# Chapter 2
# MPC with No Model Uncertainty

## 2.1 Problem Description

This section provides a review of some of the key concepts and techniques in classical MPC. Here the term "classical MPC" refers to a class of control problems involving linear time invariant (LTI) systems whose dynamics are described by a discrete time model that is not subject to any uncertainty, either in the form of unknown additive disturbances or imprecise knowledge of the system parameters. In the first instance the assumption will be made that the system dynamics can be described in terms of the LTI state-space model

$$x_{k+1} = Ax_k + Bu_k \tag{2.1a}$$

$$y_k = Cx_k \tag{2.1b}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$ are, respectively, the system state, the control input and the system output, and $k$ is the discrete time index. If the system to be controlled is described by a model with continuous time dynamics (such as an ordinary differential equation), then the implicit assumption is made here that the controller can be implemented as a sampled data system and that (2.1a) defines the discrete time dynamics relating the samples of the system state to those of its control inputs.

**Assumption 2.1** Unless otherwise stated, the state $x_k$ of the system (2.1a) is assumed to be measured and made available to the controller at each sampling instant $k = 0, 1, \ldots$

The controlled system is also assumed to be subject to linear constraints. In general these may involve both states and inputs and are expressed as a set of linear inequalities

$$Fx + Gu \leq \mathbf{1} \tag{2.2}$$



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where $F \in \mathbb{R}^{n_C \times n_x}$, $G \in \mathbb{R}^{n_C \times n_u}$ and the inequality applies elementwise. We denote by $\mathbf{1}$ a vector with elements equal to unity, the dimension of which is context dependent, i.e. $\mathbf{1} = [1 \cdots 1]^T \in \mathbb{R}^{n_C}$ in (2.2). Setting $F$ or $G$ to zero results in constraints on inputs or states alone. A feasible pair $(x_k, u_k)$ or feasible sequence $\{(x_0, u_0), (x_1, u_1), \ldots\}$ for (2.2) is any pair or sequence satisfying (2.2). The constraints in (2.2) are symmetric if $(-x_k, -u_k)$ is feasible whenever $(x_k, u_k)$ is feasible, and non-symmetric otherwise. Although the form of (2.2) does not encompass constraints involving states or inputs at more than one sampling instant (such as, for example rate constraints or more general dynamic constraints), these can be handled through a suitable and obvious extension of the results to be presented.

The classical regulation problem is concerned with the design of a controller that drives the system state to some desired reference point using an acceptable amount of control effort. For the case that the state is to be steered to the origin, the controller performance is quantified conveniently for this type of problem by a quadratic cost index of the form

$$J\,(x_0, \{u_0, u_1, u_2 \ldots\}) \doteq \sum_{k=0}^{\infty} \left( \|x_k\|_Q^2 + \|u_k\|_R^2 \right). \tag{2.3}$$

Here $\|v\|_S^2$ denotes the quadratic form $v^T S v$ for any $v \in \mathbb{R}^{n_v}$ and $S = S^T \in \mathbb{R}^{n_v \times n_v}$, and $Q$, $R$ are weighting matrices that specify the emphasis placed on particular states and inputs in the cost. We assume that $R$ is a symmetric positive-definite matrix (i.e. the eigenvalues of $R$ are real and strictly positive, denoted $R \succ 0$) and that $Q$ is symmetric and positive semidefinite (all eigenvalues of $Q$ are real and non-negative, denoted $Q \succeq 0$). This allows, for example, the choice $Q = C^T Q_y C$ for some positive-definite matrix $Q_y$, which corresponds to the case that the output vector, $y$, rather than the state, $x$, is to be steered to the origin. At time $k$, the optimal value of the cost (2.3) with respect to minimization over admissible control sequences $\{u_k, u_{k+1}, u_{k+2}, \ldots\}$ is denoted

$$J^*(x_k) \doteq \min_{u_k, u_{k+1}, u_{k+2}, \ldots} J\big(x_k, \{u_k, u_{k+1}, u_{k+2} \ldots\}\big).$$

This problem formulation leads to an optimal control problem whereby the controller is required to minimize at time $k$ the performance cost (2.3) subject to the constraints (2.2). To ensure that the optimal value of the cost is well defined, we assume that the state of the model (2.1) is stabilizable and observable.

**Assumption 2.2** In the system model (2.1) and cost (2.3), the pair $(A, B)$ is stabilizable, the pair $(A, Q)$ is observable, and $R$ is positive-definite.

Given the linear nature of the controlled system, the problem of setpoint tracking (in which the output $y$ is to be steered to a given constant setpoint) can be converted into the regulation problem considered here by redefining the state of (2.1a) in terms of the deviation from a desired steady-state value. The more general case of tracking a time-varying setpoint (e.g. a ramp or sinusoidal signal) can also be tackled within the framework outlined here provided the setpoint can itself be generated by applying a constant reference signal to a system with known LTI dynamics.

## 2.2 The Unconstrained Optimum

The problem of minimizing the quadratic cost of (2.3) in the unconstrained case (i.e. when $F = 0$ and $G = 0$ in (2.2)) is addressed by Linear Quadratic (LQ) optimal control, which forms an extension of the calculus of variations. The solution is usually obtained either using Pontryagin's Maximum Principle [1] or Dynamic Programming and the recursive Bellman equation [2]. Rather than replicating these solution methods, here we first characterize the optimal linear state feedback law that minimizes the cost of (2.3), and later show (in Sect. 2.7) through a lifting formulation that this control law is indeed optimal over all input sequences.

We first obtain an expression for the cost under linear feedback, $u = Kx$, for an arbitrary stabilizing gain matrix $K \in \mathbb{R}^{n_u \times n_x}$, using the closed-loop system dynamics

$$x_{k+1} = (A + BK)x_k$$

to write $x_k = (A + BK)^k x_0$ and $u_k = K(A + BK)^k x_0$, for all $k$. Therefore $J(x_0) = J(x_0, \{Kx_0, Kx_1, \ldots\})$ is a quadratic function of $x_0$,

$$J(x_0) = x_0^T W x_0, \tag{2.4a}$$

$$W = \sum_{k=0}^{\infty} (A + BK)^{k^T} (Q + K^T RK)(A + BK)^k. \tag{2.4b}$$

If $A + BK$ is strictly stable (i.e. each eigenvalue of $A + BK$ is strictly less than unity in absolute value), then it can easily be shown that the elements of the matrix $W$ defined in (2.4b) are necessarily finite. Furthermore, if $R$ is positive-definite and $(A, Q)$ is observable, then $J(x_0)$ is a positive-definite function of $x_0$ (since then $J(x_0) \geq 0$, for all $x_0$, and $J(x_0) = 0$ only if $x_0 = 0$), which implies that $W$ is a positive-definite matrix.

The unique matrix $W$ satisfying (2.4) can be obtained by solving a set of linear equations rather than by evaluating the infinite sum in (2.4b). This is demonstrated by the following result, which also shows that $(A + BK)$ is necessarily stable if $W$ in (2.4) exists.

**Lemma 2.1** (Lyapunov matrix equation) *Under Assumption 2.2, the matrix $W$ in (2.4) is the unique positive definite solution of the Lyapunov matrix equation*

$$W = (A + BK)^T W (A + BK) + Q + K^T RK \tag{2.5}$$

*if and only if* $A + BK$ *is strictly stable.*

*Proof* Let $W_n$ denote the sum of the first $n$ terms in (2.4b), so that

$$W_n \doteq \sum_{k=0}^{n-1} (A + BK)^{k^T} (Q + K^T RK)(A + BK)^k.$$

Then $W_1 = Q + K^T RK$ and $W_{n+1} = (A + BK)^T W_n (A + BK) + Q + K^T RK$ for all $n > 0$. Assuming that $A + BK$ is strictly stable and taking the limit as $n \to \infty$, we obtain (2.5) with $W = \lim_{n\to\infty} W_n$. The uniqueness of $W$ satisfying (2.5) is implied by the uniqueness of $W_{n+1}$ in this recursion for each $n > 0$, and $W \succ 0$ follows from the positive-definiteness of $J(x_0)$.

If we relax the assumption that $A + BK$ is strictly stable, then the existence of $W \succ 0$ satisfying (2.5) implies that there exists a Lyapunov function demonstrating that the system $x_{k+1} = (A + BK)x_k$ is asymptotically stable, since $(A, Q)$ is observable and $R \succ 0$ by Assumption 2.2. Hence $A + BK$ must be strictly stable if (2.5) has a solution $W \succ 0$. $\square$

The optimal unconstrained linear feedback control law is defined by the stabilizing feedback gain $K$ that minimizes the cost in (2.3) for all initial conditions $x_0 \in \mathbb{R}^{n_x}$. The conditions for an optimal solution to this problem can be obtained by considering the effect of perturbing the value of $K$ on the solution, $W$, of the Lyapunov equation (2.5). Let $W + \delta W$ denote the sum in (2.4b) when $K$ is replaced by $K + \delta K$. Then $W + \delta W$ and $K + \delta K$ satisfy the Lyapunov equation

$$\begin{aligned} W + \delta W = \big[A + B(K + \delta K)\big]^T (W + \delta W)\big[A + B(K + \delta K)\big] \\ + Q + (K + \delta K)^T R(K + \delta K) \end{aligned}$$

which, together with (2.5), implies that $\delta W$ satisfies

$$\begin{aligned} \delta W = {} & \delta K^T \big[B^T W(A + BK) + RK\big] + \big[(A + BK)^T W B + K^T R\big]\delta K \\ & + (A + BK)^T \delta W(A + BK) + \delta K^T (B^T W B + R)\delta K \\ & + \delta K^T B^T \delta W(A + BK) + (A + BK)^T \delta W \; B \; \delta K + \delta K^T B^T \delta W \; B \; \delta K. \end{aligned} \tag{2.6}$$

For given $\delta K_1 \in \mathbb{R}^{n_u \times n_x}$, consider a perturbation of the form

$$\delta K = \epsilon \, \delta K_1,$$

and consider the effect on $\delta W$ of varying the scaling parameter $\epsilon \in \mathbb{R}$. Clearly $K$ is optimal if and only if $x_0^T (W + \delta W)x_0 \geq x_0^T W x_0$, for all $x_0 \in \mathbb{R}^{n_x}$, for all

$\delta K_1 \in \mathbb{R}^{n_u \times n_x}$ and for all sufficiently small $\epsilon$. It follows that $K$ is optimal if and only if the solution of (2.6) has the form

$$\delta W = \epsilon^2\, \delta W_2 + \epsilon^3\, \delta W_3 + \cdots$$

for all $\epsilon \in \mathbb{R}$, where $\delta W_2$ is a positive semidefinite matrix. Considering terms in (2.6) of order $\epsilon$ and order $\epsilon^2$, we thus obtain the following necessary and sufficient conditions for optimality:

$$B^T W(A + BK) + RK = 0, \tag{2.7a}$$

$$\delta W_2 \succeq 0, \tag{2.7b}$$

$$\delta W_2 = (A + BK)^T \delta W_2 (A + BK) + \delta K_1^T (B^T W B + R) \delta K_1. \tag{2.7c}$$

Solving (2.7a) for $K$ gives $K = -(B^T W B + R)^{-1} B^T W A$ as the optimal feedback gain, whereas Lemma 2.1 and (2.7c) imply that

$$\delta W_2 = \sum_{k=0}^{\infty} (A + BK)^{k\,T} \delta K_1^T (B^T W B + R) \delta K_1 (A + BK)^k$$

and therefore (2.7b) is necessarily satisfied since $A + BK$ is strictly stable and $B^T W B + R$ is positive-definite.

These arguments are summarized by the following result.

**Theorem 2.1** (Discrete time algebraic Riccati equation) *The feedback gain matrix $K$ for which the control law*

$$u = Kx$$

*minimizes the cost of (2.3) for any initial condition $x_0$ under the dynamics of (2.1a) is given by*

$$K = -(B^T W B + R)^{-1} B^T W A, \tag{2.8}$$

*where $W \succ 0$ is the unique solution of*

$$W = A^T W A + Q - A^T W B (B^T W B + R)^{-1} B^T W A. \tag{2.9}$$

*Under Assumption 2.2, $A + BK$ is strictly stable whenever there exists $W \succ 0$ satisfying (2.9).*

*Proof* The optimality of (2.8) is a consequence of the necessity and sufficiency of the optimality conditions in (2.7a), (2.7b) and (2.7c). Equation (2.9) (which is known as the discrete time algebraic Riccati equation) is obtained by substituting $K$ in (2.8) into (2.5). From Lemma 2.1, we can conclude that, under Assumption 2.2, the solution of (2.9) for $W$ is unique and positive-definite if and only if $A + BK$ is strictly stable. □

## 2.3 The Dual-Mode Prediction Paradigm

The control law that minimizes the cost (2.3) is not in general a linear feedback law when constraints (2.2) are present. Moreover, it may not be computationally tractable to determine the optimal controller as an explicit state feedback law. Predictive control strategies overcome this difficulty by minimizing, subject to constraints, a predicted cost that is computed for a particular initial state, namely the current plant state. This constrained minimization of the predicted cost is solved online at each time step in order to derive a feedback control law. The predicted cost corresponding to (2.3) can be expressed

$$J(x_k, \{u_{0|k}, u_{1|k}, \ldots\}) = \sum_{i=0}^{\infty} \left( \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) \tag{2.10}$$

where $x_{i|k}$ and $u_{i|k}$ denote the predicted values of the model state and input, respectively, at time $k + i$ based on the information that is available at time $k$, and where $x_{0|k} = x_k$ is assumed.

The prediction horizon employed in (2.10) is infinite. Hence if every element of the infinite sequence of predicted inputs $\{u_{0|k}, u_{1|k}, \ldots\}$ were considered to be a free variable, then the constrained minimization of this cost would be an infinite-dimensional optimization problem, which is in principle intractable. However predictive control strategies provide effective approximations to the optimal control law that can be computed efficiently and in real time. This is possible because of a parameterization of predictions known as the dual-mode prediction paradigm, which enables the MPC optimization to be specified as a finite-dimensional problem.

The dual-mode prediction paradigm divides the prediction horizon into two intervals. Mode 1 refers to the predicted control inputs over the first $N$ prediction time steps for some finite horizon $N$ (chosen by the designer), while mode 2 denotes the control law over the subsequent infinite interval. The mode 2 predicted inputs are specified by a fixed feedback law, which is usually taken to be the optimum for the problem of minimizing the cost in the absence of constraints [3–6]. Therefore the predicted cost (2.10) can be written as

$$J(x_k, \{u_{0|k}, u_{1|k}, \ldots\}) = \sum_{i=0}^{N-1} \left( \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_W^2 \tag{2.11}$$

where, by Theorem 2.1, $W$ is the solution of the Riccati equation (2.9). The term $\|x_{N|k}\|_W^2$ is referred to as a terminal penalty term and accounts for the cost-to-go after $N$ prediction time steps under the mode 2 feedback law.

To simplify notation we express the predicted cost as an explicit function of the initial state of the prediction model and the degrees of freedom in predictions. Hence for the dual-mode prediction paradigm in which the control inputs over the prediction horizon of mode 1 are optimization variables, we write (2.11) as

$$J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \left( \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_W^2. \tag{2.12}$$

where $\mathbf{u}_k = \{u_{0|k}, u_{1|k}, \ldots, u_{N-1|k}\}$.

The receding horizon implementation of MPC stipulates that at each time instant $k$ the optimal mode 1 control sequence $\mathbf{u}_k^* = \{u_{0|k}^*, \ldots, u_{N-1|k}^*\}$ is computed, and only the first element of this sequence is implemented, namely $u_k = u_{0|k}^*$. Thus at each time step the most up-to-date measurement information (embodied in the state $x_k$) is employed. This creates a feedback mechanism that provides some compensation for any uncertainty present in the model of (2.1a). It also reduces the gap between the optimal value of the predicted cost $J(x_k, \mathbf{u}_k)$ in (2.12) and the optimal cost for the infinite-dimensional problem of minimizing (2.10) over the infinite sequence of future inputs $\{u_{0|k}, u_{1|k}, \ldots\}$.

The rationale behind the dual-mode prediction paradigm is as follows. Let $\{u_{0|k}^0, u_{1|k}^0, \ldots\}$ denote the optimal control sequence for the problem of minimizing the cost (2.10) over the infinite sequence $\{u_{0|k}, u_{1|k}, \ldots\}$ subject to the constraints $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$, for all $i \ge 0$, for an initial condition $x_{0|k} = x_k$ such that this problem is feasible. If the weights $Q$ and $R$ satisfy Assumption 2.2, then this notional optimal control sequence drives the predicted state of the model (2.1a) asymptotically to the origin, i.e. $x_{i|k} \to 0$ as $i \to \infty$. Since $(x, u) = (0, 0)$ is strictly feasible for the constraints $Fx + Gu \le \mathbf{1}$, there exists a neighbourhood, $\mathcal{S}$, of $x = 0$ with the property that these constraints are satisfied at all times along trajectories of the model (2.1a) under the unconstrained optimal feedback law, $u = Kx$, starting from any initial condition in $\mathcal{S}$. Hence there necessarily exists a horizon $N_\infty$ (which depends on $x_k$) such that $x_{i|k} \in \mathcal{S}$, for all $i \ge N_\infty$. Since the optimal trajectory for $i \ge N_\infty$ is necessarily optimal for the problem with initial condition $x_{N_\infty|k}$ (by Bellman's Principle of Optimality [7]), the constrained optimal sequence must therefore coincide with the unconstrained optimal feedback law, i.e. $u_{i|k}^0 = Kx_{i|k}$, for all $i \ge N_\infty$. It follows that if the mode 1 horizon is chosen to be sufficiently long, namely if $N \ge N_\infty$, then the mode 1 control sequence, $\mathbf{u}_k^*$, that minimizes the cost of (2.12) subject to the constraints $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ for $i = 0, 1, \ldots, N-1$ must be equal to the first $N$ elements of the infinite sequence that minimizes the cost (2.10), namely $u_{i|k}^* = u_{i|k}^0$ for $i = 0, \ldots, N-1$.

For completeness we next give a statement of this result; for a detailed proof and further discussion we refer the interested reader to [4, 5].

**Theorem 2.2** *There exists a finite horizon $N_\infty$, which depends on $x_k$, with the property that, whenever $N \geq N_\infty$: (i). the sequence $\mathbf{u}_k^*$ that achieves the minimum of $J(x_k, \mathbf{u}_k)$ in (2.12) subject to $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$ for $i = 0, 1, \ldots, N-1$ is equal to the first $N$ terms of the infinite sequence $\{u^0_{0|k}, u^0_{1|k}, \ldots\}$ that minimizes $J(x_k, \{u_{0|k}, u_{1|k}, \ldots\})$ in (2.10) subject to $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$, for all $i \geq 0$; and (ii). $J(x_k, \mathbf{u}_k^*) = J(x_k, \{u^0_{0|k}, u^0_{1|k}, \ldots\})$.*

It is generally convenient to consider the LQ optimal feedback law $u = Kx$ as underlying both mode 1 and mode 2, and to introduce perturbations $c_{i|k} \in \mathbb{R}^{n_u}$, $i = 0, 1, \ldots, N-1$ over the horizon of mode 1 in order to meet constraints. Then the predicted sequence of control inputs is given by

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \qquad i = 0, 1, \ldots, N-1 \tag{2.13a}$$

$$u_{i|k} = Kx_{i|k}, \qquad i = N, N+1, \ldots \tag{2.13b}$$

with $x_{0|k} = x_k$. This prediction scheme is sometimes referred to as the closed-loop paradigm because the term $Kx$ provides feedback in the horizons of both modes 1 and 2.

We argue in Sect. 3.1 (in the context of robustness to model uncertainty) that (2.13) should be classified as an open-loop prediction scheme because $K$ is fixed rather than computed on the basis of measured information (namely $x_k$). Nevertheless, the feedback term $Kx$ forms a pre-stabilizing feedback loop around the dynamics of (2.1a), which assume the form

$$x_{i+1|k} = \Phi x_{i|k} + Bc_{i|k}, \qquad i = 0, 1, \ldots, N-1 \tag{2.14a}$$

$$x_{i+1|k} = \Phi x_{i|k}, \qquad i = N, N+1, \ldots \tag{2.14b}$$

where $\Phi = A + BK$, with $x_{0|k} = x_k$. The strict stability property of $\Phi$ prevents numerical ill-conditioning that could arise in the prediction equations and the associated MPC optimization problem in the case of open-loop unstable models [8].

For the closed-loop paradigm formulation in (2.13), the predicted state trajectory can be generated by simulating (2.14a) forwards over the mode 1 prediction horizon, giving

$$\mathbf{x}_k = M_x x_k + M_c \mathbf{c}_k, \tag{2.14c}$$

where

$$\mathbf{x}_k \doteq \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix}, \quad \mathbf{c}_k \doteq \begin{bmatrix} c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}$$

$$M_x = \begin{bmatrix} \Phi \\ \Phi^2 \\ \vdots \\ \Phi^N \end{bmatrix}, \quad M_c = \begin{bmatrix} B & 0 & \cdots & 0 \\ \Phi B & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{N-1} B & \Phi^{N-2} B & \cdots & B \end{bmatrix}.$$

On the basis of these prediction equations and the fact that the predicted cost over mode 2 is given by $\|x_{N|k}\|_W^2$ (where $W$ is the solution of the Lyapunov equation (2.5)), the predicted cost of (2.11) can be written as a quadratic function of the degrees of freedom, namely the vector of predicted perturbations $\mathbf{c}_k$. The details of this computation are straightforward and will not be given here. Instead we derive an equivalent but more convenient form for the predicted cost in Sect. 2.7. For simplicity (but with a slight abuse of notation) in the following development, we denote the cost of (2.11) evaluated along the predicted trajectories of (2.13a) and (2.14a) as $J(x_k, \mathbf{c}_k)$, thus making explicit the dependence of the cost on the optimization variables $\mathbf{c}_k$.

## 2.4 Invariant Sets

The determination of the minimum prediction horizon $N$ which ensures that the predicted state and input trajectories in mode 2 meet constraints (2.2) is not a trivial matter. Instead lower bounds for this horizon were proposed in [4, 5]. However such bounds could be conservative, leading to the use of unnecessarily long prediction horizons. This in turn could make the online optimization of the predicted cost computationally intractable as a result of large numbers of free variables and large numbers of constraints in the minimization of predicted cost. In such cases it becomes necessary to use a shorter horizon $N$ while retaining the guarantee that predictions over mode 2 satisfy constraints on states and inputs. This can be done by imposing a terminal constraint which requires that the state at the end of the mode 1 horizon should lie in a set which is positively invariant under the dynamics defined by (2.13b) and (2.14b) and under the constraints (2.2).

**Definition 2.1** (*Positively invariant set*) A set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is positively invariant under the dynamics defined by (2.13b) and (2.14b) and the constraints (2.2) if and only if $(F + GK)x \leq \mathbf{1}$ and $\Phi x \in \mathcal{X}$, for all $x \in \mathcal{X}$.

The use of invariant sets within the dual prediction mode paradigm is illustrated in Fig. 2.1 for a second-order system. The predicted state at the end of mode 1 is constrained to lie in an invariant set $\mathcal{X}_T$ via the constraint $x_{N|k} \in \mathcal{X}_T$. Thereafter, in

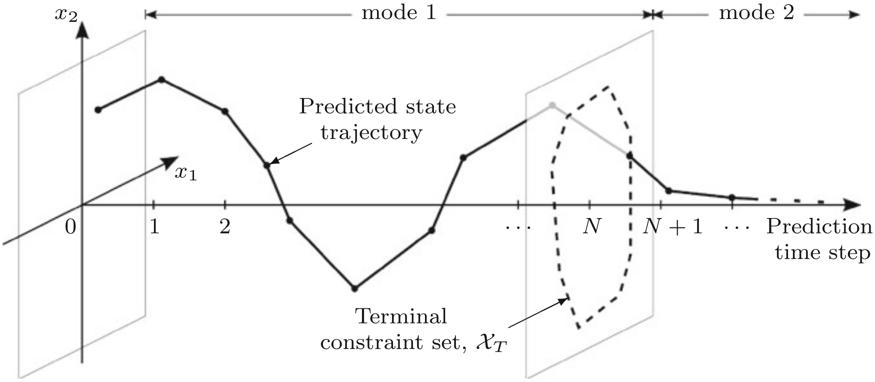


**Fig. 2.1** The dual-mode prediction paradigm with terminal constraint. The control inputs in mode 1 are chosen so as to satisfy the system constraints as well as the constraint that the $N$ step ahead predicted state should be inside the invariant set $\mathcal{X}_T$. Over the infinite mode 2 prediction horizon the predicted state trajectory is dictated by the prescribed feedback control law $u = Kx$

mode 2, the evolution of the state trajectory is that prescribed by the state feedback control law $u_k = Kx_k$.

In order to increase the applicability of the MPC algorithm, and in particular to increase the size of the set of initial conditions $x_{0|k}$ for which the terminal condition $x_{N|k} \in \mathcal{X}_T$ can be met, it is important to choose the maximal positively invariant set as the terminal constraint set. This set is defined as follows.

**Definition 2.2** (*Maximal positively invariant set*) The maximal positively invariant (MPI) set under the dynamics of (2.13b) and (2.14b) and the constraints (2.2) is the union of all sets that are positively invariant under these dynamics and constraints.

It was shown in [9] that, for the case of linear dynamics and linear constraints considered here, the MPI set is defined by a finite number of linear inequalities. This result is summarized next.

**Theorem 2.3** ([9]) *The MPI set for the dynamics defined by (2.13b) and (2.14b) and the constraints (2.2) can be expressed*

$$\mathcal{X}^{\mathrm{MPI}} \doteq \{x : (F + GK)\Phi^i x \leq \mathbf{1},\ i = 0, \ldots, \nu\} \tag{2.15}$$

*where $\nu$ is the smallest positive integer such that $(F + GK)\Phi^{\nu+1}x \leq \mathbf{1}$, for all $x$ satisfying $(F + GK)\Phi^i x \leq \mathbf{1}$, $i = 0, \ldots, \nu$. If $\Phi$ is strictly stable and $(\Phi, F + GK)$ is observable, then $\nu$ is necessarily finite.*

*Proof* Let $\mathcal{X}^{(n)} = \{x : (F + GK)\Phi^i x \leq \mathbf{1},\ i = 0, \ldots, n\}$ for $n \geq 0$, then it can be shown that (2.15) holds for some finite $\nu$ using Definition 2.2 to show that the MPI set $\mathcal{X}^{\mathrm{MPI}}$ is equal to $\mathcal{X}^{(\nu)}$ for finite $\nu$.

In particular, if $x_{0|k} \notin \mathcal{X}^{(n)}$ for given $n$, then the constraint (2.2) must be violated under the dynamics of (2.13b) and (2.14b). By Definition 2.2 therefore, any $x \notin \mathcal{X}^{(n)}$ cannot lie in $\mathcal{X}^{\text{MPI}}$, so $\mathcal{X}^{(n)}$ must contain $\mathcal{X}^{\text{MPI}}$, for all $n \geq 0$.

Furthermore, if $(F + GK)\Phi^{\nu+1}x \leq \mathbf{1}$, for all $x \in \mathcal{X}^{(\nu)}$, then $\Phi x \in \mathcal{X}^{(\nu)}$ must hold whenever $x \in \mathcal{X}^{(\nu)}$ (since $x \in \mathcal{X}^{(\nu)}$ and $(F + GK)\Phi^{\nu+1}x \leq \mathbf{1}$ imply $(F + GK)\Phi^{i}(\Phi x) \leq \mathbf{1}$ for $i = 0, \ldots \nu$). But from the definition of $\mathcal{X}^{(\nu)}$ we have $(F + GK)x \leq \mathbf{1}$ for all $x \in \mathcal{X}^{(\nu)}$, and therefore $\mathcal{X}^{(\nu)}$ is positively invariant under (2.13b), (2.14b) and (2.2). From Definition 2.2 it can be concluded that $\mathcal{X}^{(\nu)}$ is a subset of, and therefore equal to $\mathcal{X}^{\text{MPI}}$.

Finally, for $\nu \geq n_x$, the set $\mathcal{X}^{(\nu)}$ is necessarily bounded if $(\Phi, F + GK)$ is observable, and, since $\Phi$ is strictly stable, the set $\{x : (F + GK)\Phi^{(\nu+1)}x \leq \mathbf{1}\}$ must contain $\mathcal{X}^{(\nu)}$ for finite $\nu$; therefore $\mathcal{X}^{\text{MPI}}$ must be defined by (2.15) for some finite $\nu$. □

The value of $\nu$ satisfying the conditions of Theorem 2.3 can be computed by solving at most $\nu n_C$ linear programs (LPs), namely

$$\underset{x}{\text{maximize}}\ (F + GK)_j \Phi^{n+1} x \ \text{ subject to } \ (F + GK)\Phi^{i}x \leq \mathbf{1},\ i = 0, \ldots, n$$

for $j = 1, \ldots, n_C, n = 1, \ldots, \nu$, where $(F+GK)_j$ denotes the $j$th row of $F+GK$. The value of $\nu$ clearly does not depend on the system state, and this procedure can therefore be performed offline. In general $\nu \geq n_x$, and (2.15) defines the MPI set as a polytope. Therefore if $\mathcal{X}_T$ is equal to the MPI set, the terminal constraint $x_{N|k} \in \mathcal{X}_T$ can be invoked via linear inequalities on the degrees of freedom in mode 1 predictions. It will be convenient to represent the terminal set $\mathcal{X}_T$ in matrix form

$$\mathcal{X}_T = \{x : V_T x \leq \mathbf{1}\},$$

so that with $\mathcal{X}_T$ chosen as the MPI set (2.15), $V_T$ is given by

$$V_T = \begin{bmatrix} F + GK \\ (F + GK)\Phi \\ \vdots \\ (F + GK)\Phi^{\nu} \end{bmatrix}.$$

*Example 2.1* Figure 2.2 gives an illustration of the MPI set for a second-order system with state-space matrices

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{2.16a}$$

and constraints $-\mathbf{1} \leq x/8 \leq \mathbf{1}$, $-1 \leq u \leq 1$, which correspond to the following constraint matrices in (2.2),

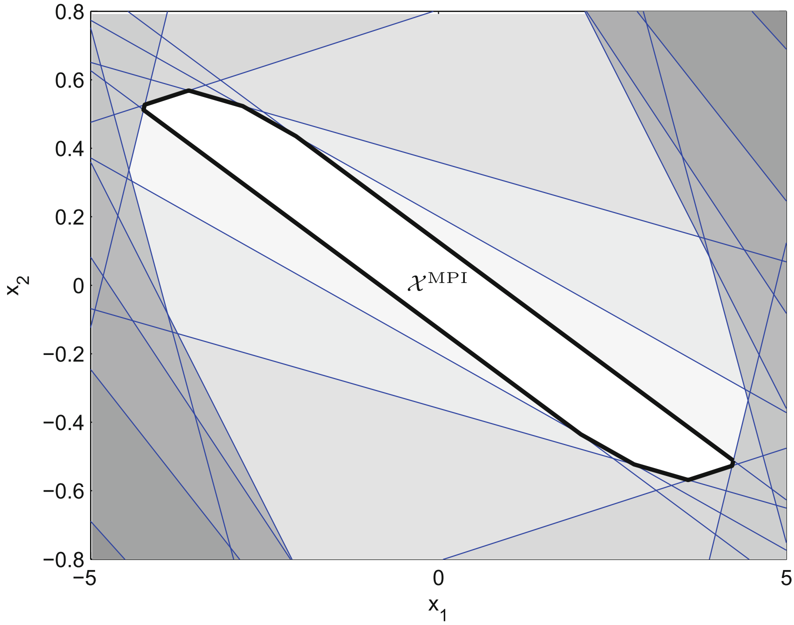


**Fig. 2.2** The maximal positively invariant (MPI) set, $\mathcal{X}^{\mathrm{MPI}}$, for the system of (2.16a), (2.16b). Each of the inequalities defining $\mathcal{X}^{\mathrm{MPI}}$ is represented by a straight line on the diagram

$$F = \begin{bmatrix} 0 & 1/8 \\ 1/8 & 0 \\ 0 & -1/8 \\ -1/8 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{2.16b}$$

The mode 2 feedback law is taken to be the optimal unconstrained linear feedback law $u = Kx$, with cost weights $Q = C^T C$ and $R = 1$, for which $K = -\big[1.19\ 7.88\big]$. The MPI set is given by (2.15) with $\nu = 5$. After removing redundant constraints, this set is defined by 10 inequalities corresponding to the 10 straight lines that intersect the boundary of the MPI set, marked $\mathcal{X}^{\mathrm{MPI}}$ in Fig. 2.2. ♢

## 2.5 Controlled Invariant Sets and Recursive Feasibility

Collecting the ideas discussed in the previous sections we can state the following MPC algorithm:

**Algorithm 2.1** (*MPC*) At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \quad J(x_k, \mathbf{c}_k) \tag{2.17a}$$

$$\text{subject to} \quad (F + GK)x_{i|k} + Gc_{i|k} \leq \mathbf{1}, i = 0, \ldots, N-1 \tag{2.17b}$$

$$V_T x_{N|k} \leq \mathbf{1} \tag{2.17c}$$

where $J(x_k, \mathbf{c}_k)$ is the cost of (2.11) evaluated for the predicted trajectories of (2.13a) and (2.14a).

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (2.17). $\lhd$

The terminal condition (2.17c) is sometimes referred to as a stability constraint because it provides a means of guaranteeing the closed-loop stability of the MPC law. It does this by ensuring that the mode 2 predicted trajectories (2.13b) and (2.14b) satisfy the constraint $(F + GK)x_{i|k} \leq \mathbf{1}$, thus ensuring that the predicted cost over mode 2 is indeed given by $\|x_{N|k}\|^2_W$, and also by guaranteeing that Algorithm 2.1 is feasible at all time instants if it is feasible at initial time. The latter property of recursive feasibility is a fundamental requirement for closed-loop stability since it guarantees that the optimization problem (2.17) is solvable and hence that the control law of Algorithm 2.1 is defined at every time instant if (2.17) is initially feasible.

Recall that the feasibility of predicted trajectories in mode 2 is ensured by constraining the terminal state to lie in a set which is positively invariant. The feasibility of Algorithm 2.1 can be similarly ensured by requiring that the state $x_k$ lies in an invariant set. However, since there are degrees of freedom in the predicted trajectories of (2.13a) and (2.14a), the relevant form of invariance is controlled positive invariance.

**Definition 2.3** (*Controlled positively invariant set*) A set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is controlled positively invariant (CPI) for the dynamics of (2.1a) and constraints (2.2) if, for all $x \in \mathcal{X}$, there exists $u \in \mathbb{R}^{n_u}$ such that $Fx + Gu \leq \mathbf{1}$ and $Ax + Bu \in \mathcal{X}$. Furthermore $\mathcal{X}$ is the maximal controlled positively invariant (MCPI) set if it is CPI and contains all other CPI sets.

To show that Algorithm 2.1 is recursively feasible, we demonstrate next that its feasible set is a CPI set. Algorithm 2.1 is feasible whenever $x_k$ belongs to the feasible set $\mathcal{F}_N$ defined by

$$\mathcal{F}_N \doteq \big\{x_k : \exists\, \mathbf{c}_k \text{ such that } (F + GK)x_{i|k} + Gc_{i|k} \leq \mathbf{1},\ i = 0, \ldots, N-1 \\ \text{and } V_T x_{N|k} \leq \mathbf{1}\big\}. \tag{2.18}$$

Clearly this is the same as the set of states of (2.1a) that can be driven to the terminal set $\mathcal{X}_T = \{x : V_T x \leq \mathbf{1}\}$ in $N$ steps subject to the constraints (2.2), and it therefore has the following equivalent definition:

$$\mathcal{F}_N = \Big\{x_0 : \exists \{u_0, \ldots, u_{N-1}\} \text{ such that } Fx_i + Gu_i \leq \mathbf{1},\ i = 0, \ldots, N-1, \\ \text{and } x_N \in \mathcal{X}_T\Big\}. \tag{2.19}$$

**Theorem 2.4** *If $\mathcal{X}_T$ in (2.19) is positively invariant for (2.13b), (2.14b) and (2.2), then $\mathcal{F}_N \subseteq \mathcal{F}_{N+1}$, for all $N > 0$, and $\mathcal{F}_N$ is a CPI set for the dynamics of (2.1a) and constraints (2.2).*

*Proof* If $x_0 \in \mathcal{F}_N$, then by definition there exists a sequence $\{u_0, \ldots, u_{N-1}\}$ such that $Fx_i + Gu_i \leq \mathbf{1}$, $i = 0, \ldots, N-1$ and $x_N \in \mathcal{X}_T$. Also, since $\mathcal{X}_T$ is positively invariant, the choice $u_N = Kx_N$ would ensure $Fx_N + Gu_N \leq \mathbf{1}$ and $x_{N+1} \in \mathcal{X}_T$, and this in turn implies $x_0 \in \mathcal{F}_{N+1}$ whenever $x_0 \in \mathcal{F}_N$. Furthermore if $x_0 \in \mathcal{F}_N$, then by definition $u_0$ exists such that $Fx_0 + Gu_0 \leq \mathbf{1}$ and $x_1 \in \mathcal{F}_{N-1}$, and since $\mathcal{F}_{N-1} \subset \mathcal{F}_N$, it follows that $\mathcal{F}_N$ is CPI. □

Although the proof of Theorem 2.4 considers the sequence of control inputs $\{u_0, \ldots, u_{N-1}\}$, the same arguments apply to the optimization variables $\mathbf{c}_k$ in (2.17), since for each feasible $u_k$, $k = 0, \ldots, N-1$, there exists a feasible $c_k$ such that $u_k = Kx_k + c_k$. Therefore, the fact that $\mathcal{F}_N$ is a CPI set for (2.1a) and (2.2) also implies that $\mathcal{F}_N$ is CPI for the dynamics (2.14a) and constraints (2.17b). Hence for any $x_k \in \mathcal{F}_N$ there must exist $c_k$ such that $(F + GK)x_k + Gc_k \leq \mathbf{1}$ and $x_{k+1} = \Phi x_k + Bc_k \in \mathcal{F}_N$. Furthermore, the proof of Theorem 2.4 shows that if $c_k = c^*_{0|k}$ (where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ in step (ii) of Algorithm 2.1), then the sequence

$$\mathbf{c}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0) \tag{2.20}$$

is necessarily feasible for the optimization (2.17) at time $k + 1$, and therefore Algorithm 2.1 is recursively feasible.

The candidate feasible sequence in (2.20) can be thought of as the extension to time $k+1$ of the optimal sequence at time $k$. It is in fact the sequence that generates, via (2.13a), the input sequence

$$\{u_{1|k}, \ldots, u_{N-1|k}, Kx_{N|k}\}$$

at time $k + 1$. For this reason, it is sometimes referred to as the *tail of the solution of the MPC optimization problem at time $k$*, or simply the *tail*. As well as demonstrating recursive feasibility, the tail is often used to construct a suboptimal solution at time $k + 1$ based on the optimal solution at time $k$. This enables a comparison of the optimal costs at successive time steps, which is instrumental in the analysis of the closed-loop stability properties of MPC laws.

Theorem 2.4 shows that the feasible sets corresponding to increasing values of $N$ are nested, so that the feasible set $\mathcal{F}_N$ necessarily grows as $N$ is increased. In practice the length of the mode 1 horizon is likely to be limited by the growth in computation that is required to solve Algorithm 2.1 (this is discussed in Sect. 2.8). However, given that $\mathcal{F}_N$ increases as $N$ grows, the question arises as to whether there exists a finite value of $N$ such that $\mathcal{F}_N$ is equal to the maximal feasible set defined by

$$\mathcal{F}_\infty \doteq \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

Here $\mathcal{F}_\infty$ is defined as the set of initial conditions that can be steered to $\mathcal{X}_T$ over an infinite horizon subject to constraints. However, $\mathcal{F}_\infty$ is independent of the choice of $\mathcal{X}_T$; this is a consequence of the fact that, for any bounded positively invariant set $\mathcal{X}_T$, the system (2.1a) can be steered from any initial state in $\mathcal{X}_T$ to the origin subject to the constraints (2.2) in finite time, as demonstrated by the following result.

**Theorem 2.5** *Let $\mathcal{F}_N^0 \doteq \big\{x_0 : \exists\{u_0, \ldots, u_{N-1}\}$ such that $Fx_i + Gu_i \le \mathbf{1}$, $i = 0, \ldots, N-1$, and $x_N = 0\big\}$. If $\mathcal{X}_T$ in (2.19) is positively invariant for (2.13b), (2.14b) and (2.2), where $\Phi$ is strictly stable and $(\Phi, F + GK)$ is observable, then $\mathcal{F}_\infty = \bigcup_{N=1}^{\infty} \mathcal{F}_N = \bigcup_{N=1}^{\infty} \mathcal{F}_N^0$.*

*Proof* First, note that any positively invariant set $\mathcal{X}_T$ must contain the origin because $\Phi$ is strictly stable. Second, strict stability of $\Phi$ and boundedness of $\mathcal{X}_T$ (which follows from observability of $(\Phi, F + GK)$) also implies that, for any $\epsilon > 0$, the set $\mathcal{B}_\epsilon \doteq \{x : \|x\| \le \epsilon\}$ is reachable from any point in $\mathcal{X}_T$ in a finite number of steps (namely for all $x_0 \in \mathcal{X}_T$ there exists a sequence $\{u_0, \ldots, u_{n-1}\}$ such that $Fx_i + Gu_i \le \mathbf{1}$ for $i = 0, \ldots, n-1$ and $x_n \in \mathcal{B}_\epsilon$) since $\|\Phi^n x\| \le \epsilon$, for all $x \in \mathcal{X}_T$ for some finite $n$. Third, since $(A, B)$ is controllable and $(0, 0)$ lies in the interior of the constraint set $\{(x, u) : Fx + Gu \le \mathbf{1}\}$, there must exist $\epsilon > 0$ such that the origin is reachable in $n_x$ steps from any point in $\mathcal{B}_\epsilon$, i.e. $\mathcal{B}_\epsilon \subseteq \mathcal{F}_{n_x}^0$. Combining these observations we obtain $\{0\} \subseteq \mathcal{X}_T \subseteq \mathcal{F}_{n+n_x}^0$ and hence $\mathcal{F}_N^0 \subseteq \mathcal{F}_N \subseteq \mathcal{F}_{n+n_x+N}^0$ for some finite $n$ and all $N \ge 0$. From this we conclude that $\bigcup_{N=1}^{\infty} \mathcal{F}_N = \bigcup_{N=1}^{\infty} \mathcal{F}_N^0$. □

A consequence of Theorem 2.5 is that replacing the terminal set $\mathcal{X}_T$ by any bounded positively invariant set (or in fact any CPI set) in (2.18) results in the same set $\mathcal{F}_\infty$. Therefore $\mathcal{F}_\infty$ is identical to the maximal CPI set or *infinite time reachability set* [10, 11], which by definition is the largest possible feasible set for any stabilizing control law for the dynamics (2.1a) and constraints (2.2). In general $\mathcal{F}_N$ does not necessarily tend to a finite limit[1] as $N \to \infty$, but the following result shows that under certain conditions $\mathcal{F}_\infty$ is equal to $\mathcal{F}_N$ for finite $N$.

[1] If for example the system (2.1a) is open-loop stable and $F = 0$, then clearly the MCPI set is the entire state space and $\mathcal{F}_N$ grows without bound as $N$ increases. In general the MCPI set is finite if and only if the system $(A, B, F, G)$, mapping input $u_k$ to output $Fx_k + Gu_k$ has no transmission zeros inside the unit circle (see, e.g. [11, 12]).

**Theorem 2.6** *If $\mathcal{F}_{N+1} = \mathcal{F}_N$ for finite $N > 0$, then $\mathcal{F}_\infty = \mathcal{F}_N$.*

*Proof* An alternative definition of $\mathcal{F}_{N+1}$ (which is nonetheless equivalent to (2.18)) is that $\mathcal{F}_{N+1}$ is the set of states $x$ for which there exists a control input $u$ such that $Fx + Gu \leq \mathbf{1}$ and $Ax + Bu \in \mathcal{F}_N$. If $\mathcal{F}_{N+1} = \mathcal{F}_N$, then it immediately follows from this definition that $\mathcal{F}_{N+2} = \mathcal{F}_{N+1}$. Applying this argument repeatedly we get $\mathcal{F}_{N+i} = \mathcal{F}_N$, for all $i = 1, 2, \ldots$ and hence $\mathcal{F}_\infty = \mathcal{F}_N$. □

*Example 2.2* Figure 2.3 shows the feasible sets $\mathcal{F}_N$ of Algorithm 2.1 for the system model and constraints of Example 2.1, for a range of values of mode 1 horizon $N$. Here the terminal set $\mathcal{X}_T$ is the maximal positively invariant set $\mathcal{X}^{\mathrm{MPI}}$ of Fig. 2.2; this is shown in Fig. 2.3 as the feasible set for $N = 0$. As expected the feasible sets $\mathcal{F}_N$ for increasing $N$ are nested. For this example, the maximal CPI set is given by $\mathcal{F}_\infty = \mathcal{F}_N$ for $N = 26$ and the minimal description of $\mathcal{F}_\infty$ involves 100 inequalities. ♢

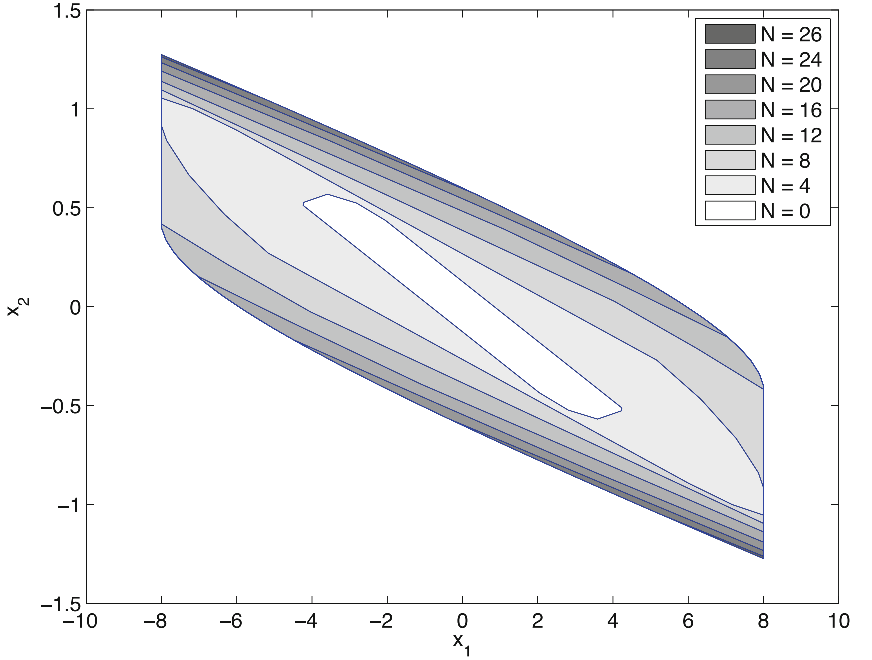


**Fig. 2.3** The feasible sets $\mathcal{F}_N$, $N = 4, 8, 12, 16, 20, 24, 26$ and the terminal set $\mathcal{F}_0 = \mathcal{X}_T$ for the example of (2.16a), (2.16b). The maximal controlled invariant set is $\mathcal{F}_\infty = \mathcal{F}_{26}$

## 2.6 Stability and Convergence

This section introduces the main tools for analysing closed-loop stability under the MPC law of Algorithm 2.1 for the ideal case of no model uncertainty or unmodeled disturbances. The control law is nonlinear because of the inequality constraints in the optimization (2.17), and the natural framework for the stability analysis is therefore Lyapunov stability theory. Using the feasible but suboptimal tail sequence that was introduced in Sect. 2.5, we show that the optimal value of the cost function in (2.17) is non-increasing along trajectories of the closed-loop system. This provides guarantees of asymptotic convergence of the state and Lyapunov stability under Assumption 2.2. Where possible, we keep the discussion in this section non-technical and refer to the literature on stability theory for technical details.

The feasibility of the tail of the optimal sequence $\mathbf{c}_k^*$ implies that the sequence $\mathbf{c}_{k+1}$ defined in (2.20) is feasible but not necessarily an optimal solution of (2.17) at time $k+1$. Using (2.20) it is easy to show that the corresponding cost $J(x_{k+1}, \mathbf{c}_{k+1})$ is equal to $J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2$. After optimization at time $k+1$, we therefore have

$$J^*(x_{k+1}) \le J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2. \tag{2.21}$$

Summing both sides of this inequality over all $k \ge 0$ gives the closed-loop performance bound

$$\sum_{k=0}^{\infty} \big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \le J^*(x_0) - \lim_{k\to\infty} J^*(x_k). \tag{2.22}$$

The quantity appearing on the LHS of this inequality is the cost evaluated along the closed-loop trajectories of (2.1) under Algorithm 2.1. Since $J^*(x_k)$ is non-negative for all $k$, the bound (2.22) implies that the closed-loop cost can be no greater than the initial optimal cost value, $J^*(x_0)$.

Given that the optimal cost is necessarily finite if (2.17) is feasible, and since each term in the sum on the LHS of (2.22) is non-negative, the closed-loop performance bound in (2.22) implies the following convergence result

$$\lim_{k\to\infty} \big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) = 0 \tag{2.23}$$

along the trajectories of the closed-loop system. We now give the basic results concerning closed-loop stability.

**Theorem 2.7** *If (2.17) feasible at $k = 0$, then the state and input trajectories of (2.1a) under Algorithm 2.1 satisfy* $\lim_{k\to\infty}(x_k, u_k) = (0, 0)$.

*Proof* This follows from (2.23) and Assumption 2.2 since $R \succ 0$ implies $u_k \to 0$ as $k \to \infty$; hence from the observability of $(Q, A)$ and $\|x_k\|_Q \to 0$ we conclude that $x_k \to 0$ as $k \to \infty$. □

**Theorem 2.8** *Under the control law of Algorithm 2.1, the origin $x = 0$ of the system (2.1a) is asymptotically stable and its region of attraction is equal to the feasible set $\mathcal{F}_N$. If $Q \succ 0$, then $x = 0$ is exponentially stable.*

*Proof* The conditions on $Q$ and $R$ in Assumption 2.2 ensure that the optimal cost $J^*(x_k)$ is a positive-definite function of $x_k$ since $J^*(x_k) = 0$ if and only if $x_k = 0$, and $J^*(x_k) > 0$ whenever $x_k \neq 0$. Therefore (2.21) implies that $J^*(x_k)$ is a Lyapunov function which demonstrates that $x = 0$ is a stable equilibrium (in the sense of Lyapunov) of the closed-loop system [13]. Combined with the convergence result of Theorem 2.7, this shows that $x = 0$ is an asymptotically stable equilibrium point, and since Theorem 2.7 applies to all feasible initial conditions, the region of attraction is $\mathcal{F}_N$.

To show that the rate of convergence is exponential if $Q \succ 0$ we first note that the optimal value of (2.17) is a continuous piecewise quadratic function of $x_k$ [14]. Therefore, $J^*(x_k)$ can be bounded above and below for all $x_k \in \mathcal{F}_N$ by

$$\alpha \|x_k\|^2 \le J^*(x_k) \le \beta \|x_k\|^2 \tag{2.24}$$

where $\alpha$ and $\beta$ are necessarily positive scalars since $J^*(x_k)$ is positive-definite. If the smallest eigenvalue of $Q$ is $\underline{\lambda}(Q)$, then from (2.24) and (2.21) we get

$$\|x_k\|^2 \le \frac{1}{\alpha} \left| 1 - \frac{\underline{\lambda}(Q)}{\beta} \right|^k J^*(x_0)$$

for all $k = 0, 1, \ldots$, and hence $x = 0$ is exponentially stable. □

*Example 2.3* For the same system dynamics, constraints and cost as in Example 2.1 the predicted and closed-loop state trajectories under the MPC law of Algorithm 2.1 with $N = 6$ and initial state $x(0) = (-7.5, 0.5)$ are shown in Fig. 2.4. Figure 2.5 gives the corresponding predicted and closed-loop input trajectories. The jump in the predicted input trajectory at $N = 6$ is due to the switch to the mode 2 feedback law at that time step.

Table 2.1 gives the variation with mode 1 horizon $N$ of predicted cost $J_0^*$ and closed-loop cost $J_{\text{cl}}(x_0) \doteq \sum_{k=0}^{\infty}(\|x_k\|_Q^2 + \|u_k\|_R^2)$ for $x(0) = (-7.5, 0.5)$. The infinite-dimensional optimal performance is obtained with $N = N_\infty$, where $N_\infty = 11$ for this initial condition, so there is no further decrease in predicted cost for values of $N > 11$. However, because of the receding horizon implementation, the closed-loop response of the MPC law for $N = 6$ is indistinguishable from the ideal optimal response for this initial condition. ◊

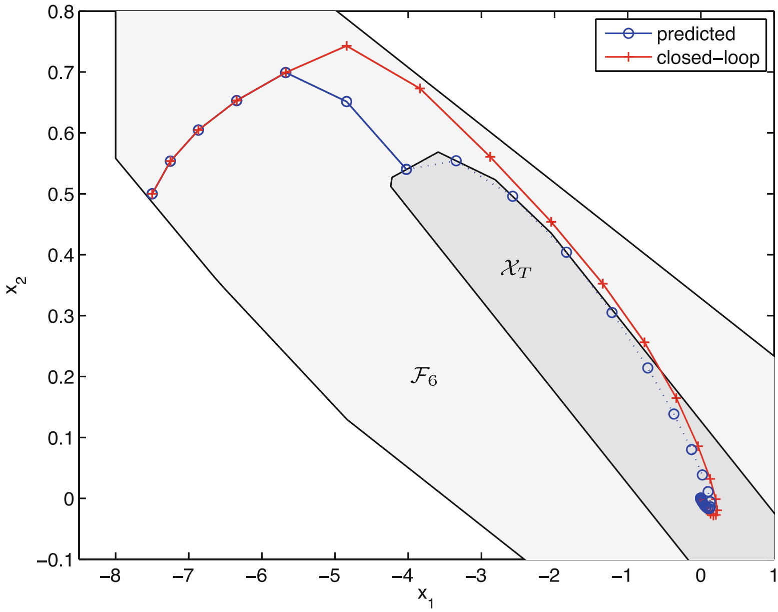


**Fig. 2.4** Predicted and closed-loop state trajectories for Algorithm 2.1 with $N = 6$

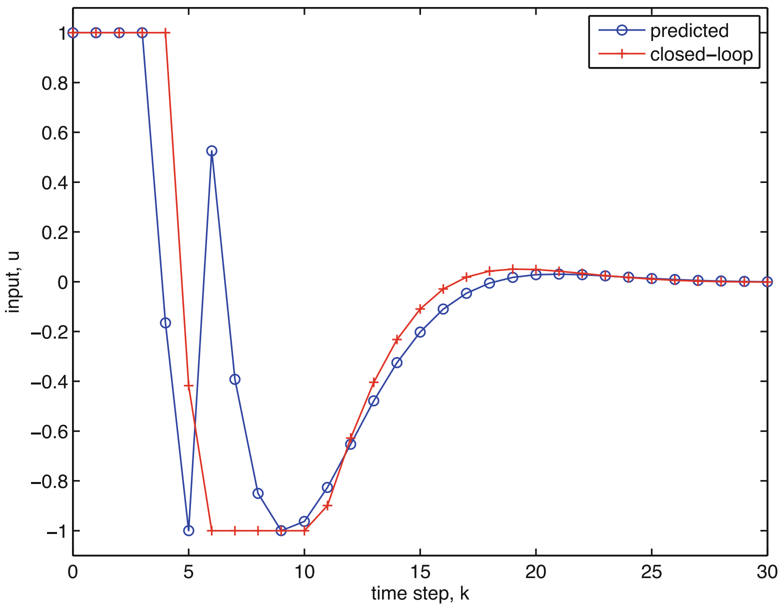


**Fig. 2.5** Predicted and closed-loop input trajectories for Algorithm 2.1 with $N = 6$

**Table 2.1** Variation of predicted and closed-loop cost with $N$ for $x_0 = (-7.5, 0.5)$ in Example 2.3

| $N$ | 6 | 7 | 8 | 11 | >11 |
|---|---|---|---|---|---|
| $J^*(x_0)$ | 364.2 | 357.0 | 356.3 | 356.0 | 356.0 |
| $J_{\mathrm{cl}}(x_0)$ | 356.0 | 356.0 | 356.0 | 356.0 | 356.0 |

## 2.7 Autonomous Prediction Dynamics

The dual-mode prediction dynamics (2.14a) and (2.14b) can be expressed in a more compact autonomous form that incorporates both prediction modes [15, 16]. This alternative prediction model, which includes the degrees of freedom in predictions within the state of an autonomous prediction system, enables the constraints on predicted trajectories to be formulated as constraints on the prediction system state at the start of the prediction horizon. With this approach the feasible sets for the model state and the degrees of freedom in predictions are determined simultaneously by computing an invariant set (rather than a controlled invariant set) for the autonomous system state. This can result in significant reductions in computation for the case that the system model is uncertain since, as discussed in Chap. 5, it greatly simplifies handling the the effects of uncertainty over the prediction horizon. In this section we show that an autonomous formulation is also convenient in the case of nominal MPC.

An autonomous prediction system that generates the predictions of (2.13a), (2.13b) and (2.14a), (2.14b) can be expressed as

$$z_{i+1|k} = \Psi z_{i|k}, \quad i = 0, 1, \ldots \tag{2.25}$$

where the initial state $z_{0|k} \in \mathbb{R}^{n_x + N n_u}$ consists of the state $x_k$ of the model (2.1a) appended by the vector $\mathbf{c}_k$ of degrees of freedom,

$$z_{0|k} = \begin{bmatrix} x_k \\ c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}.$$

The state transition matrix in (2.25) is given by

$$\Psi = \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix} \tag{2.26a}$$

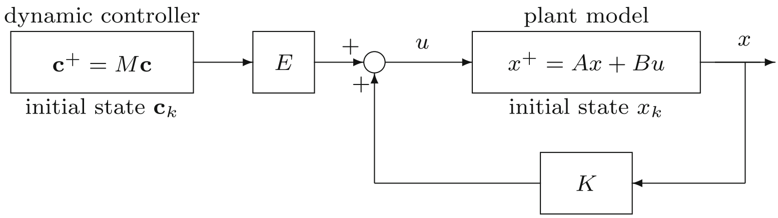


**Fig. 2.6** Block diagram representation of the autonomous prediction systems (2.25) and (2.26). The free variables in the state and input predictions at time $k$ are contained in the initial controller state $\mathbf{c}_k$; the signals marked $x$ and $u$ are the $i$ steps ahead predicted state and control input, and $x^+$, $\mathbf{c}^+$ denote their successor states

where $\Phi = A + BK$ and

$$E = \begin{bmatrix} I_{n_u} & 0 & \cdots & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & I_{n_u} & 0 & \cdots & 0 \\ 0 & 0 & I_{n_u} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{n_u} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.26b}$$

The state and input predictions of (2.13a), (2.13b) and (2.14a), (2.14b) are then given by

$$u_{i|k} = \begin{bmatrix} K & E \end{bmatrix} z_{i|k} \tag{2.27a}$$

$$x_{i|k} = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} z_{i|k} \tag{2.27b}$$

for $i = 0, 1, \ldots$. The prediction systems (2.25) and (2.26) can be interpreted as a dynamic feedback law applied to (2.1a), with the controller state at the beginning of the prediction horizon containing the degrees of freedom, $\mathbf{c}_k$, in predictions (Fig. 2.6).

### 2.7.1 Polytopic and Ellipsoidal Constraint Sets

The constraints (2.2) applied to the predictions of (2.27a), (2.27b) are equivalent to the following constraints on the initial prediction system state $z_k = z_{0|k}$:

$$\begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i z_k \le \mathbf{1}, \quad i = 0, 1, \ldots \tag{2.28}$$

Clearly this implies an infinite number of constraints that apply across an infinite prediction horizon. However, analogously to the definition of terminal invariant sets in Sect. 2.4, a feasible set for $z_k$ satisfying (2.28) can be constructed by determining a positively invariant set for the dynamics $z_{k+1} = \Psi z_k$ and constraints

$\begin{bmatrix} F + GK & GE \end{bmatrix} z_k \leq \mathbf{1}$. Theorem 2.3 shows that the maximal positively invariant set for these dynamics and constraints is given by

$$\mathcal{Z} \doteq \{z : \begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i z \leq \mathbf{1}, \ \ i = 0, 1, \ldots, \nu_z\} \tag{2.29}$$

where $\nu_z$ is a positive integer such that $\begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^{\nu_z+1} z \leq \mathbf{1}$, for all $z$ satisfying $\begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i z \leq \mathbf{1}$, $i = 0, 1, \ldots, \nu_z$. Since $\mathcal{Z}$ is the MPI set, every state $z_k$ for which (2.28) is satisfied must lie in $\mathcal{Z}$. Given that a mode 1 prediction horizon of $N$ steps is implicit in the augmented prediction dynamics (2.25), the projection of $\mathcal{Z}$ onto the $x$-subspace is therefore equal to the feasible set $\mathcal{F}_N$ defined in (2.18), i.e.

$$\mathcal{F}_N = \left\{ x : \exists\, \mathbf{c} \text{ such that } \begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i \begin{bmatrix} x \\ \mathbf{c} \end{bmatrix} \leq \mathbf{1}, \ i = 0, 1, \ldots, \nu_z \right\}.$$

The value of $\nu_z$ defining the MPI set in (2.29) grows as the mode 1 prediction horizon $N$ is increased. Furthermore, it can be seen from (2.26) that every eigenvalue of $\Psi$ is equal either to 0 or to an eigenvalue of $\Phi$, so if one or more of the eigenvalues of $\Phi$ lies close to the unit circle in the complex plane, then $\nu_z$ in (2.29) could be large even for short horizons $N$. The equivalence of (2.27a), (2.27b) with (2.13a), (2.13b) and (2.14a), (2.14b) implies that the online MPC optimization in (2.17) is equivalent to

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \ J(x_k, \mathbf{c}_k) \ \ \text{subject to} \ \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{Z}. \tag{2.30}$$

which is a quadratic programming problem with $\nu_z n_C$ constraints.

A large value of $\nu_z$ could therefore make the implementation of Algorithm 2.1 computationally demanding. If this is the case, and in particular for applications with very high sampling rates, it may be advantageous to replace the polyhedral invariant set $\mathcal{Z}$ with an ellipsoidal invariant set, $\mathcal{E}_z$:

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \ J(x_k, \mathbf{c}_k) \ \ \text{subject to} \ \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{E}_z. \tag{2.31}$$

This represents a simplification of the online optimization to a problem that involves just a single constraint, thus allowing for significant computational savings. Furthermore, using an ellipsoidal set that is positively invariant for the autonomous prediction dynamics (2.25) and constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z \leq \mathbf{1}$, the resulting MPC law retains the recursive feasibility and stability properties of Algorithm 2.1. Approximating the MPI set $\mathcal{Z}$ (which is by definition maximal) using a smaller ellipsoidal set necessarily introduces suboptimality into the resulting MPC law; but as discussed in Sect. 2.8, the degree of suboptimality is in many cases negligible.

The invariant ellipsoidal set $\mathcal{E}_z$ can be computed offline by solving an appropriate convex optimization problem. The design of these sets is particularly convenient computationally because the conditions for invariance with respect to the linear autonomous dynamics (2.25) and linear constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z_k \leq \mathbf{1}$

may be written in terms of linear matrix inequalities (LMIs), which are necessarily convex and can be handled using semidefinite programming (SDP) [17]. Linear matrix inequalities and the offline optimization of $\mathcal{E}_z$ are considered in more detail in Sect. 2.7.3; here we simply summarize the conditions for invariance of $\mathcal{E}_z$ in the following theorem:

**Theorem 2.9** *The ellipsoidal set defined by $\mathcal{E}_z \doteq \{z : z^T P_z z \le 1\}$ for $P_z \succ 0$ is positively invariant for the dynamics $z_{k+1} = \Psi z_k$ and constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z_k \le \mathbf{1}$ if and only if $P_z$ satisfies*

$$P_z - \Psi^T P_z \Psi \succeq 0 \tag{2.32}$$

*and*

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GE \end{bmatrix} \\ \begin{bmatrix} (F + GK)^T \\ (GE)^T \end{bmatrix} & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, 2, \ldots, n_C \tag{2.33}$$

*for some symmetric matrix $H$, where $e_i$ is the $i$th column of the identity matrix.*

*Proof* The inequality in (2.32) implies $z^T \Psi^T P \Psi z \le z^T P z \le 1$, which is a sufficient condition for invariance of the ellipsoidal set $\mathcal{E}_z$ under $z_{k+1} = \Psi z_k$. Conversely, (2.32) is also necessary for invariance since if $P_z - \Psi^T P_z \Psi \not\succeq 0$, then there would exist $z$ satisfying $z^T \Psi^T P_z \Psi z > z^T P_z z$ and $z^T P_z z = 1$, which would imply that $\Psi z \notin \mathcal{E}_z$ for some $z \in \mathcal{E}_z$.

We next show that (2.33) provides necessary and sufficient conditions for satisfaction of the constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z \le \mathbf{1}$, for all $z \in \mathcal{E}_z$. To simplify notation, let $\tilde{F} \doteq \begin{bmatrix} F + GK & GE \end{bmatrix}$ and let $\tilde{F}_i$ denote the $i$th row of $\tilde{F}$. Since

$$\max_z \bigl\{ \tilde{F}_i z \ \text{ subject to } \ z^T P_z z \le 1 \bigr\} = \bigl( \tilde{F}_i P_z^{-1} \tilde{F}_i^T \bigr)^{1/2}$$

it follows that $\tilde{F} x \le \mathbf{1}$, for all $x \in \mathcal{E}_z$ if and only if $\tilde{F}_i P_z^{-1} \tilde{F}_i^T \le 1$ for each row $i = 1, \ldots, n_C$. These conditions can be expressed equivalently in terms of a condition on a positive-definite diagonal matrix:

$$\begin{bmatrix} H_{1,1} - \tilde{F}_1 P_z^{-1} \tilde{F}_1^T & & \\ & \ddots & \\ & & H_{n_C,n_C} - \tilde{F}_{n_C} P_z^{-1} \tilde{F}_{n_C}^T \end{bmatrix} \succeq 0$$

for some scalars $H_{i,i} \le 1$, $i = 1, \ldots, n_C$, and this in turn is equivalent to

$$H - \tilde{F} P^{-1} \tilde{F}^T \succeq 0$$

for some symmetric matrix $H$ with $e_i^T H e_i \le 1$, for all $i$. Using Schur complements (as discussed in Sect. 2.7.3), this condition is equivalent to

$$\begin{bmatrix} H & \tilde{F} \\ \tilde{F}^T & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots, n_C$$

which implies the necessity and sufficiency of (2.33). □

### *2.7.2 The Predicted Cost and MPC Algorithm*

Given the autonomous form of the prediction dynamics of (2.25) it is possible to use a Lyapunov equation similar to (2.5) to evaluate the predicted cost $J(x_k, \mathbf{c}_k)$ of (2.11) along the predicted trajectories of (2.27a), (2.27b). The stage cost (namely the part of the cost incurred at each prediction time step) has the general form

$$\begin{aligned} \|x\|_Q^2 + \|u\|_R^2 &= \|x\|_Q^2 + \|Kx + c\|_R^2 = x^T(Q + K^T RK)x + \mathbf{c}^T E^T RE\mathbf{c} \\ &= \|z\|_{\hat{Q}}^2, \quad \hat{Q} = \begin{bmatrix} Q + K^T RK & K^T RE \\ E^T RK & E^T RE \end{bmatrix}. \end{aligned}$$

Hence $J(x_k, \mathbf{c}_k)$ can be written as

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \bigl(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\bigr) = \sum_{i=0}^{\infty} \|z_{i|k}\|_{\hat{Q}}^2 = \|z_{0|k}\|_W^2$$

where, by Lemma 2.1, $W$ is the (positive-definite) solution of the Lyapunov equation

$$W = \Psi^T W \Psi + \hat{Q}. \tag{2.34}$$

The special structure of $\Psi$ and $\hat{Q}$ in this Lyapunov equation implies that its solution also has a specific structure, as we describe next.

**Theorem 2.10** *If $K$ is the optimal unconstrained linear feedback gain for the dynamics of (2.1a), then the cost (2.11) for the predicted trajectories of (2.27a), (2.27b) can be written as*

$$\begin{gathered} J(x_k, \mathbf{c}_k) = x_k^T W_x x_k + \mathbf{c}_k^T W_c \mathbf{c}_k \\ W_c = \begin{bmatrix} B^T W_x B + R & 0 & \cdots & 0 \\ 0 & B^T W_x B + R & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & B^T W_x B + R \end{bmatrix} \end{gathered} \tag{2.35}$$

*where $W_x$ is the solution of the Riccati equation (2.9).*

*Proof* Let $W = \begin{bmatrix} W_x & W_{xc} \\ W_{cx} & W_c \end{bmatrix}$, then substituting for $W$, $\Psi$ and $\hat{Q}$ in (2.34) gives

$$W_x = \Phi^T W_x \Phi + Q + K^T R K \tag{2.36a}$$

$$W_{cx} = M^T W_{cx} \Phi + E^T (B^T W_x \Phi + RK) \tag{2.36b}$$

$$W_c = (BE)^T W_x (BE) + (BE)^T W_{xc} M + M^T W_{cx} BE + M^T W_c M + E^T RE \tag{2.36c}$$

The predicted cost for $\mathbf{c}_k = 0$ is $\|x_k\|^2_{W_x}$, and since $K$ is the unconstrained optimal linear feedback gain, it follows from (2.36a) and Theorem 2.1 that $W_x$ is the solution of the Riccati equation (2.9). Furthermore, from Theorem 2.1 we have $K = -(B^T W_x B + R)^{-1} B^T W_x A$, so that $B^T W_x \Phi + RK = 0$ and hence (2.36b) gives $W_{cx} - M^T W_{cx} \Phi = 0$, which implies that $W_{cx} = 0$. Therefore,

$$W = \begin{bmatrix} W_x & 0 \\ 0 & W_c \end{bmatrix}, \tag{2.37}$$

and from (2.36c) we have $W_c - M^T W_c M = E^T (B^T W_x B + R) E$. Hence from the structure of $M$ and $E$ in (2.26b), $W_c$ is given by (2.35). □

**Corollary 2.1** *The unconstrained LQ optimal control law is given by the feedback law $u = Kx$, where $K = -(B^T W_x B + R)^{-1} B^T W_x A$ and $W_x$ is the solution of the Riccati equation (2.9).*

*Proof* Theorem 2.1 has already established that the unconstrained optimal linear feedback gain is as given in the corollary. The question remains as to whether it is possible to obtain a smaller cost by perturbing this feedback law. Equation (2.35) implies that this cannot be the case because the minimum cost is obtained for $\mathbf{c}_k = 0$. This argument applies for arbitrary $N$ and hence for perturbation sequences of any length. □

Using the autonomous prediction system formulation of this section, Algorithm 2.1 can be restated as follows:

**Algorithm 2.2** At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|\mathbf{c}_k\|^2_{W_c} \ \text{subject to} \ \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{S} \tag{2.38}$$

where $\mathcal{S} = \mathcal{Z}$ defined in (2.29) ($\nu_z n_C$ linear constraints), or $\mathcal{S} = \mathcal{E}_z$ defined by the solution of (2.32) and (2.33) (a single quadratic constraint).

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (2.38). $\triangleleft$

**Theorem 2.11** *Under the MPC law of Algorithm 2.2, the origin $x = 0$ of system (2.1a) is an asymptotically stable equilibrium with a region of attraction equal to the set of states that are feasible for the constraints in (2.38).*

*Proof* The constraint set in (2.38) is by assumption positively invariant. Therefore, the tail $\mathbf{c}_{k+1} = M\mathbf{c}^*_k$ provides a feasible but suboptimal solution for (2.38) at time $k + 1$. Stability and asymptotic convergence of $x_k$ to the origin is then shown by applying the arguments of the proofs of Theorems 2.7 and 2.8 to the optimal value of the cost $J(x_k, \mathbf{c}^*_k)$ at the solution of (2.38). $\square$

### 2.7.3 Offline Computation of Ellipsoidal Invariant Sets

In order to determine the invariant ellipsoidal set $\mathcal{E}_z$ for the autonomous prediction dynamics (2.25), the matrices $P_z$ and $H$ must be considered as variables in the conditions of Theorem 2.9. These conditions then constitute Linear Matrix Inequalities (LMIs) in the elements of $P_z$ and $H$. Linear matrix inequalities are used extensively throughout this book; for an introduction to the properties of LMIs and LMI-based techniques that are commonly used in systems analysis and control design problems, we refer the reader to [17].

In its most general form a linear matrix inequality is a condition on the positive definiteness of a linear combination of matrices, where the coefficients of this combination are considered as variables. Thus a (strict) LMI in the variable $x \doteq (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be expressed

$$M(x) \doteq M_0 + M_1 x_1 + \ldots + M_n x_n \succ 0 \tag{2.39}$$

where $M_0, \ldots, M_n$ are given matrices.[2] The convenience of LMIs lies in the convexity of (2.39) (see also Questions 1–3 on page 233). This property makes it possible to include conditions, such as those defining an invariant ellipsoidal set in Theorem 2.9, in convex optimization problems that can be solved efficiently using semidefinite programming.

[2] A non-strict LMI is similarly defined by $M(x) \succeq 0$. Any non-strict LMI can be expressed equivalently as a combination of a linear equality constraint and a strict LMI (see, e.g. [17]). However, none of the non-strict LMIs encountered in this chapter or in Chap. 5 carry implicit equality constraints, and hence non-strict LMIs may be assumed to be either strictly feasible or infeasible. We therefore make use of both strict and non-strict LMIs with the understanding that $M(x) \succeq 0$ can be replaced with $M(x) \succ 0$ for the purposes of numerical implementation.

A suitable criterion for selecting $P_z$ is to maximize the region of attraction of Algorithm 2.2, namely the feasible set for the constraint $z_k^T P_z z_k \leq 1$. This region is equal to the projection of $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ onto the $x$-subspace:

$$\{x : \exists\, \mathbf{c} \text{ such that } x^T P_{xx} x + 2\mathbf{c}^T P_{cx} x + \mathbf{c}^T P_{cc} \mathbf{c} \leq 1\}$$

where the matrices $P_{xx}$, $P_{xc}$, $P_{cx}$, $P_{cc}$ are blocks of $P_z$ partitioned according to

$$P_z = \begin{bmatrix} P_{xx} & P_{xc} \\ P_{cx} & P_{cc} \end{bmatrix}. \tag{2.40}$$

By considering the minimum value of $z^T P_z z$ over all $\mathbf{c}$ for given $x$, it is easy to show that the projection of $\mathcal{E}_z$ onto the $x$-subspace is given by

$$\mathcal{E}_x \doteq \left\{x : x^T (P_{xx} - P_{xc} P_{cc}^{-1} P_{xc}) x \leq 1\right\}.$$

Inverting the partitioned matrix $P_z$ we obtain

$$P_z^{-1} = S \doteq \begin{bmatrix} S_{xx} & S_{xc} \\ S_{cx} & S_{cc} \end{bmatrix},$$

where

$$S_{xx} = \left(P_{xx} - P_{xc} P_{cc}^{-1} P_{xc}\right)^{-1},$$

and hence the volume of the projected ellipsoidal set $\mathcal{E}_x$ is proportional to $1/\det(S_{xx}^{-1}) = \det(S_{xx})$. The volume of the region of attraction of Algorithm 2.2 is therefore maximized by the optimization

$$\underset{S, P_z, H}{\text{maximize}} \ \det(S_{xx}) \text{ subject to (2.32), (2.33)} \tag{2.41}$$

Maximizing the objective in (2.41) is equivalent to maximizing $\log\det(S_{xx})$, which is a concave function of the elements of $S$ (see, e.g. [18]). But this is not yet a semidefinite programming problem since (2.32) and (2.33) are LMIs in $P_z$ rather than $S$. These constraints can however be expressed as Linear Matrix Inequalities in $S$ using Schur complements.

In particular, the positive definiteness of a partitioned matrix

$$\begin{bmatrix} U & V^T \\ V & W \end{bmatrix} \succ 0$$

where $U$, $V$, $W$ are real matrices of conformal dimensions, is equivalent to positive definiteness of the Schur complements

$$U \succ 0 \ \text{ and } \ W - VU^{-1}V^T \succ 0,$$

or

$$W \succ 0 \ \text{ and } \ U - V^T W^{-1} V \succ 0$$

(the proof of this result is discussed in Question 1 in Chap. 5 on page 233). Therefore, after pre- and post-multiplying (2.32) by $S$, using Schur complements we obtain the following condition:

$$\begin{bmatrix} S & \Psi S \\ S\Psi^T & S \end{bmatrix} \succeq 0, \tag{2.42}$$

which is an LMI in $S$. Similarly, pre- and post-multiplying the matrix inequality in (2.33) by $\left[\begin{smallmatrix} I & 0 \\ 0 & S \end{smallmatrix}\right]$ yields the condition

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GE \end{bmatrix} S \\ S \begin{bmatrix} (F+GK)^T \\ (GE)^T \end{bmatrix} & S \end{bmatrix} \succeq 0 \tag{2.43}$$

which is an LMI in $S$ and $H$. Therefore $\mathcal{E}_z$ can be computed by solving the SDP problem

$$\begin{aligned} \underset{S,H}{\text{maximize}} \ \ \log\det(S_{xx}) \ \ \text{subject to} \ \ & (2.42), (2.43) \\ \text{and } & e_i^T H e_i \le 1, \ i = 1, \ldots, n_C. \end{aligned} \tag{2.44}$$

*Example 2.4* For the system model, constraints and cost of Example 2.1, Fig. 2.7 shows the ellipsoidal regions of attraction $\mathcal{E}_x$ of Algorithm 2.2 for values of $N$ in the range 5–40 and compares these with the polytopic feasible set $\mathcal{F}_N$ for $N = 10$. As expected, the ellipsoidal feasible sets are smaller than the polytopic feasible sets of Fig. 2.3, but the difference in area is small; the area of $\mathcal{E}_x$ for $N = 40$ is 13.4 while that of $\mathcal{F}_{10}$ is 13.6, a difference of only 1 %. On the other hand 36 linear constraints are needed to define the polytopic set $\mathcal{Z}$ for $N = 10$ whereas $\mathcal{E}_z$ is a single (quadratic) constraint.

Figure 2.8 shows closed-loop state and input responses for Algorithm 2.2, comparing the responses obtained with the ellipsoidal constraint $z_k \in \mathcal{E}_z$ against the responses obtained with the linear constraint set $z_k \in \mathcal{Z}$ for $N = 10$. The difference in the closed-loop costs of the two controllers for the initial condition $x_0 = (-7.5, 0.5)$ is 17 %. ◊

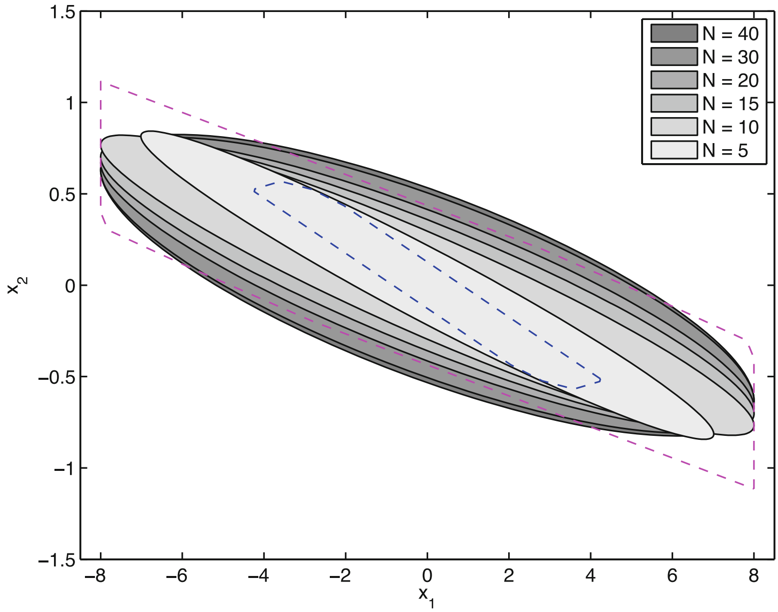


**Fig. 2.7** The ellipsoidal regions of attraction of Algorithm 2.2 for $N = 5, 10, 15, 20, 30, 40$. The polytopic sets $\mathcal{F}_{10}$ and $\mathcal{X}_T$ are shown (*dashed lines*) for comparison

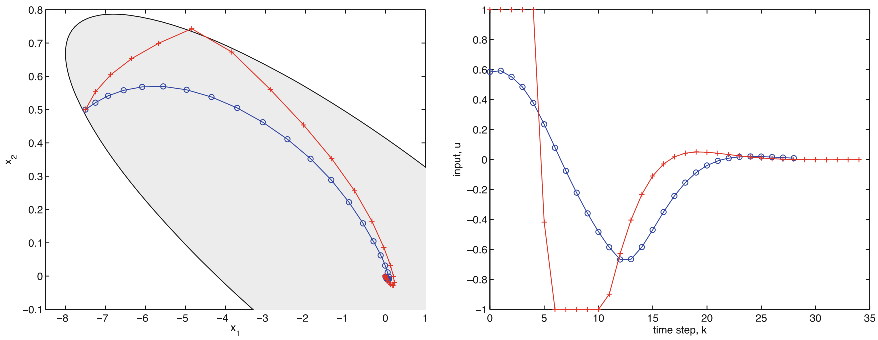


**Fig. 2.8** Closed-loop responses of Algorithm 2.2 for the example of (2.16a), (2.16b) for the quadratic constraint $z_k \in \mathcal{E}_z$ with $N = 20$ (*blue* o) and the linear constraints $z_k \in \mathcal{Z}$ with $N = 10$ (*red* +). *Left* state trajectories and the feasible set $\mathcal{E}_x$ for $N = 20$. *Right* control inputs

## 2.8 Computational Issues

The optimization problem to be solved online in Algorithm 2.1 has a convex quadratic objective function and linear constraints, and is therefore a convex Quadratic Program (QP). Likewise if Algorithm 2.2 is formulated in terms of linear constraints, then this also requires the online solution of a convex QP problem. A variety of general QP solvers (based on active set methods [19] or interior point methods [20]), can therefore be used to perform the online MPC optimization required by these algorithms.

However algorithms for general quadratic programming problems do not exploit the special structure of the MPC problem considered here, and as a result their computational demand may exceed allowable limits. In particular they may not be applicable to problems with high sample rates, high-dimensional models, or long prediction horizons. For example the computational load of both interior point and active set methods grows approximately cubically with the mode 1 prediction horizon $N$.

The rate of growth with $N$ of the required computation can be reduced however if the predicted model states are considered to be optimization variables. Thus redefining the vector of degrees of freedom as $d_k \in \mathbb{R}^{Nn_x+Nn_u}$:

$$d_k = (c_{0|k}, x_{1|k}, c_{1|k}, x_{2|k}, \ldots, c_{N-1|k}, x_{N|k})$$

and introducing the predicted dynamics of (2.14) as equality constraints results in an online optimization of the form

$$\underset{d_k}{\text{minimize}}\ d_k^T H_d d_k \ \text{ subject to }\ D_d d_k = h_h,\ C_c d_k \le h_c.$$

Although the number of optimization variables has increased from $Nn_u$ to $Nn_u + Nn_x$, the key benefit is that the matrices $H_d$, $D_d$, $C_c$ are sparse and highly structured. This structure can be exploited to reduce the online computation so that it grows only linearly with $N$ (e.g. see [19, 20]).

An alternative to reducing the online computation is to use multiparametric programming to solve the optimization problem offline for initial conditions that lie in different regions of the state space. Thus, given that $x_k$ is a known constant, the minimization of the cost of (2.35) is equivalent to the minimization of

$$J(d) = d^T H_0 d \tag{2.45}$$

where for simplicity, the vector of degrees of freedom $\mathbf{c}$ has been substituted by $d$ and the cost is renamed as simply $J$. The minimization of $J$ is subject to the linear constraints implied by the dynamics (2.14) and system constraints (2.2), together with the terminal constraints of (2.35); the totality of these constraints can be written as

$$C_0 d \le h_0 + V_0 x \tag{2.46}$$

Then adjoining the constraints (2.46) with the cost of (2.45) through the use of a vector of Lagrange multipliers $\lambda$, we obtain the first-order Karush–Kuhn–Tucker (KKT) conditions [19]

$$H_0 d + C_0^T \lambda = 0 \tag{2.47a}$$

$$\lambda^T (C_0 d - h_0 - V_0 x) = 0 \tag{2.47b}$$

$$C_0 d \leq h_0 + V_0 x \tag{2.47c}$$

$$\lambda \geq 0 \tag{2.47d}$$

Now suppose that at the given $x$ only a subset of (2.46) is active, so that gathering all these active constraints and the corresponding Lagrange multipliers we can write

$$\tilde{C}_0 d - \tilde{h}_0 - \tilde{V}_0 x = 0 \tag{2.48a}$$

$$\tilde{\lambda} \geq 0 \tag{2.48b}$$

In addition, the Lagrange multipliers corresponding to inactive constraints will be zero so that from (2.47) it follows that

$$d = -H_0^{-1} \tilde{C}_0^T \tilde{\lambda}. \tag{2.49}$$

The solution for $\tilde{\lambda}$ can be derived by substituting (2.49) into (2.48a) as

$$\tilde{\lambda} = -(\tilde{C}_0 H_0^{-1} \tilde{C}^T)^{-1} (\tilde{h}_0 + \tilde{V}_0 x). \tag{2.50}$$

and substituting this into (2.49) produces the optimal solution as

$$d = H_0^{-1} \tilde{C}_0^T - (\tilde{C}_0 H_0^{-1} \tilde{C}_0^T)^{-1} (\tilde{h}_0 + \tilde{V}_0 x). \tag{2.51}$$

Thus for given active constraints, the optimal solution is a known affine function of the state. Clearly the optimal solution must satisfy the constraints (2.46) as well as the Lagrange multipliers of (2.50) must satisfy (2.48a):

$$C_o [H_o^{-1} \tilde{C}_o^T - (\tilde{C}_o H_o^{-1} \tilde{C}_o^T)^{-1} (\tilde{h}_o + \tilde{V}_o x)] \leq h_o + V_o x$$

and

$$-(\tilde{C}_o H_o^{-1} \tilde{C}_o^T)^{-1} (\tilde{h}_o + \tilde{V}_o x) > 0.$$

These two conditions give a characterization of the polyhedral region in which $x$ must lie in order that (2.48a) is the active constraint set.

A procedure based on these considerations is given in [14] for partitioning the controllable set of Algorithms 2.1 and 2.2 into the union of a number of non-overlapping polyhedral regions. Then the MPC optimization can be implemented online by identifying the particular polyhedral region in which the current state lies. In this approach the associated optimal solution (2.51) is then recovered from a lookup table, and the first element of this is used to compute and implement the current optimal control input.

A disadvantage of this multiparametric approach is that the number of regions grows exponentially with the dimension of the state and the length of the mode 1 prediction horizon $N$, and this can make the approach impractical for anything other than small-scale problems with small values of $N$. Indeed in most other cases, the computational and storage demands of the multiparametric approach exceed those required by the QP solvers that exploit the MPC structure described above. Methods have been proposed (e.g. [21]) for improving the efficiency with which the polyhedral state-space partition is computed by merging regions that have the same control law, however the complexity of the polyhedral partition remains prohibitive in this approach.

*Example 2.5* For the second-order system defined in (2.16a), (2.16b), with the cost and terminal constraints of Example 2.3 the MPC optimization problem (2.17) can be solved using multiparametric programming. For a mode 1 horizon of $N = 10$ this results in a partition of the state space into 243 polytopic regions (Fig. 2.9), each of which corresponds to a different active constraint set at the solution of the MPC optimization problem (2.17). $\diamondsuit$

A further alternative [15, 16] which results in significant reduction in the online computation replaces the polytopic constraints $z_k \in \mathcal{Z}$ defined (2.29) by the ellipsoidal constraint $z_k \in \mathcal{E}_z$ defined in (2.44) and thus addresses the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|z_k\|_W^2 \quad \text{subject to} \ \ z_k^T P_z z_k \le 1, \quad z_k = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \tag{2.52}$$

As discussed in Sect. 2.7, this results in a certain degree of conservativeness because the ellipsoidal constraint $z_k \in \mathcal{E}_z$ gives an inner approximation to the polytopic constraint $z_k \in \mathcal{Z}$ of (2.29). The problem defined in (2.52) can be formulated as a second-order cone program (SOCP) in $Nn_u + 1$ variables.[3] If a generic solution method is employed, then this problem could turn out to be more computationally demanding than the QP that arises when the constraints are linear. However, the simple form of the cost and constraint in (2.38) allow for a particularly efficient solution, which is to be discussed next.

[3] Second-order cone programs are convex optimization problems that can be solved using interior point methods. See [22] for details and further applications of SOCP.

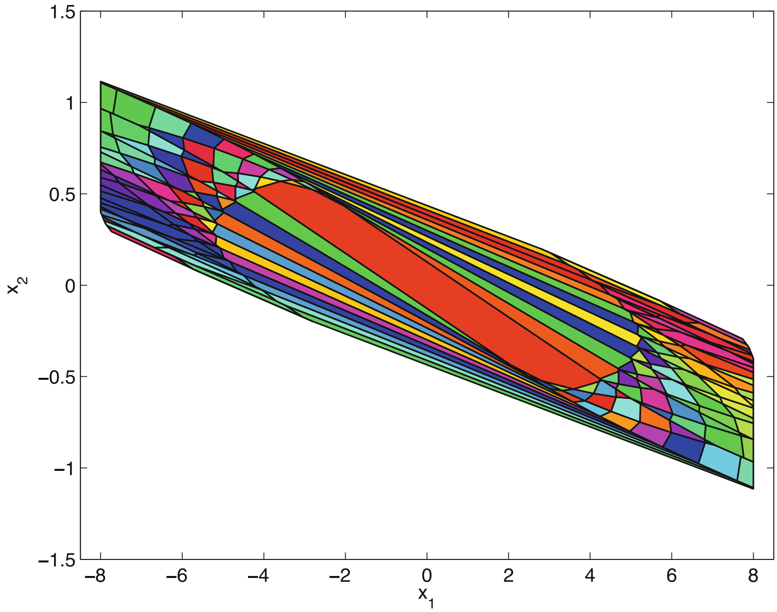


**Fig. 2.9** The partition of the state space of the system of Example 2.5 into regions in which different constraint sets are active at the solution of the online MPC optimization problem

To exploit the structure of the cost and constraint in (2.52), we use the partitions of (2.37) and (2.40) to write $z_k^T W z_k = x_k^T W_x x + \mathbf{c}_k^T W_c \mathbf{c}_k$ and $z_k^T P_z z_k = x_k^T P_{xx} x_k + 2\mathbf{c}_k^T P_{cx} x_k + \mathbf{c}_k^T P_{cc} \mathbf{c}_k$, where use has been made of the fact that $P_{xc} = P_{cx}^T$. The minimizing value of $\mathbf{c}_k$ in (2.52) can only occur at a point at which the two ellipsoidal boundaries, $\partial\mathcal{E}_J \doteq \{z_k : z_k^T W z_k = \alpha\}$ and $\partial\mathcal{E}_z \doteq \{z_k : z_K^T P_z z_k = 1\}$, are tangential to one another for some constant $\alpha > 0$, namely when the gradients (with respect to $\mathbf{c}$) are parallel, i.e.

$$W_c \mathbf{c}_k = \mu(P_{cx} x_k + P_{cc} \mathbf{c}_k), \quad \mu \le 0 \tag{2.53}$$

for some scalar $\mu$, or equivalently

$$\mathbf{c}_k = \mu M_\mu P_{cx} x_k, \quad M_\mu = (W_c - \mu P_{cc})^{-1}. \tag{2.54}$$

At the solution therefore, the inequality constraint in (2.52) will hold with equality so that $\mu$ can be obtained as the solution of $x_k^T P_{xx} x_k + 2\mathbf{c}_k^T P_{cx} + \mathbf{c}_k^T P_{cc} \mathbf{c}_k = 1$ , which after some algebraic manipulation gives $\mu$ as a root of

$$\phi(\mu) = x_k^T P_{xc} \left( M_\mu W_c P_{cc}^{-1} W_c M_\mu - P_{cc}^{-1} \right) P_{cx} x_k + x_k^T P_{xx} x_k - 1 = 0. \tag{2.55}$$

Equation (2.55) is equivalent to a polynomial equation in $\mu$ which can be shown (using straightforward algebra) to have $2N$ roots, all corresponding to points of tangency of $\partial\mathcal{E}_J$ and $\partial\mathcal{E}_z$. However (2.52) has a unique minimum, and it follows that only one of these roots can be negative, as is required by (2.53).

By repeatedly differentiating $\phi(\mu)$ with respect to $\mu$ it is easy to show that the derivatives of this polynomial satisfy

$$\frac{d^r\phi}{d\mu^r} > 0 \ \ \forall\mu \le 0.$$

This implies that the Newton–Raphson method, when initialised at $\mu = 0$, is guaranteed to converge to the unique negative root of (2.55), and that the rate of its convergence is quadratic.

Thus the optimal solution to (2.52) is obtained extremely efficiently by substituting the negative root of (2.55) into (2.54); in fact the computation required is equivalent to solving a univariate polynomial with monotonic derivatives. The price that must be paid for this gain in computational efficiency is a degree of suboptimality that results from the use of the ellipsoidal constraint $z_k \in \mathcal{E}_z$, which provides only an inner approximation to the actual polytopic constraint of (2.29). However, simulation results [16] show that in most cases the degree of suboptimality is not significant. Furthermore predicted performance can be improved by a subsequent univariate search over $\alpha \in [0, 1]$ with $z_k = (x_k, \alpha\mathbf{c}_k^*)$ where $\mathbf{c}_k^*$ is the solution of (2.52). To retain the guarantee of closed-loop stability this is performed subject to the constraints that the vector $\Psi z_k$ defining the tail of the predicted sequence at time $k$ should lie in the ellipsoid $\mathcal{E}_z$ and subject to the constraint $Fx_k + Gu_k \le 1$. This modification requires negligible additional computation.

## 2.9 Optimized Prediction Dynamics

The MPC algorithms described thus far parameterize the predicted inputs in terms of a projection onto the standard basis vectors $e_i$, so, for example

$$\mathbf{c}_k = \sum_{i=0}^{N-1} c_{i|k}e_{i+1}$$

in the case that if $n_u = 1$. As a consequence the degrees of freedom have a direct effect on the predictions only over the $N$-step mode 1 prediction horizon, which therefore has to be taken to be sufficiently long to ensure that constraints are met during the transients of the prediction system response. Combined with the additional requirement that the terminal constraint is met at the end of the mode 1 horizon for as large a set of initial conditions as possible, this places demands on $N$ that can make the computational load of MPC prohibitive for applications with high sampling rates.

To overcome this problem an extra mode can be introduced into the predicted control trajectories, as is done for example in triple mode MPC [23]. This additional mode introduces degrees of freedom into predictions after the end of the mode 1 horizon but allows efficient handling of the constraints at these prediction instants, thus allowing the mode 1 horizon to be shortened without adversely affecting optimality and the size of the feasible set. Alternatively in the context of dual-mode predictions it is possible to consider parameterizing predicted control trajectories as an expansion over a finite set of basis functions. Exponential basis functions, which allow the use of arguments based on the tail for analysing stability and convergence (e.g. [24]), are most commonly employed in MPC, a special case being expansion over Laguerre functions (e.g. [25]).

A framework that encompasses projection onto a general set of exponential basis functions was developed in [26]. In this approach, the matrices $E$ and $M$ appearing in the transition matrix $\Psi$ of the augmented prediction dynamics (2.25) are not chosen as prescribed by (2.26b), but instead are replaced by variables, denoted $A_c$ and $C_c$ that are optimized offline as we discuss later in this section. With this modification the prediction dynamics are given by

$$z_{i+1|k} = \Psi z_{i|k}, \quad i = 0, 1, \ldots \tag{2.56a}$$

where

$$z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Phi & BC_c \\ 0 & A_c \end{bmatrix} \tag{2.56b}$$

and the predicted state and control trajectories are generated by

$$u_{i|k} = \begin{bmatrix} K & C_c \end{bmatrix} z_{i|k} \tag{2.56c}$$

$$x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} z_{i|k}. \tag{2.56d}$$

As in Sect. 2.7, the predicted control law of (2.56c) has the form of a dynamic feedback controller, the initial state of which is given by $\mathbf{c}_k$. However in Sect. 2.7 the matrix $M$ of (2.26) is nilpotent, so that $M^N \mathbf{c}_k = 0$ and hence $u_{i|k} = Kx_{i|k}$, for all $i = N, N+1, \ldots$. For the general case considered in (2.56), $A_c$ is not necessarily nilpotent, which implies that the direct effect of the elements of $\mathbf{c}_k$ can extend beyond the initial $N$ steps of the prediction horizon in this setting.

Following a development analogous to that of Sect. 2.7, the predicted cost (2.11) can be expressed as $J(x_k, \mathbf{c}_k) = \|z_{0|k}\|_W^2$ where $W$ satisfies the Lyapunov matrix equation

$$W = \Psi^T W \Psi + \hat{Q}, \quad \hat{Q} = \begin{bmatrix} Q + K^T RK & K^T RC_c \\ C_c^T RK & C_c^T RC_c \end{bmatrix}. \tag{2.57}$$

By examining the partitioned blocks of this equation, it can be shown (using the same approach as the proof of Theorem 2.10) that its solution is block diagonal

$$W = \begin{bmatrix} W_x & 0 \\ 0 & W_c \end{bmatrix}$$

whenever $K$ is the unconstrained optimal feedback gain. Here $W_x$ is the solution of the Riccati equation (2.9) and $W_c$ is the solution of the Lyapunov equation $W_c = A_c^T W A_c + C_c^T (B^T W_x B + R) C_c$. By Lemma 2.1, the solution is unique and satisfies $W_c \succ 0$ whenever $A_c$ is strictly stable.

The constraints (2.2) applied to the predictions of (2.56) require that $z_{0|k}$ lies in the polytopic set

$$\mathcal{Z} = \{z : \begin{bmatrix} F + GK & GC_c \end{bmatrix} \Psi^i z \leq \mathbf{1}, \quad i = 0, 1, \ldots, \nu_z\}, \tag{2.58}$$

where $\begin{bmatrix} F + GK & GC_c \end{bmatrix} \Psi^{\nu_z + 1} z \leq \mathbf{1}$, for all $z$ satisfying $\begin{bmatrix} F + GK & GC_c \end{bmatrix} \Psi^i z \leq \mathbf{1}$, $i = 0, 1, \ldots, \nu_z$. By Theorem 2.3 this is the MPI set for the dynamics of (2.56) and constraints (2.2), and its projection onto the $x$-subspace is therefore equal to the feasible set for $x_k$ for the prediction system (2.56) and constraints $\begin{bmatrix} F + GK & GC_c \end{bmatrix} z \leq \mathbf{1}$. The MPC law of Algorithm 2.2 with the cost matrix $W$ defined in (2.57) and constraint set $\mathcal{Z}$ defined in (2.58) has the stability and convergence properties stated in Theorem 2.11.

Alternatively, and similarly to the discussion in Sect. 2.7, it is possible to replace the linear constraints $z_{0|k} \in \mathcal{Z}$ by a single quadratic constraint $z_{0|k} \in \mathcal{E}_z$ in order to reduce the online computational load of Algorithm 2.2. As in Sect. 2.7, we require that $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ is positively invariant for the dynamics $z_{k+1} = \Psi z_k$ and constraints $\begin{bmatrix} F + GK & GC_c \end{bmatrix} z_k \leq \mathbf{1}$, which by Theorem 2.9 requires that there exists a symmetric matrix $H$ such that $P_z$, $A_c$ and $C_c$ satisfy

$$P_z - \Psi^T P_z \Psi \succeq 0 \tag{2.59a}$$

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GC_c \end{bmatrix} \\ \begin{bmatrix} (F + GK)^T \\ (GC_c)^T \end{bmatrix} & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots n_C. \tag{2.59b}$$

Under these conditions the stability and convergence properties specified by Theorem 2.11 again apply.

Using $\mathcal{E}_z$ as the constraint set in the online optimization in place of $\mathcal{Z}$ reduces the region of attraction of the MPC law. However, to compensate for this effect it is possible to design the prediction system parameters $A_c$ and $C_c$ so as to maximize the projection of $\mathcal{E}_z$ onto the $x$-subspace. Analogously to (2.44), this is achieved by maximizing the determinant of $[I_{n_x} \ \ 0] P_z^{-1} [I_{n_x} \ \ 0]^T$ subject to (2.59a), (2.59b). Unlike the case considered in Sect. 2.7, this is performed with $A_c$ and $C_c$ as optimization variables. Viewed as inequalities in these variables, (2.59a), (2.59b) represent non-convex constraints. The problem can however be convexified provided the dimension of $\mathbf{c}_k$ is at least as large as that of $n_x$ [26] using a technique introduced by [27] in the context of $\mathcal{H}_\infty$ control, as we discuss next.

Introducing variables $U, V \in \mathbb{R}^{n_x \times \nu_c}$ (where $\nu_c$ is the length of $\mathbf{c}_k$), $\Xi \in \mathbb{R}^{n_x \times n_x}$, $\Gamma \in \mathbb{R}^{n_u \times n_x}$ and symmetric $X, Y \in \mathbb{R}^{n_x \times n_x}$, we re-parameterize the problem by defining

$$P_z = \begin{bmatrix} X^{-1} & X^{-1}U \\ U^T X^{-1} & \bullet \end{bmatrix} \quad P_z^{-1} = \begin{bmatrix} Y & V \\ V^T & \bullet \end{bmatrix}, \quad \Xi = U A_c V^T, \quad \Gamma = C_c V^T \tag{2.60}$$

(where $\bullet$ indicates blocks of $P_z$ and $P_z^{-1}$ that are determined uniquely by $X$, $Y$, $U$, $V$). Since $P_z P_z^{-1} = I$, we also require that

$$U V^T = X - Y. \tag{2.61}$$

The constraints (2.59a), (2.59b) can then be expressed as LMIs in $\Xi$, $\Gamma$, $X$ and $Y$. Specifically, using Schur complements, (2.59a) is equivalent to

$$\begin{bmatrix} P_z & P_z \Psi \\ \Psi^T P_z & P_z \end{bmatrix} \succeq 0,$$

and multiplying the LHS of this inequality by $\operatorname{diag}\{\Pi^T, \Pi^T\}$ on the left and $\operatorname{diag}\{\Pi, \Pi\}$ on the right, where $\Pi = \begin{bmatrix} Y & X \\ V^T & 0 \end{bmatrix}$, yields the equivalent condition

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi Y + B\Gamma & \Phi X \\ \Xi + \Phi Y + B\Gamma & \Phi X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0 \tag{2.62a}$$

(where the block marked $\star$ is omitted as the matrix is symmetric). Similarly, pre- and post-multiplying the matrix inequality in (2.59b) by $\operatorname{diag}\{I, \Pi^T\}$ and $\operatorname{diag}\{I, \Pi\}$, respectively, yields

$$\begin{bmatrix} H & \begin{bmatrix} (F + GK)Y + G\Gamma & (F + GK)X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, \ldots, n_C. \tag{2.62b}$$

Therefore matrices $P_z$, $A_c$ and $C_c$ can exist satisfying (2.59a), (2.59b) only if the conditions (2.62a), (2.62b) are feasible. Moreover, (2.62a), (2.62b) are both necessary and sufficient for feasibility of (2.59a), (2.59b) if $\nu_c \geq n_x$ since (2.61) then imposes no additional constraints on $X$ and $Y$ (in the sense that $U$ and $V$ then exist satisfying (2.61), for all $X, Y \in \mathbb{R}^{n_x \times n_x}$). The volume of the projection of $\mathcal{E}_z$ onto the $x$-subspace is proportional to $\det(Y)$, which is maximized by solving the convex optimization:

$$\underset{\Xi,\Gamma,X,Y}{\text{maximize}} \ \log\det(Y) \ \text{ subject to (2.62a), (2.62b).} \tag{2.63}$$

Finally, we note that the conditions (2.62a), (2.62b) do not depend on the value of $\nu_c$, and since there is no advantage to be gained using a larger value, we set $\nu_c = n_x$. From the solution of (2.63), $A_c$ and $C_c$ are given uniquely by

$$A_c = U^{-1}\Xi V^{-T}, \quad C_c = \Gamma V^{-T}.$$

while $P_z$ can be recovered from (2.60).

A remarkable property of the optimized prediction dynamics is that the maximal projection of $\mathcal{E}_z$ onto the $x$-subspace is as large as the maximal positively invariant ellipsoidal set under any linear state feedback control law [26]. The importance of this is that it overcomes the trade-off that exists in the conventional MPC formulations of Sects. 2.7 and 2.5 between performance and the size of the feasible set. Thus, in the interests of enlarging the terminal invariant set (and hence the overall region of attraction), it may be tempting to de-tune the terminal control law. But this has an adverse effect on predicted performance, and potentially also reduces closed-loop performance. Such loss of performance is however avoided if the optimized prediction dynamics are used since $K$ can be chosen to be the unconstrained LQ optimal gain, without any detriment to the size of the region of attraction.

*Example 2.6* The maximal ellipsoidal region of attraction of Algorithm 2.2 for the same system model, constraints and cost as Example 2.1 is shown in Fig. 2.10. Since this is obtained by optimizing the prediction dynamics using (2.63), the number of degrees of freedom in the resulting prediction system (i.e. the length of $\mathbf{c}_k$ in (2.56)) is the same as $n_x$, which here is 2. The area of this maximal ellipsoid is 13.5, whereas the area of the ellipsoidal region of attraction obtained from (2.44) for the non-optimized prediction system (2.25) and the same number of degrees of freedom in predictions (i.e. $N = 2$) is just 2.3.

Figure 2.10 also shows the polytopic feasible set for $x_k$ in Algorithm 2.2 when the optimized prediction dynamics are used to define the polytopic constraint set $\mathcal{Z}$ in (2.58). Despite having only 2 degrees of freedom, the optimized prediction dynamics result in a polytopic feasible set covering 97 % of the area of the maximal feasible set $\mathcal{F}_\infty$, which for this example is equal to the polytopic feasible set for the non-optimized dynamics with $N = 26$ degrees of freedom (also shown in Fig. 2.10). For the initial condition $x_0 = (-7.5, 0.5)$, the closed-loop cost of Algorithm 2.2 with the optimized prediction dynamics containing 2 degrees of freedom and polytopic constraint set $\mathcal{Z}$ is 357.7, which from Table 2.1 is only 0.5 % suboptimal relative to the ideal optimal cost with $N = 11$. $\diamondsuit$

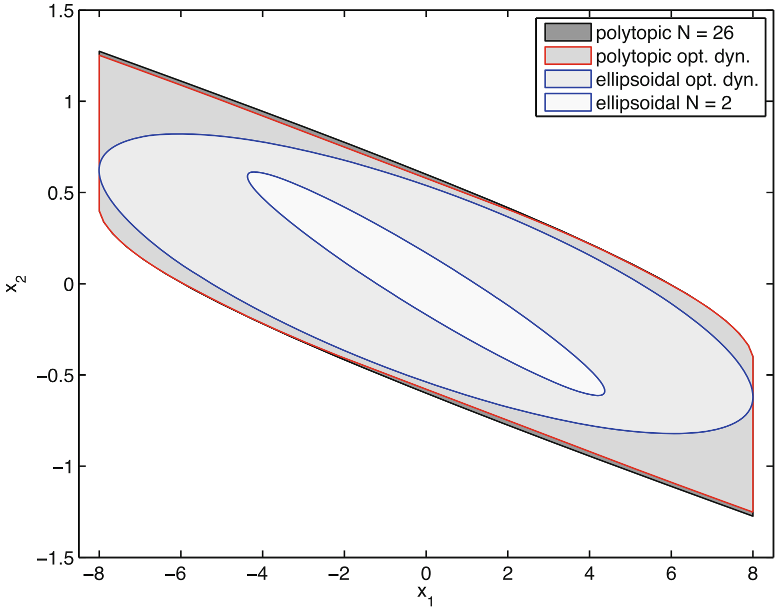


**Fig. 2.10** Ellipsoidal region of attraction for optimized dynamics (with 2 degrees of freedom) and ellipsoidal region of attraction for $N = 2$. Also shown are the maximal polytopic region of attraction ($\mathcal{F}_{26}$) and the polytopic region of attraction for the optimized dynamics

## 2.10 Early MPC Algorithms

Perhaps the earliest reference to MPC strategies is [28], although the ideas of rolling horizons and decision making based on forecasts had been used earlier in different contexts (e.g. production scheduling). There have since been thousands of MPC papers published in the open literature, including a plethora of reports on applications of MPC to industrial problems. Early contributions (e.g. [29, 30]) were based on finite horizon predictive costs and as such did not carry guarantees of closed-loop stability.

The most cited of the early papers on predictive control is the seminal work [31, 32] on Generalized Predictive Control (GPC). This uses an input–output model to express the vector of output predictions as an affine function of the vector of predicted inputs

$$\mathbf{y}_k = \begin{bmatrix} y_{1|k} \\ \vdots \\ y_{N|k} \end{bmatrix} = C_G \Delta \mathbf{u}_k + \mathbf{y}_k^f, \quad \Delta \mathbf{u}_k = \begin{bmatrix} \Delta u_{0|k} \\ \vdots \\ \Delta u_{N_u-1|k} \end{bmatrix}$$

Here $N_u$ denotes an input prediction horizon which is chosen to be less than or equal to the prediction horizon $N$. The matrix $C_G$ is the block striped (Toeplitz) lower triangular matrix comprising the coefficients of the system step response, $C_G \Delta \mathbf{u}_k$ denotes the predicted forced response at time $k$, and $\mathbf{y}_k^f$ denotes the free response at time $k$ due to non-zero initial conditions. The notation $\Delta u$ is used to denote the control increments (i.e. $\Delta u_{i|k} = u_{i|k} - u_{i-1|k}$). Posing the problem in terms of control increments implies the automatic inclusion in the feedback loop of integral action which rejects (in the steady state) constant additive disturbances.

The GPC algorithm minimizes a cost, subject to constraints, which penalizes predicted output errors (deviations from a constant reference vector $r$) and predicted control increments

$$J_k = (\mathbf{r} - \mathbf{y}_k)^T \hat{Q} (\mathbf{r} - \mathbf{y}_k) + \Delta \mathbf{u}_k^T \hat{R} \Delta \mathbf{u}_k \tag{2.64}$$

where $\mathbf{r} = [r^T \ \cdots \ r^T]^T$, $\hat{Q} = \operatorname{diag}\{Q, \ldots, Q\}$ and $\hat{R} = \operatorname{diag}\{R, \ldots, R\}$. By setting the derivative of this cost with respect to $\Delta \mathbf{u}_k$ equal to zero, the unconstrained optimum vector of predicted control increments can be derived as

$$\Delta \mathbf{u}_k = \left( C_G^T \hat{Q} C_G + \hat{R} \right)^{-1} C_G^T \hat{Q} (\mathbf{r} - \mathbf{y}_k^f) \tag{2.65}$$

The optimal current control move $\Delta u_{0|k}$ is then computed from the first element of this vector, and the control input $u_k = \Delta u_{0|k} + u_{k-1}$ is applied to the plant.

GPC has proven effective in a wide range of applications and is the basis of a number of commercially successful MPC algorithms. There are several reasons for the success of the approach, principal among these are: the simplicity and generality of the plant model, and the lack of sensitivity of the controller to variable or unknown plant dead time and unknown model order; the fact that the approach lends itself to self-tuning and adaptive control, output feedback control and stochastic control problems; and the ability of GPC to approximate various well-known control laws through appropriate definition of the cost (2.64), for example LQ optimal control, minimum variance and dead-beat control laws. For further discussion of these aspects of GPC and its industrial applications we refer the reader to [31–34].

Although widely used in industry, the original formulation of GPC did not guarantee closed-loop stability except in limiting cases of the input and output horizons (for example, in the limit as both the prediction and control horizons tend to infinity, or when the control horizon is $N_u = 1$, the prediction horizon is $N = \infty$ and the open-loop system is stable). However, the missing stability guarantee can be established by imposing a suitable terminal constraint on predictions.

Terminal equality constraints that force the predicted tracking errors to be zero at all prediction times beyond the $N$-step prediction horizon were proposed for receding horizon controllers in the context of continuous time, time-varying unconstrained systems in [35], time invariant discrete time unconstrained systems [36], and nonlinear constrained systems [37]. This constraint effectively turns the cost of (2.64) into an infinite horizon cost which can be shown to be monotonically non-increasing using an argument based on the prediction tail. As a result it can be shown that tracking errors are steered asymptotically to zero. The terminal equality constraint need only to be applied over $n_x$ prediction steps after the end of an initial $N$-step horizon. Under the assumption that $N > n_x$, the general solution of the equality constraints will contain, implicitly, $(N - n_x)n_u$ degrees of freedom and these can be used to minimize the resulting predicted cost (i.e. the cost of (2.64) after the expression for the general solution of the equality constraints has been substituted into (2.64)). A closely related algorithm to GPC that addresses the case of constrained systems is Stable GPC (SGPC) [38], which establishes closed-loop stability by ensuring that optimal predicted cost is a Lyapunov function for the closed-loop system. Related approaches [36, 39] use terminal equality constraints explicitly, however SGPC implements the equality constraints implicitly while preserving an explicit representation of the degrees of freedom in predictions.

The decision variables in the SGPC predicted control trajectories appear as perturbations of a stabilizing feedback law, and in terms of a left factorization of transfer function matrices, the predicted control sequence is given by

$$u_k = \tilde{Y}^{-1}(z^{-1}) \left( c_k - z^{-1} \tilde{X}(z^{-1}) y_{k+1} \right). \tag{2.66}$$

Here $z$ is the $z$-transform variable ($z^{-1}$ can be thought of as the backward shift operator, namely $z^{-1} f_k = f_{k-1}$), and $\tilde{X}(z^{-1})$, $\tilde{Y}(z^{-1})$ are polynomial solutions (expressed in powers of $z^{-1}$) of the matrix Bezout identity

$$\tilde{Y}(z^{-1}) A(z^{-1}) + z^{-1} \tilde{X}(z^{-1}) B(z^{-1}) = I. \tag{2.67}$$

For simplicity, we use $u_k$ instead of $\Delta u_k$ and consider the regulation rather than the setpoint tracking problem (i.e. we take $r = 0$). Here $B(z^{-1})$, $A(z^{-1})$ are the polynomial matrices (in powers of $z^{-1}$) defining right coprime factors of the system transfer function matrix, $G(z^{-1})$, where

$$y_{k+1} = G(z^{-1}) u_k = B(z^{-1}) A^{-1}(z^{-1}) u_k \tag{2.68}$$

The determination of the coprime factors can be achieved through the computation of the Smith–McMillan form of the transfer function matrix, $G(z^{-1}) = L(z^{-1}) S(z^{-1}) R(z^{-1})$ where $S(z^{-1}) = \mathcal{E}(z^{-1}) \Psi^{-1}(z^{-1})$ with both $\mathcal{E}(z^{-1})$ and $\Psi(z^{-1})$ being diagonal polynomial matrix functions of $z^{-1}$. The right coprime factors can then be chosen as $B(z^{-1}) = L(z^{-1}) \mathcal{E}(z^{-1})$, $A(z^{-1}) = R^{-1}(z^{-1}) \Psi(z^{-1})$. Alternatively, $B(z^{-1})$, $A(z^{-1})$ can be computed through an iterative procedure, which we describe now.

Assuming that $G(z^{-1})$ is given as

$$G(z^{-1}) = \frac{1}{d\left(z^{-1}\right)} N(z^{-1})$$

we need to find the solution, $A(z^{-1})$, $B(z^{-1})$, of the Bezout identity

$$N(z^{-1})A(z^{-1}) = B(z^{-1})d(z^{-1}) \tag{2.69}$$

for which (2.67) admits a solution for $\tilde{X}(z^{-1})$, $\tilde{Y}(z^{-1})$. This solution can be shown to be unique under the assumption that the coefficient of $z^0$ in $A(z^{-1})$ is the identity, and that $A(z^{-1})$ and $B(z^{-1})$ are of minimal degree. Equation (2.69) defines a set of under-determined linear conditions on the coefficients of $B(z^{-1})$, $A(z^{-1})$. Thus the coefficients of $B(z^{-1})$, $A(z^{-1})$ can be expressed as an affine function of a matrix, say $R$, where $R$ defines the degrees of freedom which are to be given up so that (2.67) admits a solution. The determination of $R$ constitutes a nonlinear problem which, nevertheless, can be solved to any desired degree of accuracy by solving (2.67) iteratively. The iteration consists of using the least squares solution for $R$ of (2.67) to update the choice for the coefficients of $A(z^{-1})$, $B(z^{-1})$; these updated values are then used in (2.67) to update the solution for $\tilde{Y}(z^{-1})$, $\tilde{X}(z^{-1})$, and so on. Each cycle of this iteration reduces the norm of the error in the solution of (2.67) and the iterative process can be terminated when the norm of the error is below a practically desirable threshold.

Substituting (2.68) into (2.66), pre-multiplying by $\tilde{Y}(z^{-1})$ and using the Bezout identity (2.67) provides the prediction model:

$$\begin{aligned} y_{k+1} &= B(z^{-1})c_k + y^f_{k+1} \\ u_k &= A(z^{-1})c_k + u^f_k. \end{aligned} \tag{2.70}$$

Here $y^f_k$ and $u^f_k$ denote the components of the predicted output and input trajectories corresponding to the free response of the model due to non-zero initial conditions. Consider now the dual coprime factorizations $B(z^{-1})A^{-1}(z^{-1}) = \tilde{A}^{-1}(z^{-1})\tilde{B}(z^{-1})$, $X(z^{-1})Y^{-1}(z^{-1}) = \tilde{Y}^{-1}(z^{-1})\tilde{X}(z^{-1})$ satisfying the Bezout identity

$$\begin{bmatrix} z^{-1}\tilde{X}(z^{-1}) & \tilde{Y}(z^{-1}) \\ \tilde{A}(z^{-1}) & -\tilde{B}(z^{-1}) \end{bmatrix} \begin{bmatrix} B(z^{-1}) & Y(z^{-1}) \\ A(z^{-1}) & -z^{-1}X(z^{-1}) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.71}$$

Detailed calculation, based on simulating forward in time the relationships $\tilde{Y}(z^{-1})u_k = c_k - z^{-1}\tilde{X}(z^{-1})y_{k+1}$ and $\tilde{A}(z^{-1})y_{k+1} = \tilde{B}(z^{-1})u_k$, leads to the following affine relationship from the vector of predicted controller perturbations, $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ (with $c_{i|k} = 0$, for all $i \geq \nu$), to the vectors of predicted outputs, $\mathbf{y}_k = (y_{1|k}, \ldots, y_{N|k})$, and inputs, $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$:

$$\begin{bmatrix} C_{z^{-1}\tilde{X}} & C_{\tilde{Y}} \\ C_{\tilde{A}} & -C_{\tilde{B}} \end{bmatrix} \begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \mathbf{c}_k \\ 0 \end{bmatrix} - \begin{bmatrix} H_{z^{-1}\tilde{X}} & C_{\tilde{Y}} \\ H_{\tilde{A}} & -H_{\tilde{B}} \end{bmatrix} \begin{bmatrix} \mathbf{y}^p_k \\ \mathbf{u}^p_k \end{bmatrix} \tag{2.72}$$

where $N = \nu + n_A$, $\mathbf{y}^p_k = (y_{k-n_X-1}, \ldots, y_k)$ and $\mathbf{u}^p_k = (u_{k-n_Y}, \ldots, u_{k-1})$ denote vectors of past input and output values and $n_A, n_X, n_Y$ are the degrees of the polynomials $A(z^{-1})$, $X(z^{-1})$, $Y(z^{-1})$. The $C$ and $H$ matrices are block Toeplitz convolution matrices, which are defined for any given matrix polynomial $F(z^{-1}) = F_0 + F_1 z^{-1} + \cdots + F_m z^{-m}$ by

$$C_F \doteq \begin{bmatrix} F_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ F_1 & F_0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F_m & F_{m-1} & \cdots & F_0 & 0 & \cdots & 0 \\ 0 & F_m & \cdots & F_1 & F_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_m & F_{m-1} & \cdots & F_0 \end{bmatrix}, \quad H_F \doteq \begin{bmatrix} F_m & F_{m-1} & \cdots & F_1 \\ 0 & F_m & \cdots & F_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the row-blocks of $C_F$ and $H_F$ consist, respectively, of $N$ and $m$ blocks.

The solution of (2.72) for the vectors, $\mathbf{y}_k$ and $\mathbf{u}_k$, of output and input predictions is affine in the vector of the degrees of freedom $\mathbf{c}_k$, and hence the predicted cost is quadratic in $\mathbf{c}_k$. In particular the Bezout identity (2.71) implies an explicit expression for the inverse of the matrix on the LHS of (2.72), which in turn implies the solution

$$\begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} C_B \\ C_A \end{bmatrix} \mathbf{c}_k - \begin{bmatrix} C_B & C_Y \\ C_A & -C_{z^{-1}X} \end{bmatrix} \begin{bmatrix} H_{z^{-1}\tilde{X}} & H_{\tilde{Y}} \\ H_{\tilde{A}} & -H_{\tilde{B}} \end{bmatrix} \begin{bmatrix} \mathbf{y}_k^p \\ \mathbf{u}_k^p \end{bmatrix}.$$

The second term on the RHS of this expression corresponds to the free responses of the output and input predictions, and, on account of the structure of the convolution matrices in (2.72) and the Bezout identity (2.71), these free responses are zero at the end of the prediction horizon consisting of $N = \nu + N_A$ steps. From this observation and the finite impulse response of the filters $B(z^{-1})$ and $A(z^{-1})$ in (2.70), it follows that SGPC imposes an implicit terminal equality constraint, namely that both the predicted input and output vectors reach the steady value of zero at the end of the horizon of $N = \nu + N_A$ prediction time steps, and this gives the algorithm a guarantee of closed-loop stability.

Equality terminal constraints can be overly stringent but it is possible to modify SGPC so that the predicted control law of (2.66) imitates what is obtained using the predicted control law, $u_{i|k} = K x_{i|k} + c_{i|k}$, of the closed-loop paradigm. This can be achieved through the use of the Bezout identity

$$\tilde{Y}(z^{-1})A(z^{-1}) + z^{-1}\tilde{X}(z^{-1})B(z^{-1}) = A_{cl}(z^{-1}) \tag{2.73}$$

where $A_{cl}(z^{-1})$ is such that $B(z^{-1})$ and $A_{cl}(z^{-1})$ define right coprime factors of the closed-loop transfer function matrix (under the control law $u = Kx + c$). The fact that the same $B(z^{-1})$ can be used for both the open and closed-loop transfer function matrices can be argued as follows. Let $\hat{B}(z^{-1})$, $\hat{A}(z^{-1})$ be the right coprime factors of $(zI - A)^{-1}B$ such that $B\hat{A}(z^{-1}) = (zI - A)\hat{B}(z^{-1})$. The consistency condition for this equation is $N(zI - A)\hat{B}(z^{-1}) = 0$ where $N$ is the full-rank left annihilator of $B$ (satisfying the condition $NB = 0$). This is however is also the consistency condition for the equation

$$BA_{cl}(z^{-1}) = (zI - A - BK)^{-1}\,\hat{B}(z^{-1}),$$

which implies that $\hat{B}(z^{-1})$ can also be used in the right coprime factorization of $(zI - A - BK)^{-1}B$. Thus the same $B(z^{-1}) = C\hat{B}(z^{-1})$ can be used for both the open and closed-loop transfer function matrices given that these transfer function matrices are obtained by the pre-multiplication by $C$ of $(zI - A)^{-1}B$ and $(zI - A - BK)^{-1}B$, respectively. The property that a common $B(z^{-1})$ can be used in the factorization of the open and closed-loop transfer function matrices can also be used to prove that the control law of (2.66) guarantees the internal stability of the closed-loop system [40] (when $\tilde{Y}(z^{-1})$, $\tilde{X}(z^{-1})$ satisfy either of (2.67) or (2.73)).

SGPC introduced a Youla parameter into the MPC problem and this provides an alternative way to that described in Sect. 2.9 to endow the prediction structure with control dynamics. This can be achieved by replacing the polynomial matrices $\tilde{Y}(z^{-1})$, $\tilde{X}(z^{-1})$, respectively by

$$\begin{aligned}\tilde{M}(z^{-1}) &= \tilde{Y}(z^{-1}) - z^{-1}Q(z^{-1})B(z^{-1})\\ \tilde{N}(z^{-1}) &= \tilde{X}(z^{-1}) + A(z^{-1})Q(z^{-1})\end{aligned}$$

where $Q(z^{-1})$ represents a free parameter (which can be chosen to be any polynomial matrix, or stable transfer function matrix). If $\tilde{Y}(z^{-1})$ and $\tilde{X}(z^{-1})$ satisfy the Bezout identity (either (2.67) or (2.73)), then so will $\tilde{M}(z^{-1})$ and $\tilde{N}(z^{-1})$, which therefore can be used in the control law of (2.66) in place of $\tilde{Y}(z^{-1})$ and $\tilde{X}(z^{-1})$. The advantage of this is that the degrees of freedom in $Q(z^{-1})$ can be used to enhance the robustness of the closed-loop system to model parameter uncertainty or to enlarge the region of attraction of the algorithm [38].

At first sight it may appear that the relationships above will not hold in the presence of constraints. However this is not so, because the perturbations $c_k$ have been introduced in order to ensure that constraints are respected and therefore the predicted trajectories are generated by the system operating within its linear range. These prediction equations can be used to express the vector of predicted outputs and inputs as functions of the vector of predicted degrees of freedom, $\mathbf{c}_k = (c_{0k}, \ldots, c_{N-1|k}, c_\infty, c_\infty, \ldots)$ where $c_\infty$ denotes the constant value of $c$ which ensures that the steady-state predicted output is equal to the desired setpoint vector $r$ and the vector $\mathbf{c}_k$ contains $Nn_u$ degrees of freedom. Clearly for a regulation problem with $r = 0$, $c_\infty$ would be chosen to be zero. SGPC then proceeds to minimize the cost of (2.65) over the degrees of freedom $(c_{0k}, \ldots, c_{N-1|k})$ subject to constraints and implements the control move indicated by (2.66).

The algorithms discussed in this section are based on output feedback and are appropriate in cases where the assumption that the states are measurable and available for the purposes of feedback does not hold true. In instances like this one can, instead, revert to a state-space system representation constructed using current and past inputs and outputs as states (e.g. [41]) or a state-space description of the combination of the system dynamics together with the dynamics of a state observer (e.g. [42], which established invariance using low-complexity polytopes, namely polytopes with $2n_x$ vertices).

## 2.11 Exercises

**1** A first-order system with the discrete time model

$$x_{k+1} = 1.5x_k + u_k$$

is to be controlled using a predictive controller that minimizes the predicted performance index

$$J(x_k, u_{0|k}, u_{1|k}) = \sum_{i=0}^{1} \left( x_{i|k}^2 + 10u_{i|k}^2 \right) + qx_{2|k}^2$$

where $q$ is a positive constant.

(a) Show that the unconstrained predictive control law is $u_k = -0.35x_k$ if $q = 1$.
(b) The unconstrained optimal control law with respect to the infinite horizon cost $\sum_{k=0}^{\infty}(x_k^2 + 10u_k^2)$ is $u_k = -0.88x_k$. Determine the value of $q$ so that the unconstrained predictive control law coincides with this LQ optimal control law.
(c) The predicted cost is to be minimized subject to input constraints

$$-0.5 \le u_{i|k} \le 1.$$

If the predicted inputs are defined as $u_{i|k} = -0.88x_{i|k}$, for all $i \ge 2$, show that the MPC optimization problem is guaranteed to be recursively feasible if $u_{i|k}$ satisfies these constraints for $i = 0, 1$ and 2.

**2** (a) A discrete time system is defined by

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} x_k, \quad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

where $\alpha$ is a constant. Show that $-1 \le y_k \le 1$, for all $k \ge 0$ if and only if $|\alpha| < 1$ and

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq x_0 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) A model predictive control strategy is to be designed for the system

$$x_{k+1} = \begin{bmatrix} \beta & 1 \\ 0 & \alpha \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k, \quad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k, \quad -1 \leq u_k \leq 1$$

where $\alpha$ and $\beta$ are constants, with $|\alpha| < 1$. Assuming that, for $i \geq N$, the $i$ steps ahead predicted input is defined as

$$u_{i|k} = \begin{bmatrix} -\beta & 0 \end{bmatrix} x_{i|k},$$

show that:

(i) $\displaystyle\sum_{i=0}^{\infty}(y_{i|k}^2 + u_{i|k}^2) = \sum_{i=0}^{N-1}(y_{i|k}^2 + u_{i|k}^2) + (\beta^2 + 1)x_{N|k}^T \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{1-\alpha^2} \end{bmatrix} x_{N|k}.$

(ii) $-1 \leq u_{i|k} \leq 1$ for all $i \geq N$ if

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq |\beta|\, x_{N|k} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) Comment on the suggestion that an MPC law based on minimizing the cost in (b)(i) subject to $-1 \leq u_{i|k} \leq 1$ for $i = 0, \ldots, N-1$ and the terminal constraint $x_{N|k} = 0$ would be stable. Why would it be preferable to use the terminal inequality constraints of (b)(ii) instead of this terminal equality constraint.

**3** A system has the model

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} u_k, \quad y_k = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} x_k.$$

(a) Show that, if $u_k = \frac{1}{\sqrt{2}} y_k$, then

$$\sum_{k=0}^{\infty} \tfrac{1}{2}\left(y_k^2 + u_k^2\right) = \|x_0\|^2.$$

(b) A predictive control law is defined at each time step $k$ by $u_k = u_{0|k}^*$, where $(u_{0|k}^*, \ldots, u_{N-1|k}^*)$ is the minimizing argument of

$$\min_{u_{0|k},\ldots,u_{N-1|k}} \sum_{i=0}^{N-1} \tfrac{1}{2}\left(y_{i|k}^2 + u_{i|k}^2\right) + \|x_{N|k}\|^2.$$

Show that the closed-loop system is stable.

(c) The system is now subject to the constraint $-1 \le y_k \le 1$, for all $k$. Will the closed-loop system necessarily be stable if the optimization in part (b) includes the constraints $-1 \le y_{i|k} \le 1$, for $i = 1, 2, \ldots, N+1$?

**4** A discrete time system is described by the model $x_{k+1} = Ax_k + Bu_k$ with

$$A = \begin{bmatrix} 0.3 & -0.9 \\ -0.4 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

where $u_k = Kx_k$ for $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$, and for all $k = 0, 1 \ldots$ the state $x_k$ is subject to the constraints

$$\left| \begin{bmatrix} 1 & -1 \end{bmatrix} x_k \right| \le 1.$$

(a) Describe a procedure based on linear programming for determining the largest invariant set compatible with constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x| \le 1$.

(b) Demonstrate by solving a linear program that the maximal invariant set is defined by

$$\{x : Fx \le \mathbf{1} \text{ and } F\Phi x \le \mathbf{1}\},$$

where $F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\Phi = \begin{bmatrix} 0.42 & -0.025 \\ -0.16 & -0.35 \end{bmatrix}$.

**5** Consider the system of Question 4 with the cost $\sum_{k=0}^{\infty} \big(\|x_k\|_Q^2 + \|u_k\|_R^2\big)$, with $Q = I$ and $R = 1$.

(a) For $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$, solve the Lyapunov matrix equation (2.5) to find $W$ and hence verify using Theorem 2.1 that $K$ is the optimal unconstrained feedback gain.

(b) Use the maximal invariant set given in Question 4(b) to prove that $x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} \Psi^i z_k$ satisfies the constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x_{i|k}| \le 1$, for all $i \ge 0$ if $\begin{bmatrix} F & 0 \end{bmatrix} \Psi^i z_k \le \mathbf{1}$ for $i = 0, 1, \ldots, N+1$, where

$$F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} A + BK & BE \\ 0 & M \end{bmatrix}, \quad z_k = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times N}, \quad M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

(c) Show that the predicted cost is given by

$$J(x_k, \mathbf{c}_k) = \|x_k\|_W^2 + \rho \|\mathbf{c}_k\|^2, \quad W = \begin{bmatrix} 1.33 & 0.58 \\ 0.58 & 4.64 \end{bmatrix}, \quad \rho = 6.56.$$

(d) For the initial condition $x_0 = (3.8, 3.8)$, the optimal predicted cost,

$$J_N^*(x_0) \doteq \min_{\mathbf{c}\in\mathbb{R}^N} J(x_0, \mathbf{c}) \ \text{ subject to } \begin{bmatrix} F & 0 \end{bmatrix} \Psi^i \begin{bmatrix} x_0 \\ \mathbf{c} \end{bmatrix} \leq \mathbf{1}, \ i = 1, \ldots, N+1$$

varies with $N$ as follows:

| $N$ | 8 | 9 | 10 | 11 |
|---|---|---|---|---|
| $J_N^*(x_0)$ | $\infty$ | 826.6 | 826.6 | 826.6 |

(the problem is infeasible for $N \leq 8$). Suggest why $J_N^*(x_0)$ is likely to be equal to 826.6, for all $N > 9$ and state the likely value of the infinite horizon cost for the closed loop state and control sequence starting from $x_0$ under $u_k = Kx_k + c_{0|k}^*$ if $N = 9$.

**6** For the system and constraints of Question 4 with $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$:

(a) Taking $N = 2$, solve the optimization (2.41) to determine, for the prediction dynamics $z_{k+1} = \Psi z_k$, the ellipsoidal invariant set $\{z : z^T P_z z \leq 1\}$ that has the maximum area projection onto the $x$-subspace. Hence show that the greatest scalar $\alpha$ such that $x_0 = (\alpha, \alpha)$ satisfies $z_0^T P_z z_0 \leq 1$ for $z_0 = (x_0, \mathbf{c}_0)$, for some $\mathbf{c}_0 \in \mathbb{R}^2$, is $\alpha = 1.79$.
(b) Show that, for $N = 2$, the greatest $\alpha$ such that $x_0 = (\alpha, \alpha)$ is feasible for the constraints $\begin{bmatrix} F & 0 \end{bmatrix} \Psi^i z_0 \leq \mathbf{1}$, $i = 0, \ldots, N+1$, for $z_0 = (x_0, \mathbf{c}_0)$, for some $\mathbf{c}_0 \in \mathbb{R}^2$, is $\alpha = 2.41$. Explain why this value is necessarily greater than the value of $\alpha$ in (a).
(c) Determine the optimized prediction dynamics by solving (2.63) and verify that

$$C_c = \begin{bmatrix} -1.22 & -0.45 \end{bmatrix}, \quad A_c = \begin{bmatrix} 0.96 & 0.32 \\ -0.015 & -0.063 \end{bmatrix},$$

and also that the maximum scaling $\alpha$ such that $x_0 = (\alpha, \alpha)$ is feasible for $z_0^T P_z z_0 \leq 1$ for $z_0 = (x_0, \mathbf{c}_0)$, for some $\mathbf{c}_0 \in \mathbb{R}^2$, is $\alpha = 2.32$.
(d) Using the optimized prediction dynamics computed in part (c), define

$$\hat{\Psi} = \begin{bmatrix} A + BK & BC_c \\ 0 & A_c \end{bmatrix}$$

and show that $x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} \hat{\Psi}^i z_k$ satisfies constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x_{i|k}| \leq 1$, for all $i \geq 0$ if $\begin{bmatrix} F & 0 \end{bmatrix} \hat{\Psi}^i z_k \leq \mathbf{1}$ for $i = 0, \ldots, 5$. Hence show that the maximum scaling $\alpha$ such that $x_0 = (\alpha, \alpha)$ satisfies these constraints for some $\mathbf{c}_0 \in \mathbb{R}^2$ is $\alpha = 3.82$.
(e) Show that the optimal value of the predicted cost for the prediction dynamics and constraints determined in (d) with $x_0 = (3.8, 3.8)$ is $J^*(x_0) = 1686$.

Explain why this value is greater than the predicted cost in Question 5(d) for $N = 9$. What is the advantage of the MPC law based on the optimized prediction dynamics?

**7** With $K = \begin{bmatrix}0.067 & 2\end{bmatrix}$, the model of Question 4 gives

$$A + BK = \begin{bmatrix} 0.33 & 0.1 \\ -0.33 & -0.1 \end{bmatrix}.$$

(a) Explain the significance of this for the size of the feasible initial condition set of an MPC law which is subject to the state constraints $|\begin{bmatrix}1 & 1\end{bmatrix} x| \le 1$ rather than the constraints of Question 4?
(b) Explain why the feasible set of the MPC algorithm in Question 5(d) (which is subject to the constraints $|\begin{bmatrix}1 & -1\end{bmatrix} x| \le 1$) is finite for all $N$.

**8** GPC can be cast in terms of state-space models, through which the predicted output sequence $\mathbf{y}_k = (y_{1|k}, \ldots, y_{N|k})$ can be expressed as an affine function of the predicted input sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N_u-1|k})$ as $\mathbf{y}_k = C_x x_k + C_u \mathbf{u}_k$. Using this expression show that the unconstrained optimum for the minimization of the regulation cost $J_k = \mathbf{y}_k^T \hat{Q} \mathbf{y}_k + \mathbf{u}_k^T \hat{R} \mathbf{u}_k$, with $\hat{Q} = \mathrm{diag}\{Q, \ldots, Q\}$ and $\hat{R} = \mathrm{diag}\{R, \ldots, R\}$, is given by

$$\mathbf{u}_k^* = -\left(\hat{R} + C_u^T \hat{Q} C_u\right)^{-1} C_u^T \hat{Q} C_x x_k.$$

Hence show that for

$$A = \begin{bmatrix} 0.83 & -0.46 \\ -0.05 & 0.86 \end{bmatrix}, \quad B = \begin{bmatrix} 0.26 \\ 0.55 \end{bmatrix}, \quad C = \begin{bmatrix} 0.67 & 0.71 \end{bmatrix},$$

and in the absence of constraints, GPC results in an unstable closed loop system for all prediction horizons $N \le 9$ and input horizons $N_u \le N$. Confirm that the open-loop system is stable but that its zero is non-minimum phase. Construct an argument which explains the instability observed above.

**9** (a) Compute the transfer function of the system of Question 8 and show that the polynomials

$$\tilde{X}(z^{-1}) = 21.0529 z^{-1} - 32.2308, \quad \tilde{Y}(z^{-1}) = 19.8907 z^{-1} + 1$$

are solutions of the Bezout identity (2.67).
(b) It is proposed to use SGPC to regulate the system of part (a) about the origin (i.e. the reference setpoint is taken to be $r = 0$) using two degrees of freedom, $\mathbf{c}_k = (c_{0|k}, c_{1|k})$, in the predicted state and input sequences, the implicit assumption being that $c_{i|k} = 0$, for all $i \ge 2$. Form the $4 \times 4$ convolution matrices $C_{z^{-1}\tilde{X}}$, $C_{\tilde{Y}}$, $C_{\tilde{A}}$, $C_{\tilde{B}}$ and confirm that

$$\begin{bmatrix} C_{z^{-1}\tilde{X}} & C_{\tilde{Y}} \\ C_{\tilde{A}} & -C_{\tilde{B}} \end{bmatrix}^{-1} = \begin{bmatrix} C_A & C_Y \\ C_B & -C_{z^{-1}X} \end{bmatrix}.$$

Hence show that the prediction equation giving the vectors of predicted outputs $\mathbf{y}_k = (y_{1|k}, \ldots, y_{4|k})$ and inputs $\mathbf{u}_k = (u_{0|k}, \ldots, u_{3|k})$ is

$$\begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} C_B \\ C_A \end{bmatrix} \begin{bmatrix} \mathbf{c}_k \\ 0_{2\times 1} \end{bmatrix} - \begin{bmatrix} 12.6 & -19.9 & 11.9 \\ & 0_{3\times 3} & \\ 21.1 & -32.2 & 19.9 \\ -13.31 & 21.1 & -12.6 \\ & 0_{2\times 3} & \end{bmatrix} \begin{bmatrix} \mathbf{y}_k^p \\ \mathbf{u}_k^p \end{bmatrix}.$$

(c) Show that the predicted sequences in (b) implicitly satisfy a terminal constraint. Hence explain why the closed-loop system under SGPC is necessarily stable.

**10** For the data of Question 9 plot the frequency response of the modulus of $K(z^{-1})/\big(1 + G(z^{-1})K(z^{-1})\big)$ where

$$K(z^{-1}) = \frac{\tilde{X}(z^{-1}) + A(z^{-1})Q(z^{-1})}{\tilde{Y}(z^{-1}) - z^{-1}B(z^{-1})Q(z^{-1})}$$

for the following two cases:

(a) $Q(z^{-1}) = 0$
(b) $Q(z^{-1}) = -11.7z^{-1} + 43$

Hence suggest what might be the benefit of introducing a Youla parameter into SGPC in terms of robustness to additive uncertainty in the system transfer function.

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