Preface

C'est une grande folie de vouloir être sage tout seul.
(François de La Rochefoucauld, Réflexions)

Metastability is a wide-spread phenomenon in the dynamics of non-linear systems—physical, chemical, biological or economic—subject to the action of temporal random forces typically referred to as *noise*. In the narrower perspective of statistical physics, metastable behaviour can be seen as the dynamical manifestation of a first-order *phase transition*, i.e., a crossover that involves a jump in some intrinsic physical parameter such as the energy density or the magnetisation. Attempts to understand and model metastable systems mathematically go back to the early 20th century, notably through the work of H. Eyring and H.A. Kramers, who were concerned with metastable phenomena occurring in chemical reactions.

The modern mathematical approach to metastability was pioneered by M.I. Freidlin and A.D. Wentzell in the late 1960's and early 1970's. They introduced the theory of *large deviations on path-space* in order to analyse the long-term behaviour of dynamical systems under the influence of weak random perturbations. Their realisation that metastable behaviour is controlled by large deviations of the random processes driving the dynamics has permeated most of the mathematical literature on the subject since. A comprehensive account of this development, referred to as the *pathwise approach to metastability*, is given in their 1984 monograph *Random Perturbations of Dynamical Systems* [115]. At around the same time the application of these ideas in a statistical physical context was initiated in a paper by M. Cassandro, A. Galves, E. Olivieri and M.E. Vares [51], which in turn triggered a whole series of papers on metastability of Markovian lattice models. This further development is treated at length in the 2005 monograph *Large Deviations and Metastability* by E. Olivieri and M.E. Vares [198], which provides the key elements of the symbiosis between statistical physics, large deviation theory and metastability.

The present book is concerned with an alternative way to tackle metastability, initiated around 2000 by A. Bovier, M. Eckhoff, V. Gayrard and M. Klein [33], referred to now as the *potential-theoretic approach to metastability*. Here, the pathwise view taken in the Freidlin-Wentzell theory is largely discarded. Instead of aiming at identifying the most likely paths and estimating their probabilities, it interprets the metastability phenomenon as a sequence of visits of the path to different *metastable sets*, and focuses on the precise analysis of the respective *hitting probabilities and hitting times* of these sets with the help of potential theory. The fact that this requires the solution of *Dirichlet problems* in typically high-dimensional spaces has probably acted as a deterrent for a long time, and has prevented an efficient use of the ensuing methods at a much earlier stage. The key point in the potential-theoretic approach is the realisation that, in the specific setting related to metastability, most questions of interest can be reduced to the computation of *capacities*, and that these capacities in turn can be estimated by exploiting powerful *variational principles*. In this way, the metastable *dynamics* of the system can essentially be understood via an analysis of its *statics*. This constitutes a major simplification, and acts as a guiding principle. In addition, potential theory also allows to deduce detailed information on the spectral characteristics of the generator of the dynamics, which are typically assumed in the so-called *spectral approach to metastability* initiated by Davies [73, 74] in the 1980's.

The setting of this book is the theory of Markov processes, for the most part, *reversible* Markov processes. Within this limitation, however, there is a wide range of models that are adequate to describe a variety of different real-world systems. The models we aim at range from finite-state Markov chains, finite-dimensional diffusions and stochastic partial differential equations, via mean-field dynamics with and without disorder, to stochastic spin-flip and particle-hopping dynamics and probabilistic cellular automata. Our main aim is to unveil the *common universal features* of these systems with respect to their metastable behaviour.

The book is divided into nine parts:

- *Part I* presents the metastability phenomenon in its various manifestations, with emphasis on its universal aspects. A brief overview of the history of the subject is given, including a comparison of the pathwise, the spectral, the potential-theoretic and the computational approach. Two paradigmatic models are presented: the Kramers model of Brownian motion in a double-well potential and the two-state Markov chain. These models serve as a *red thread* through the book, in the sense that the much more complex and real-world models treated later still exhibit a metastable behaviour that is in many respects similar. An outline of which models will be treated in the book and which main techniques will be used to analyse them is provided, as well as a brief perspective on metastability in areas other than interacting particle systems.
- *Part II* provides the necessary background on Markov processes (and can be skipped by readers with a basic knowledge of probability theory). Here, the central theme is the relation between Markov processes, martingales, and Dirichlet problems. A brief outline of large deviation theory is provided, as well as a description of three *variational principles* for capacities that play a key role in the study of metastability: the Dirichlet principle, the Thomson principle and the Berman-Konsowa principle.
- *Part III* contains the core of the theory. Here, we give the definition of metastable systems and metastable sets in terms of properties of capacities, and we describe the consequences of these definitions for the distribution of metastable hitting

times and for the spectral properties of the associated Markov generators. We also introduce and discuss the basic techniques that can be used to compute capacities and equilibrium potentials, and to estimate harmonic functions.

Parts IV–VIII highlight the key models that can be treated with the help of these techniques. It is here that the potential-theoretic approach to metastability fully comes to life.

- *Part IV* studies *diffusions with small noise*, both finite-dimensional (random walks and stochastic differential equations) and infinite-dimensional (stochastic partial differential equations).
- *Part V* describes *coarse-graining techniques* applied to the Curie-Weiss model *in large volumes at positive temperatures*, both for a non-random and a random magnetic field.
- *Part VI* focusses on *lattice systems in small volumes at low temperatures*. In this setting, energy dominates entropy. An abstract set-up is put forward, and universal metastability theorems are derived under general hypotheses. These hypotheses are subsequently proved for Ising spins subject to Glauber dynamics and lattice gases subject to Kawasaki dynamics.
- *Part VII* extends the results in Part VI to *lattice systems in large volumes at low temperatures*. In large volumes, spatial entropy comes into play, which complicates the analysis. Both for Glauber dynamics and Kawasaki dynamics the key quantities controlling metastable behaviour can be identified, but at the cost of a severe restriction on the starting measure of the dynamics.
- *Part VIII* looks at metastable behaviour of *lattice systems in small volumes at high densities*, in particular the zero-range process.
- *Part IX* lists a number of challenges for future research, both within metastability and beyond. It describes systems that are presently too hard to deal with in detail, but are expected to come within reach in the next few years. In particular, we look at post-nuclear growth for Ising spins subject to Glauber dynamics (limiting shape of large droplets) and at continuum particle systems with pair interactions (crystallisation), for which a number of results are already available.

Along the way we will encounter a variety of ideas and techniques from probability theory, analysis and combinatorics, including martingale theory, variational calculus and isoperimetric inequalities. It is the combination of physical insight and mathematical tools that allows for making progress, in the best of the tradition of mathematical physics.

Throughout the book we only consider classical stochastic dynamics. It would be interesting to consider quantum stochastic dynamics as well, but this is beyond the scope of the book. We also do not address issues related to numerical simulation, which is rather delicate due to the extremely long time scales involved.

It is a pleasure to thank the colleagues with whom we have worked on metastability over the past 15 years: Florent Barret, Alessandra Bianchi, Michael Eckhoff, Alessandra Faggionato, Alexandre Gaudillière, Véronique Gayrard, Dmitry Ioffe, Sabine Jansen, Oliver Jovanovski, Markus Klein, Roman Kotecký, Francesco

Manzo, Sylvie Méléard, Patrick Müller, Francesca Nardi, Rebecca Neukirch, Enzo Olivieri, Elena Pulvirenti, Elisabetta Scoppola, Martin Slowik, Cristian Spitoni, Siamak Taati and Alessio Troiani. Special thanks are due to Aernout van Enter for reading the entire text and providing a host of valuable comments.

Bonn, Germany
Leiden, The Netherlands
June 4, 2015

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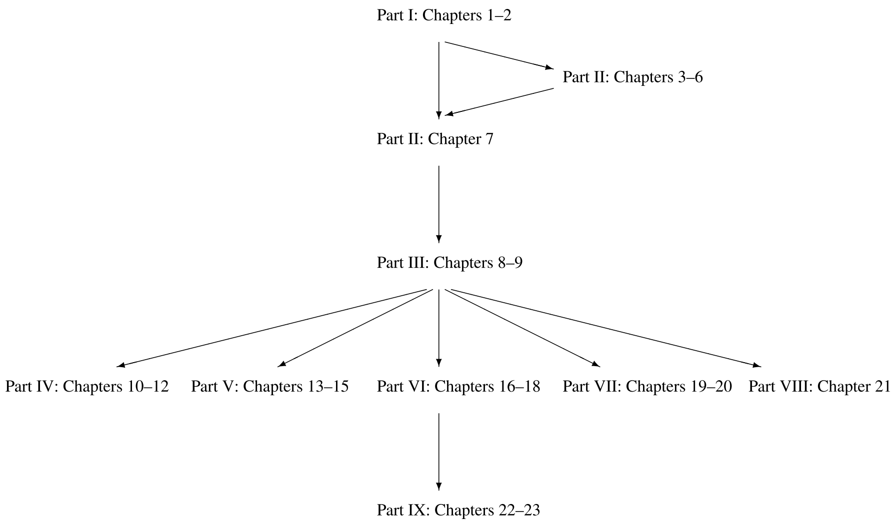
Logical structure of the monograph



http://www.springer.com/978-3-319-24775-5

Metastability
A Potential-Theoretic Approach
Bovier, A.; den Hollander, F.
2015, XXI, 581 p. 96 illus., 14 illus. in color., Hardcover
ISBN: 978-3-319-24775-5