$, $E^i$, $iB^i$, $i\sigma^i$, with parity requiring an even number of factors of $D^i$ and $E^i$. Terms at $1/M^4$ with two field strength factors $E^i$ or $B^i$ are straightforward to tabulate; note that we have used $[\partial_t \boldsymbol{B}] = -[\boldsymbol{\partial} \times \boldsymbol{E}]$ and the assumption of canonical form to eliminate time derivatives of the magnetic field. Remaining terms at $1/M^4$ involve one factor of electric field $E^i$ and three spatial derivatives $D^i$. Spin-independent terms are straightforward to tabulate; the basis of operators parameterized by $c_{X1}, c_{X2}, c_{X3}$ differs from other possible choices by terms involving commutators $[D^i, D^j]$, i.e., terms with two field strengths. For spin-dependent terms we use $[\boldsymbol{\partial} \times \boldsymbol{E}] = -[\partial_t \boldsymbol{B}]$ and the assumption of canonical form to eliminate occurrences of $[\boldsymbol{\partial} \times \boldsymbol{E}]$. The three-vector identity,

$$D^i(\boldsymbol{E} \times \boldsymbol{\sigma})^j + (\boldsymbol{\sigma} \times \boldsymbol{D})^j E^i + \sigma^i(\boldsymbol{D} \times \boldsymbol{E})^j = \boldsymbol{D} \cdot \boldsymbol{E} \times \boldsymbol{\sigma} \delta^{ij}\,, \tag{2.78}$$

applied to remaining terms of the form $\psi^\dagger D^i(\dots)D^j\psi$, leaves the basis of operators parameterized by $c_{X5}, c_{X6}$.

## 2.6 NRQED Example: Relativistic Invariance

The Lagrangian (2.77) is invariant, by construction, under rotations and spacetime translations. The remaining constraints of relativity are enforced by demanding invariance under boosts. Here we derive these additional constraints, first by a variational calculation in Sect. 2.6.1, and then by an equivalent invariant operator construction in Sect. 2.6.2.

### *2.6.1 Variational Method*

As detailed in Sect. 2.2, under infinitesimal boosts, with infinitesimal boost parameter $\boldsymbol{\eta} = -\boldsymbol{q}/M$, we may choose the heavy fermion to transform as

$$\psi \to e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left\{ 1 + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2} + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}\boldsymbol{D}^2}{4M^4} - \frac{\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{D}}{4M^2}\left[1 + \frac{\boldsymbol{D}^2}{4M^2}\right] + \frac{ic_D g}{8M^3}\boldsymbol{q}\cdot\boldsymbol{E} + \frac{c_S g}{8M^3}\boldsymbol{q}\cdot\boldsymbol{\sigma}\times\boldsymbol{E} + \mathcal{O}(g/M^4, 1/M^6) + \dots \right\} \psi\,, \tag{2.79}$$

while derivatives and gauge fields transform as Lorentz vectors:

$$\boldsymbol{B} \to \boldsymbol{B} - \frac{1}{M}\boldsymbol{q}\times\boldsymbol{E}\,, \quad \boldsymbol{E} \to \boldsymbol{E} + \frac{1}{M}\boldsymbol{q}\times\boldsymbol{B}\,, \quad \boldsymbol{D} \to \boldsymbol{D} + \frac{1}{M}\boldsymbol{q}D_t\,, \quad D_t \to D_t + \frac{1}{M}\boldsymbol{q}\cdot\boldsymbol{D}\,. \tag{2.80}$$

Field strength-dependent terms in (2.79) have been chosen to maintain canonical form. Since we are interested in the canonical Lagrangian through order $1/M^4$, we need not specify the explicit form of the order $1/M^4$ field strength-dependent terms, denoted by $\mathcal{O}(g/M^4)$. A straightforward computation yields

$$\delta\mathcal{L} = \frac{1}{M}\delta\mathcal{L}_1 + \frac{1}{M^2}\delta\mathcal{L}_2 + \frac{1}{M^3}\delta\mathcal{L}_3 + \frac{1}{M^4}\delta\mathcal{L}_4 + \dots\,, \tag{2.81}$$

where

$$\begin{aligned}
\delta\mathcal{L}_1 &= \psi^\dagger\left[(1-c_2)i\boldsymbol{q}\cdot\boldsymbol{D}\right]\psi\,,\\
\delta\mathcal{L}_2 &= \psi^\dagger\left[-\frac{1}{2}(1-c_2)\{\boldsymbol{q}\cdot\boldsymbol{D},D_t\} + \frac{g}{4}(Z-2c_F+c_S)\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{E}\right]\psi\,,\\
\delta\mathcal{L}_3 &= \psi^\dagger\left[\frac{g}{8}\boldsymbol{q}\cdot[\boldsymbol{\partial}\times\boldsymbol{B}]\,(c_F-c_D+2c_M) + \frac{i}{4}\{\boldsymbol{q}\cdot\boldsymbol{D},\boldsymbol{D}^2\}\,(c_2-c_4)\right.\\
&\quad + \frac{ig}{8}\{\boldsymbol{q}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{B}\}\,(c_2Z+2c_F-c_S-2c_{W1}+2c_{W2})\\
&\quad + \frac{ig}{8}\{\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{q}\cdot\boldsymbol{B}\}\,(-c_2Z+c_F-c_{p'p})\\
&\quad \left. + \frac{ig}{8}\boldsymbol{q}\cdot\boldsymbol{\sigma}(\boldsymbol{D}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{D})\,(-c_F+c_S-c_{p'p})\right]\psi.
\end{aligned} \tag{2.82}$$

From $\delta\mathcal{L}_1$, $\delta\mathcal{L}_2$ and $\delta\mathcal{L}_3$, we find [58, 86][11]

$$c_2=1\,,\; c_S=2c_F-Z\,,\; c_4=1\,,\; 2c_M=c_D-c_F\,,\; c_{W2}=c_{W1}-Z\,,\; c_{p'p}=c_F-Z\,. \tag{2.83}$$

Employing the above relations, the variation $\delta\mathcal{L}_4$ takes the form

$$\begin{aligned}
\delta\mathcal{L}_4 &= \psi^\dagger\left[\frac{ig}{8}[\boldsymbol{D}^2,\boldsymbol{q}\cdot\boldsymbol{E}]\left(\frac{5Z}{4}-c_F+c_D-32c_{X1}\right)\right.\\
&\quad + \frac{ig}{8}\{\boldsymbol{q}\cdot\boldsymbol{D},[\boldsymbol{\partial}\cdot\boldsymbol{E}]\}\left(-\frac{Z}{4}+c_F-16c_{X2}\right)\\
&\quad + \frac{g^2}{8}\boldsymbol{q}\cdot\boldsymbol{E}\times\boldsymbol{B}\left(\frac{Z^2}{2}+2c_F(Z-c_F)-2Zc_D+c_{A2}+16c_{X4}\right)\\
&\quad + \frac{g}{8}[\boldsymbol{q}\cdot\boldsymbol{\sigma}\times\boldsymbol{\partial}\boldsymbol{\partial}\cdot\boldsymbol{E}]\left(-Z+c_F-\frac{1}{4}c_D+c_{W1}+8c_{X6}\right)\\
&\quad \left. +\frac{g}{8}D^i\left(q^i(\boldsymbol{E}\times\boldsymbol{\sigma})^j+(\boldsymbol{E}\times\boldsymbol{\sigma})^iq^j+\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{E}\delta^{ij}\right)D^j\left(\frac{Z}{2}-2c_F+16c_{X5}\right)\right]\psi\,,
\end{aligned} \tag{2.84}$$

[11] As noted in [58], we find the opposite sign in the relation for $c_M$ in (2.83) compared to [86].

where we have suppressed terms that are removed by field strength-dependent modifications of the boost generator, denoted by $\mathcal{O}(g/M^4)$ in (2.79). We readily find,

$$\begin{aligned}
32c_{X1} &= \frac{5Z}{4} - c_F + c_D\,, \\
32c_{X2} &= -\frac{Z}{2} + 2c_F\,, \\
32c_{X4} &= -Z^2 - 4c_F(Z - c_F) + 4Zc_D - 2c_{A2}\,, \\
32c_{X5} &= -Z + 4c_F\,, \\
32c_{X6} &= 4(Z - c_F) + c_D - 4c_{W1}\,,
\end{aligned} \tag{2.85}$$

while coefficients $c_{X3}$ and $c_{X7\ldots X12}$ are not constrained by Lorentz invariance. We thus find that seven new quantities are required at order $1/M^4$ to describe the proton's response to arbitrary background electromagnetic fields. The above relations following from relativistic symmetry are non-renormalizable.

### *2.6.2 Invariant Operators*

An alternate method for enforcing Lorentz invariance is to construct the Lagrangian from explicitly invariant operators. We summarize here the main points; the details are presented in Sect. 2.3.

The basic building block in the construction is the field $\Psi_v = \Gamma(v, iD)\psi_v$, where $\psi_v$ is a Dirac spinor field with $\not{v}\psi_v = \psi_v$. The matrix-valued operator $\Gamma(v, iD)$ is determined such that under an infinitesimal boost $\Lambda$, where $\Lambda^\mu{}_\nu v^\nu = v^\mu + q^\mu/M$, the field $\Psi_v$ has a simple transformation law: $\Psi_v \to e^{iq\cdot x}\Psi_v$. Noting that $e^{-iq\cdot x}(iD^\mu + Mv^\mu + q^\mu)e^{iq\cdot x} = iD^\mu + Mv^\mu$, we may thus build invariant bilinears from contractions of polynomials of $\gamma^\mu$ and $\mathcal{V}^\mu \equiv v^\mu + iD^\mu/M$, between $\overline{\Psi}_v$ and $\Psi_v$.

The function $\Gamma(v, iD)$ is a solution to the invariance equation,

$$\Gamma(v + q/M, iD - q)\Lambda^{-1}W(\Lambda, iD + Mv) = \Gamma(v, iD), \tag{2.86}$$

where $W(\Lambda, p)$ is an element of the little group for timelike invariant vector $v^\mu$, following from the theory of induced representations of the Lorentz group. Up to the relevant order for determining the $1/M^4$ Lagrangian we have [57]

$$\begin{aligned}
\Gamma = 1 + \frac{i\not{D}_\perp}{2M} + \frac{1}{M^2}\left\{-\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\not{D}_\perp iv\cdot D\right\} + \frac{1}{M^3}\left\{\frac{1}{4}(iD_\perp)^2 iv\cdot D\right. \\
\left. + \frac{i\not{D}_\perp}{2}\left[-\frac{3}{8}(iD_\perp)^2 + (iv\cdot D)^2\right] + \frac{gZ}{8}F_{\mu\nu}v^\mu D_\perp^\nu + \frac{gZ}{16}\sigma_\perp^{\mu\nu}F_{\mu\nu}i\not{D}_\perp\right\} + \ldots,
\end{aligned} \tag{2.87}$$

where we have defined $D_\perp^\mu \equiv D^\mu - v^\mu v\cdot D$, and for Abelian gauge fields $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the last two terms of (2.87) are absent in the ansatz for reparameterization invariance given in [84], leading to incorrect Lorentz invariance constraints at $1/M^4$ and beyond. This subtlety is explained in Sects. 2.1–2.3 above.

A complete basis of invariant bilinears required through order $1/M^4$ is

$$
\begin{aligned}
\mathcal{L} = \overline{\Psi}_v \Big\{ & M(\not{\mathcal{V}} - 1) - a_F g \frac{\sigma^{\mu\nu}F_{\mu\nu}}{4M} + i a_D g \frac{\{\mathcal{V}_\mu, [M\mathcal{V}_\nu, F^{\mu\nu}]\}}{16M^2} \\
& - a_{W1} g \frac{[M\mathcal{V}^\alpha, [M\mathcal{V}_\alpha, \sigma^{\mu\nu}F_{\mu\nu}]]}{16M^3} \\
& + a_{A1} g^2 \frac{F_{\mu\nu}F^{\mu\nu}}{16M^3} + a_{A2} g^2 \frac{\mathcal{V}_\alpha F^{\mu\alpha}F_{\mu\beta}\mathcal{V}^\beta}{16M^3} \Big\} \Psi_v + a_{X3}\mathcal{B}_{X3} + \sum_{i=7}^{12} a_{Xi}\mathcal{B}_{Xi}.
\end{aligned}
\tag{2.88}
$$

The bilinears $\mathcal{B}_{Xi}$ for $i = 3, 7 \ldots 12$ are chosen to reduce to the respective operators multiplying $c_{Xi}$ in (2.77) upon setting $v^\mu = (1, 0, 0, 0)$ and neglecting $1/M$ suppressed corrections. Since we are concerned only with the Lagrangian through order $1/M^4$ we do not specify an explicit choice for these $\mathcal{B}_{Xi}$. A computation shows that the field redefinition to recover canonical form is

$$
\begin{aligned}
\psi_v = \Big\{ & 1 + \frac{1}{4M^2}(iD_\perp)^2 \left(1 - \frac{iv\cdot D}{M}\right) - \frac{gZ}{16M^2}\sigma_\perp^{\mu\nu}F_{\mu\nu} \\
& - \frac{gZ}{4M^3} D_\perp^\mu v^\alpha F_{\alpha\mu} + \frac{igZ}{4M^3}\sigma_{\mu\nu} D_\perp^\mu v_\alpha F^{\alpha\nu} \\
& - \frac{gZ}{8M^3} v^\alpha F_{\alpha\mu} D_\perp^\mu + \frac{g a_F}{4M^3}\left[-D_\perp^\mu v^\alpha F_{\alpha\mu} + i\sigma_{\mu\nu}D_\perp^\mu v_\alpha F^{\alpha\nu}\right] - \frac{g a_D}{8M^3} v^\alpha F_{\alpha\mu} D_\perp^\mu \\
& + \frac{i g a_{W1}}{8M^3}\sigma_{\mu\nu}[D_\perp^\mu, v_\alpha F^{\alpha\nu}] \Big\} \psi_v' \,.
\end{aligned}
\tag{2.89}
$$

Upon setting $v^\mu = (1, 0, 0, 0)$, the resulting Lagrangian, expressed in terms of $\psi_v'$, is identical to the previous result (2.77) with constraints (2.83) and (2.85).

## 2.7 NRQED Example: One-Photon Matching

This section relates the matching conditions in the one-fermion sector to standard form factors of the nucleon. The coefficient relations of the previous section, derived from relativistic invariance, are verified explicitly. We focus here on operators contributing to the one-photon matrix element. Coefficient relations for operators contributing to the two-photon matrix element may be similarly verified.

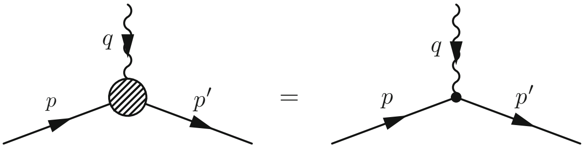


**Fig. 2.1** Tree level matching of the one-photon amplitude in the full theory and NRQED. The *black dot* in the diagram on the right-hand side represents single-photon NRQED vertices

Consider first the operators contributing to the one-photon matrix element of the nucleon. The matching is performed in terms of standard invariant form factors,

$$\langle N(p')|J^{\rm em}_\mu|N(p)\rangle = \bar{u}(p')\Gamma_\mu(q)u(p)\,, \quad \Gamma_\mu(q) \equiv \gamma_\mu F_1^N(q^2) + \frac{i\sigma_{\mu\nu}}{2M_N}F_2^N(q^2)q^\nu\,, \tag{2.90}$$

where $q = p' - p$ and $N$ denotes a proton or neutron; we suppress the superscript $N$ in the following. Equating the effective theory with the full theory,[12] we find (cf. Fig. 2.1)

$$\begin{aligned}
c_F &= \bar{F}_1 + \bar{F}_2 \equiv Z + a_N + \mathcal{O}(\alpha)\,, \\
c_D &= \bar{F}_1 + 2\bar{F}_2 + 8\bar{F}_1' \equiv Z + \frac{4}{3}M^2(r_E^N)^2 + \mathcal{O}(\alpha)\,, \\
c_{W1} &= \bar{F}_1 + \frac{1}{2}\bar{F}_2 + 4\bar{F}_1' + 4\bar{F}_2'\,, \\
c_{X3} &= \frac{1}{8}\bar{F}_1' + \frac{1}{4}\bar{F}_2' + \frac{1}{2}\bar{F}_1''\,,
\end{aligned} \tag{2.91}$$

where $Z$ denotes the electric charge, $a_N$ is the anomalous magnetic moment of the nucleon, and $r_E^N$ is the nucleon charge radius. We have introduced dimensionless barred quantities to denote derivatives with respect to $q^2/M^2$ at $q^2 = 0$: $\bar{F}_1 \equiv F_1(0) = Z$, $\bar{F}_2 \equiv F_2(0) = a_N$, $\bar{F}_i' \equiv M^2F_i'(0)$, etc. The new quantity $F_1''$ appears at $1/M^4$. Expressions for other Wilson coefficients up to $1/M^3$ in terms of form factors can be found using (2.83). At $1/M^4$, we also find

$$\begin{aligned}
c_{X1} &= \frac{5}{128}\bar{F}_1 + \frac{1}{32}\bar{F}_2 + \frac{1}{4}\bar{F}_1'\,, \\
c_{X2} &= \frac{3}{64}\bar{F}_1 + \frac{1}{16}\bar{F}_2\,, \\
c_{X5} &= \frac{3}{32}\bar{F}_1 + \frac{1}{8}\bar{F}_2\,, \\
c_{X6} &= -\frac{3}{32}\bar{F}_1 - \frac{1}{8}\bar{F}_2 - \frac{1}{4}\bar{F}_1' - \frac{1}{2}\bar{F}_2'\,,
\end{aligned} \tag{2.92}$$

[12] The nonrelativistic normalization of states in NRQED is obtained using $\bar{u}(p)u(p) = M/E_p$ in (2.90).

and it is readily verified that these expressions satisfy the constraints (2.85). In the presence of radiative corrections, the form factors on the right hand sides of (2.91) and (2.92) should be interpreted in an appropriate infrared regularization scheme; alternatively, the matching may be performed with infrared finite observables. The corresponding infrared subtractions and ultraviolet renormalizations must be performed to obtain the Wilson coefficients including radiative corrections.[13]

## 2.8 NRQED Example: Photon and Four-Fermion Sectors

So far our analysis has focused on the one-fermion sector. We have derived the form of the Lagrangian appropriate, e.g., to a proton in a background electromagnetic field. Let us consider the complete QED theory including dynamical photon, as well as a lepton (electron or muon) field. The case of a nonrelativistic lepton is appropriate to bound state hydrogen studies, or very low-energy lepton-nucleon (e.g. muon-proton) scattering, where $E \ll m_\ell, M$. We first consider this case, constructing the operator basis, deriving coefficient relations and identifying redundant operators. We then turn to a brief discussion of the case of a relativistic lepton, appropriate to e.g. low-energy electron-proton scattering with $m_\ell, E \ll M$.

### *2.8.1 Pure Photon Operators*

The pure gauge sector for NRQED is the well known Euler-Heisenberg Lagrangian. Enforcing parity and time reversal symmetry and neglecting total derivatives we find

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_{V2}\frac{F_{\mu\nu}[\partial^2 F^{\mu\nu}]}{M^2} + c_{V4}\frac{F_{\mu\nu}[\partial^4 F^{\mu\nu}]}{M^4} + c_{E1}g^2\frac{(F_{\mu\nu}F^{\mu\nu})^2}{M^4} + c_{E2}g^2\frac{F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu}{M^4} + \dots . \tag{2.93}$$

The coefficients $c_{V2}$ and $c_{V4}$ may be set to zero through field redefinitions on $A^\mu$, as discussed in Sect. 2.8.3 below.

[13] The expressions on the right hand side of (2.91) and (2.92) correspond to those referred to as $c_i^{\rm QED}$ in [77]. The renormalization procedure in dimensional regularization is described in [86].

### 2.8.2 Four-Fermion Operators

Consider four-fermion operators relevant for processes in the one-nucleon, one-lepton sector. We enforce hermiticity and invariance under parity, time-reversal and rotational symmetries. We use the notation $\overleftarrow{D}$ for a covariant derivative acting to the left, $X\overleftarrow{D}^i \equiv [\partial^i X] + igZXA^i$, and define $D_+ \equiv D + \overleftarrow{D}$, $D_- \equiv D - \overleftarrow{D}$. Having performed field redefinitions to eliminate operators with time derivatives acting on heavy fermions, the Lagrangian in this sector, through $1/M^4$, is

$$
\begin{aligned}
\mathcal{L}_{\psi\chi} = &\frac{d_1}{M^2}\psi^\dagger\sigma^i\psi\,\chi^\dagger\sigma^i\chi + \frac{d_2}{M^2}\psi^\dagger\psi\,\chi^\dagger\chi + \frac{d_3}{M^4}\psi^\dagger D_+^i\psi\,\chi^\dagger D_+^i\chi \\
&+ \frac{d_4}{M^4}\psi^\dagger D_-^i\psi\,\chi^\dagger D_-^i\chi \\
&+ \frac{d_5}{M^4}\psi^\dagger(\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2)\psi\,\chi^\dagger\chi + \frac{d_6}{M^4}\psi^\dagger\psi\,\chi^\dagger(\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2)\chi \\
&+ \frac{gd_7}{M^4}\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi\,\chi^\dagger\chi + \frac{id_8}{M^4}\epsilon^{ijk}\psi^\dagger\sigma^i D_-^j\psi\,\chi^\dagger D_+^k\chi \\
&+ \frac{id_9}{M^4}\epsilon^{ijk}\psi^\dagger\sigma^i D_+^j\psi\,\chi^\dagger D_-^k\chi \\
&+ \frac{gd_{10}}{M^4}\psi^\dagger\psi\,\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{B}\chi + \frac{id_{11}}{M^4}\epsilon^{ijk}\psi^\dagger D_+^k\psi\,\chi^\dagger\sigma^i D_-^j\chi \\
&+ \frac{id_{12}}{M^4}\epsilon^{ijk}\psi^\dagger D_-^k\psi\,\chi^\dagger\sigma^i D_+^j\chi \\
&+ \frac{d_{13}}{M^4}\psi^\dagger\sigma^i D_+^j\psi\,\chi^\dagger\sigma^i D_+^j\chi + \frac{d_{14}}{M^4}\psi^\dagger\sigma^i D_-^j\psi\,\chi^\dagger\sigma^i D_-^j\chi \\
&+ \frac{d_{15}}{M^4}\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{D}_+\psi\,\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{D}_+\chi \\
&+ \frac{d_{16}}{M^4}\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{D}_-\psi\,\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{D}_-\chi + \frac{d_{17}}{M^4}\psi^\dagger\sigma^i D_-^j\psi\,\chi^\dagger\sigma^j D_-^i\chi \\
&+ \frac{d_{18}}{M^4}\psi^\dagger\sigma^i(\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2)\psi\,\chi^\dagger\sigma^i\chi + \frac{d_{19}}{M^4}\psi^\dagger\sigma^i(D^iD^j + \overleftarrow{D}^j\overleftarrow{D}^i)\psi\,\chi^\dagger\sigma^j\chi \\
&+ \frac{d_{20}}{M^4}\psi^\dagger\sigma^i\psi\,\chi^\dagger\sigma^i(\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2)\chi + \frac{d_{21}}{M^4}\psi^\dagger\sigma^i\psi\,\chi^\dagger\sigma^j(D^iD^j + \overleftarrow{D}^j\overleftarrow{D}^i)\chi\,.
\end{aligned}
\tag{2.94}
$$

Here $\chi$ is the nonrelativistic lepton field with mass $M_\chi$ and for notational simplicity we write all operators in terms of the common mass scale $M$.[14] Covariant derivatives appearing within a fermion bilinear in (2.94) are understood to act only on fields

[14]Note that the coefficients $d_{1,2}$ in (2.94) are related to those of Caswell and Lepage [21] by a factor $M_\chi/M$.

in that bilinear. The heavy field $\psi$ transforms under boosts as in (2.79). Recalling that $\boldsymbol{q}$ in (2.79) is related to the mass-independent infinitesimal boost parameter by $\boldsymbol{\eta} = -\boldsymbol{q}/M$, the transformation law for $\chi$ is obtained by the replacement $M \to rM$ and $q \to rq$, where we define $r \equiv M_\chi/M$. We thus find

$$
\begin{aligned}
\delta\mathcal{L}_{\psi\chi} =& \frac{1}{M^4}\Big\{ \psi^\dagger i\boldsymbol{q}\cdot\boldsymbol{D}_-\psi\, \chi^\dagger\chi \left[\frac{d_2}{2} - 2rd_4 - 2d_5\right] \\
&+ \psi^\dagger\psi\, \chi^\dagger i\boldsymbol{q}\cdot\boldsymbol{D}_-\chi \left[\frac{d_2}{2r} - 2d_4 - 2rd_6\right] \\
&+ \psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{D}_+\psi\, \chi^\dagger\chi \left[-\frac{d_2}{4} + \frac{d_1}{4r} - 2d_8 - 2rd_9\right] \\
&+ \psi^\dagger i\boldsymbol{q}\cdot\boldsymbol{D}_-\sigma^i\psi\, \chi^\dagger\sigma^i\chi \left[\frac{d_1}{2} - 2rd_{14} - 2d_{18}\right] \\
&+ \psi^\dagger\psi\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{D}_+\chi \left[-\frac{d_2}{4r} + \frac{d_1}{4} - 2rd_{11} - 2d_{12}\right] \\
&+ \psi^\dagger\sigma^i\psi\chi^\dagger i\boldsymbol{q}\cdot\boldsymbol{D}_-\sigma^i\chi \left[\frac{d_1}{2r} - 2d_{14} - 2rd_{20}\right] \\
&+ \psi^\dagger i\boldsymbol{\sigma}\cdot\boldsymbol{D}_-\psi\, \chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{q}\chi \left[\frac{d_1}{4} - 2rd_{16} - d_{19}\right] \\
&+ \psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{q}\psi\, \chi^\dagger i\boldsymbol{\sigma}\cdot\boldsymbol{D}_-\chi \left[\frac{d_1}{4r} - 2d_{16} - rd_{21}\right] \\
&+ \psi^\dagger i\boldsymbol{\sigma}\cdot\boldsymbol{q}D^i_-\psi\, \chi^\dagger\sigma^i\chi \left[-\frac{d_1}{4} - 2rd_{17} - d_{19}\right] \\
&+ \psi^\dagger\sigma^i\psi\chi^\dagger i\boldsymbol{\sigma}\cdot\boldsymbol{q}D^i_-\chi \left[-\frac{d_1}{4r} - 2d_{17} - rd_{21}\right]\Big\} \\
&+ \mathcal{O}(1/M^5)\,.
\end{aligned}
\tag{2.95}
$$

This enforces the relations

$$
\begin{aligned}
rd_4 + d_5 = \frac{d_2}{4}\,, \quad d_5 = r^2 d_6\,, \quad 8r(d_8 + rd_9) = -rd_2 + d_1\,, \\
8r(rd_{11} + d_{12}) = -d_2 + rd_1\,, \\
rd_{14} + d_{18} = \frac{d_1}{4}\,, \quad d_{18} = r^2 d_{20}\,, \quad 2rd_{16} + d_{19} = \frac{d_1}{4}\,, \\
r(d_{16} + d_{17}) + d_{19} = 0\,, \quad d_{19} = r^2 d_{21},
\end{aligned}
\tag{2.96}
$$

implying a total of 12 independent four-fermion operators through $1/M^4$, including two at order $1/M^2$. By performing field redefinitions on the gauge field $A^\mu$, some of these four-fermion operators are found to mix with one-heavy particle sector operators, as discussed in Sect. 2.8.3 below.

Equation (2.94), with constraints (2.96), applies to the case of distinct heavy particles represented by $\psi$, $\chi$, with arbitrary mass ratio $M_\chi/M$. For certain applications, e.g. positronium or heavy quarkonium bound states, the fields $\psi$ and $\chi$ can be taken to represent particle-antiparticle pairs with $r = M_\chi/M = 1$. Charge conjugation symmetry is then implemented by enforcing invariance under $\psi \leftrightarrow \chi$, thus reducing the basis of operators. This case has been investigated for QCD through $\mathcal{O}(1/M^4)$ by Brambilla et al. [17]. We find that our basis of four-fermion operators (2.94) and constraints (2.96) are equivalent to those found in [17] for this special case.[15]

### *2.8.3 Field Redefinitions and Redundant Operators*

With a dynamical photon field, we may perform field redefinitions that maintain reality and gauge, parity, time reversal and rotational symmetries. In order to avoid upsetting the previously determined coefficient relations, we must also maintain the transformation law for $A^\mu$ as a four-vector under Lorentz transformations, i.e.,

$$A^0 \to A^0 - \frac{1}{M}\boldsymbol{q}\cdot\boldsymbol{A}\,, \quad \boldsymbol{A} \to \boldsymbol{A} - \frac{1}{M}\boldsymbol{q}A^0\,. \tag{2.97}$$

Let us write

$$A_\mu = A'_\mu + \Delta_\gamma A_\mu + \Delta_\psi A_\mu + \Delta_\chi A_\mu + \dots\,. \tag{2.98}$$

For the pure gauge field terms the most general expression is

$$\Delta_\gamma A^\mu = a_{\gamma 1}\frac{\partial_\nu F^{\nu\mu}}{M^2} + a_{\gamma 2}\frac{\partial^2\partial_\nu F^{\nu\mu}}{M^4} + \mathcal{O}(1/M^6)\,. \tag{2.99}$$

Terms involving the heavy fermion $\psi$ take the form

$$\begin{aligned}\frac{\Delta_\psi A^\mu}{g} &= \tilde{a}_{\psi 1}\frac{\overline{\Psi}_v\gamma^\mu\Psi_v}{M^2} + \tilde{a}_{\psi 2}\frac{\partial_\alpha(\overline{\Psi}_v\sigma^{\alpha\mu}\Psi_v)}{M^3}\\ &\quad + \tilde{a}_{\psi 3}g\frac{\overline{\Psi}_v\{\gamma^\mu,\,\sigma^{\alpha\beta}F_{\alpha\beta}\}\Psi_v}{M^4} + \tilde{a}_{\psi 4}\frac{\partial^2(\overline{\Psi}_v\gamma^\mu\Psi_v)}{M^4}\\ &\quad + \tilde{a}_{\psi 5}g\frac{\overline{\Psi}_v\sigma^{\mu\alpha}\{\mathcal{V}^\beta, F_{\alpha\beta}\}\Psi_v}{M^4} + \mathcal{O}(1/M^5)\,,\end{aligned} \tag{2.100}$$

[15]The difference between Abelian and nonAbelian gauge fields is trivial for four-fermion operators through this order.

where we have employed the invariant operator formalism of Sect. 2.6.2. In particular, $\Psi_v = \Gamma\psi_v$ with $\Gamma$ from (2.87) and $\psi_v$ from (2.89), expressed in terms of the field $\psi'_v \equiv \psi$ with canonical Lagrangian (2.77). As an alternative to the invariant operator formalism employed in (2.100) we may expand $\Delta_\psi A^0$ and $\Delta_\psi \boldsymbol{A}$ in a series of rotationally invariant operators with arbitrary coefficients, and subsequently constrain these coefficients using (2.97). The result is equivalent to (2.100), with five free parameters through $\mathcal{O}(1/M^4)$,

$$\begin{aligned}\frac{\Delta_\psi A^0}{g} &= a_{\psi 1}\frac{\psi^\dagger\psi}{M^2} + a_{\psi 2}\frac{\boldsymbol{\partial}^2(\psi^\dagger\psi)}{M^4} - i\Big(\frac{a_{\psi 1}}{4} - a_{\psi 4}\Big)\frac{\psi^\dagger\boldsymbol{\sigma}\cdot\overleftarrow{\boldsymbol{D}}\times\boldsymbol{D}\psi}{M^4} \\ &\quad + a_{\psi 3}g\frac{\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi}{M^4} + \mathcal{O}(1/M^5), \\ \frac{\Delta_\psi\boldsymbol{A}}{g} &= -a_{\psi 1}\frac{\psi^\dagger i\boldsymbol{D}_-\psi}{2M^3} + a_{\psi 4}\frac{\boldsymbol{\partial}\times(\psi^\dagger\boldsymbol{\sigma}\psi)}{M^3} + a_{\psi 5}g\frac{\psi^\dagger\boldsymbol{\sigma}\times\boldsymbol{E}\psi}{M^4} + \mathcal{O}(1/M^5)\,. \end{aligned} \tag{2.101}$$

The expansion of $\Delta_\chi A^\mu$ is obtained from (2.101) with the replacements $\psi \to \chi$, $M \to M_\chi$, $Z \to Z_\chi$ and $a_{\psi\, i} \to a_{\chi\, i}$. In terms of the field $A'_\mu$ in (2.98), we find in the pure photon sector,

$$\delta c_{V2} = -\frac{1}{2}a_{\gamma 1}\,, \quad \delta c_{V4} = -\frac{1}{2}a_{\gamma 2} - \frac{1}{4}a_{\gamma 1}^2 + 2a_{\gamma 1}c_{V2}\,, \tag{2.102}$$

while for the $\psi$ sector,

$$\begin{aligned} \delta c_D &= -8Za_{\gamma 1} + 8a_{\psi 1}\,, \quad \delta c_{W1} = -4c_F a_{\gamma 1} + 8a_{\psi 4}\,, \\ \delta c_{A2} &= -16Z^2 a_{\gamma 1} + 16Za_{\psi 1}\,, \\ \delta c_{X3} &= -\frac{c_D a_{\gamma 1}}{8} + Za_{\gamma 2} - a_{\gamma 1}a_{\psi 1} + 4c_{V2}a_{\psi 1} + a_{\psi 2}\,, \quad \delta c_{X7} = -\frac{c_S Z a_{\gamma 1}}{4} + a_{\psi 3}\,, \\ \delta c_{X8} &= c_F Z a_{\gamma 1} - \frac{c_F a_{\psi 1}}{2} - Za_{\psi 4}\,, \quad \delta c_{X9} = -\frac{c_F^2 a_{\gamma 1}}{2} + c_F a_{\psi 4}\,, \\ \delta c_{X11} &= \frac{c_F^2 a_{\gamma 1}}{2} - c_F a_{\psi 4}\,, \delta c_{X12} = \frac{c_S Z a_{\gamma 1}}{2} + a_{\psi 5}\,. \end{aligned} \tag{2.103}$$

Similar relations hold for the Wilson coefficients $c_i^{(\chi)}$ in the $\chi$ Lagrangian, defined as in (2.77), with $\psi \to \chi$, $Z \to Z_\chi$, $M \to M_\chi$, $c_i \to c_i^{(\chi)}$. Finally, for the four-fermion operator coefficients,

$$\begin{aligned} \frac{\delta d_2}{g^2} &= -Z_\chi a_{\psi 1} - \frac{Za_{\chi 1}}{r^2}\,, \quad \frac{\delta d_3}{g^2} = \frac{c_D^{(\chi)}a_{\psi 1}}{8r^2} + \frac{c_D a_{\chi 1}}{8r^2} + Z_\chi a_{\psi 2} + \frac{Za_{\chi 2}}{r^4}\,, \\ \frac{\delta d_4}{g^2} &= -\frac{Z_\chi a_{\psi 1}}{4r} - \frac{Za_{\chi 1}}{4r^3}\,, \quad \frac{\delta d_7}{g^2} = -\frac{ZZ_\chi}{4}(a_{\psi 1} - 4a_{\psi 4}) - Z_\chi a_{\psi 3}\,, \end{aligned}$$

$$\frac{\delta d_8}{g^2} = \frac{Z_\chi}{8}\left(a_{\psi 1} - 4a_{\psi 4}\right) - \frac{c_S a_{\chi 1}}{8r^2}\,, \quad \frac{\delta d_{10}}{g^2} = -\frac{ZZ_\chi}{4r^4}\left(a_{\chi 1} - 4a_{\chi 4}\right) - \frac{Za_{\chi 3}}{r^4}\,,$$

$$\frac{\delta d_{11}}{g^2} = -\frac{c_S^{(\chi)} a_{\psi 1}}{8r^2} + \frac{Z}{8r^4}\left(a_{\chi 1} - 4a_{\chi 4}\right)\,, \quad \frac{\delta d_{13}}{g^2} = -\frac{\delta d_{15}}{g^2} = \frac{c_F^{(\chi)} a_{\psi 4}}{2r} + \frac{c_F a_{\chi 4}}{2r^3}\,. \tag{2.104}$$

The coefficient relations (2.83), (2.85) and (2.96) are preserved, since by construction the Lorentz transformation properties of $A^\mu$ are unchanged and hence the boost transformation rules (2.79) and (2.80) still apply.

We may use (2.102) to eliminate vacuum polarization terms $c_{V2}$ and $c_{V4}$ in favor of compensating terms in (2.103). Similarly, (2.103), together with the analogous relations for $c_i^{(\chi)}$, and (2.104), can be used to eliminate ten linear combinations of Wilson coefficients for two-fermion and four-fermion operators. Different applications may favor elimination of different operators.[16]

### 2.8.4 Relativistic Lepton

For applications such as lepton-nucleon scattering at energies $m_\ell, E \ll M$ (e.g., low-energy electron-proton scattering), the relevant effective theory involves a heavy fermion (e.g., the proton) interacting with an electromagnetically charged relativistic fermion (e.g., the electron). Let us briefly discuss this case. Enforcing parity, time-reversal, gauge, Lorentz as well as chiral symmetry at $m_\ell = 0$, we find the leptonic interactions with the photon,

$$\mathcal{L}_\ell = \bar{\ell}\left[i\not{D} - m_\ell + gc_F^{(\ell)} m_\ell \frac{\sigma^{\mu\nu}F_{\mu\nu}}{M^2} + gc_2^{(\ell)} m_\ell \frac{D^2}{M^2} + gc_D^{(\ell)} \frac{[\partial^\mu F_{\mu\nu}]\gamma^\nu}{M^2} + \mathcal{O}(1/M^4)\right]\ell, \tag{2.105}$$

where we assume field redefinitions have been performed to remove power suppressed terms involving $(i\not{D} - m_\ell)\ell$.

Having performed field redefinitions to eliminate operators with time derivatives acting on fermion fields, the Lagrangian for the nucleon-relativistic lepton sector through $\mathcal{O}(1/M^3)$ is

$$\begin{aligned}\mathcal{L}_{\psi\ell} &= \frac{b_1}{M^2}\psi^\dagger\psi\; \bar{\ell}\gamma^0\ell + \frac{b_2}{M^2}\psi^\dagger\sigma^i\psi\; \bar{\ell}\gamma^i\gamma_5\ell + \frac{b_3}{M^3}\psi^\dagger\psi\; m_\ell\bar{\ell}\ell + \frac{b_4}{M^3}\psi^\dagger iD_-^i\psi\; \bar{\ell}\gamma^i\ell \\ &+ \frac{b_5}{M^3}\psi^\dagger\psi\bar{\ell}i\gamma\cdot\boldsymbol{D}_-\ell + \frac{b_6}{M^3}\epsilon^{ijk}\psi^\dagger\sigma^i\psi\; m_\ell\bar{\ell}\sigma^{jk}\ell + \frac{b_7}{M^3}\epsilon^{ijk}\psi^\dagger\sigma^i\psi\; \bar{\ell}\gamma^j D_+^k\ell\end{aligned}$$

[16] We have not specified gauge fixing and source terms, which are also affected by field redefinitions.

$$+ \frac{b_8}{M^3}\psi^\dagger\sigma^i\psi\,\bar{\ell}\gamma^0\gamma_5 iD^i\_\ell + \frac{b_9}{M^3}\psi^\dagger\sigma^i iD^i\_\psi\,\bar{\ell}\gamma^0\gamma_5\ell + \mathcal{O}(1/M^4), \tag{2.106}$$

where $\ell$ is the relativistic lepton field with mass $m_\ell$ and $\sigma^{ij} \equiv \frac{i}{2}[\gamma^i, \gamma^j]$. The heavy field $\psi$ transforms under boosts as in (2.79), while $\ell$ transforms under finite dimensional representations of the Lorentz group in the usual way. Under Lorentz transformation, we thus find

$$\delta\mathcal{L}_{\psi\ell} = -\frac{1}{M^3}\psi^\dagger\psi\bar{\ell}q^i\gamma^i\ell\,(b_1 + 2b_4) - \frac{1}{M^3}\psi^\dagger\sigma^i q^i\psi\bar{\ell}\gamma^0\gamma_5\ell\,(b_2 + 2b_9) + \mathcal{O}(1/M^4). \tag{2.107}$$

This enforces the relations

$$b_4 = \frac{1}{2}b_1\,, \quad b_9 = -\frac{1}{2}b_2\,, \tag{2.108}$$

leaving seven operators in this sector through order $1/M^3$, including two at order $1/M^2$.

By performing field redefinitions on the gauge field $A^\mu$, some of these four-fermion operators are found to mix with one-heavy particle operators. In addition to the contributions $\Delta_\gamma A^\mu$ and $\Delta_\psi A^\mu$ from (2.98) we may employ

$$\Delta_\ell A^\mu = g a_{\ell 1}\frac{\bar{\ell}\gamma^\mu\ell}{M^2} + \mathcal{O}(1/M^4)\,. \tag{2.109}$$

We thus find the modified couplings in $\mathcal{L}_\ell$,

$$\delta c_D^{(\ell)} = -Z_\ell a_{\gamma 1} + a_{\ell 1}\,, \tag{2.110}$$

and for the four fermion operators in $\mathcal{L}_{\psi\ell}$,

$$\frac{\delta b_1}{g^2} = -Z a_{\ell 1} - Z_\ell a_{\psi 1}\,, \quad \frac{\delta b_7}{g^2} = -Z_\ell a_{\psi 4} - \frac{1}{2}c_F a_{\ell 1}\,, \tag{2.111}$$

with relation (2.108) remaining intact.

## 2.9 Discussion

The usual procedure of implementing Lorentz invariance via finite dimensional representations of the Lorentz group is insufficient for application to heavy particle effective theories. We have adapted the formalism of induced representations for application to heavy particle field transformation laws. Returning to the questions posed at the beginning of the chapter, we see that the parameter $v$ enters as an arbitrary reference vector in the effective theory construction. Rules identifiable with "reparameterization invariance" (2.48) are obtained by a rewriting of the transformation law for generalized boosts, and the class of reparameterization transformations consistent with Lorentz and gauge invariance is identified through a systematic solution of the invariance equation (2.52).

Let us compare our formalism to previous work. A naive ansatz for implementing Lorentz invariance via reparameterization invariance breaks down for $\Gamma(v, iD)$ starting at order $1/M^3$, corresponding to new effects at order $1/M^4$ in the canonical Lagrangian. The transformation law defined by $W(\Lambda, iD)$ is corrected at order $1/M^4$. These subtleties were not treated in the classic work of Luke and Manohar [84, 86], and the ansatz proposed there would lead to inconsistencies at the orders in $1/M$ specified above. Brambilla et al. [16] recognized that Wilson-coefficient dependent corrections to $W(\Lambda)$ must be included when deriving an invariant Lagrangian in canonical form. However, there the constraints of Lorentz invariance are derived (through order $1/M^2$) at the level of canonically quantized charges, a procedure that becomes increasingly cumbersome at high orders in the $1/M$ expansion. In Sect. 2.2 we have used general properties of commutators of the $S$ matrix with conserved charges to derive constraints at the Lagrangian level that implement Lorentz invariance for heavy particle effective theories in canonical form. In Sect. 2.3 we have derived consistent reparameterization transformations that allow solution to the invariance equation (2.52), and hence the construction of manifestly invariant Lagrangians to arbitrary order.

At a practical level, the main results for building heavy fermion Lagrangians are contained in (2.34), or for the invariant operator method, in (2.59) and (2.60). We have illustrated the utility of these results by constructing the NRQED Lagrangian to order $1/M^4$. This provides the rigorous framework for a range of applications such as computing radiative corrections to low-energy lepton-nucleon scattering, and understanding a sharp discrepancy in proton charge radius measurements through scrutinizing proton structure effects in atomic bound states. In the next chapter, we will construct heavy particle Lagrangians for WIMPs interacting with Standard Model particles using the formalism developed here.

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