Chapter 2
The biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$

In this chapter we introduce bigraded rings and their properties, we introduce the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ and its subvarieties, and we introduce the required background on Cohen-Macaulay rings. In particular, we build the correspondence between the bihomogeneous ideals of the bigraded ring $R = k[x_0,x_1,y_0,y_1]$ and the varieties of $\mathbb{P}^1 \times \mathbb{P}^1$, mimicking the well-known correspondence between graded ideals of a polynomial ring and the varieties of $\mathbb{P}^n$. While many of the results of this chapter extend quite naturally to any multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, we will focus primarily on the case of $\mathbb{P}^1 \times \mathbb{P}^1$ (see the discussion at the end of the chapter for what is known in the general setting). Similarly, our discussion of Cohen-Macaulay rings takes place within the context of the bigraded ring $R = k[x_0,x_1,y_0,y_1]$.

Throughout this monograph, k denotes an algebraically closed field of characteristic zero. In addition, $\mathbb{N} := \{0,1,2,\ldots\}$ denotes the set of non-negative integers. We let $\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}$ and let $\preceq$ denote the natural partial order on the elements of $\mathbb{N}^2$ defined by $(a,b) \preceq (c,d)$ in $\mathbb{N}^2$ if and only if $a \leq c$ and $b \leq d$.

2.1 Bigraded rings

Let $R := k[x_0,x_1,y_0,y_1]$ be a polynomial ring with coefficients in k. We let $\mathbf{m} := (x_0,x_1,y_0,y_1)$. Set $\deg x_0 = \deg x_1 = (1,0)$ and $\deg y_0 = \deg y_1 = (0,1)$. A monomial $m = x_0^a x_1^b y_0^c y_1^d \in R$ has *bidegree* (or simply, *degree*) $\deg m = (a+b,c+d)$. We make the convention that 0, the additive identity, has $\deg 0 = (i,j)$ for all $(i,j) \in \mathbb{N}^2$. Note that the elements of k all have degree $(0,0)$.

For each $(i,j) \in \mathbb{N}^2$, let $R_{i,j}$ denote the vector space over k spanned by all the monomials of degree (i,j). (Because $0 \in R_{i,j}$ for all $(i,j) \in \mathbb{N}^2$, $R_{i,j}$ satisfies all the axioms of a vector space). The ring R is then a *bigraded ring* because there exists a direct sum decomposition

E. Guardo, A. Van Tuyl, *Arithmetically Cohen-Macaulay Sets of Points in $P^1 \times P^1$*,
SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-24166-1_2

$$R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{i,j}$$

such that $R_{i,j}R_{k,l} \subseteq R_{i+k,j+l}$ for all $(i,j),(k,l) \in \mathbb{N}^2$.

An element $F \in R$ is *bihomogeneous* if $F \in R_{i,j}$ for some $(i,j) \in \mathbb{N}^2$. If F is bihomogeneous, we say its *degree* is $\deg F = (i,j)$. Any polynomial $F \in R$ can be written uniquely as $F = F_1 + \cdots + F_t$ where each F_i is bihomogeneous. We call the F_i's the *bihomogeneous terms* of F.

Example 2.1. The element $F = x_0^2x_1y_0^3 + 17x_0x_1^2y_0^2y_1 - 34x_0^3y_1^3$ is a bihomogeneous element of R because it is an element of $R_{3,3}$. In particular, $\deg F = (3,3)$. On the other hand, the element $G = -16x_0x_1y_0y_1 + 14x_1^4$ is not bihomogeneous because $\deg(-16x_0x_1y_0y_1) = (2,2)$ but $\deg 14x_1^4 = (4,0)$. Note that if R has the standard grading , i.e., $\deg x_0 = \deg x_1 = \deg y_0 = \deg y_1 = 1$, then G is a homogeneous element of degree 4.

Suppose that $I = (F_1,\ldots,F_r) \subseteq R$ is an ideal. If each F_i is bihomogeneous, then we say that I is a *bihomogeneous ideal*. Just as in the standard graded case, it can be shown that I is a bihomogeneous ideal if and only if for every $F \in I$, all of the bihomogeneous terms of F also belong to I.

If $I \subseteq R$ is any ideal, then set $I_{i,j} := I \cap R_{i,j}$ for all $(i,j) \in \mathbb{N}^2$. Each $I_{i,j}$ is a subvector space of $R_{i,j}$, and furthermore, $I \supseteq \bigoplus_{(i,j)\in\mathbb{N}^2} I_{i,j}$. If I is bihomogeneous, then we have an equality, i.e., $I = \bigoplus_{(i,j)\in\mathbb{N}^2} I_{i,j}$, because the bihomogeneous terms of F belong to I if $F \in I$. When I is a bihomogeneous ideal of R, then the quotient ring R/I also inherits a bigraded ring structure. In particular, let $(R/I)_{i,j} := R_{i,j}/I_{i,j}$ for all $(i,j) \in \mathbb{N}^2$. Then

$$R/I = \bigoplus_{(i,j)\in\mathbb{N}^2} (R/I)_{i,j}.$$

An R-module M is a *bigraded R-module* if it has a direct sum decomposition

$$M = \bigoplus_{(i,j)\in\mathbb{N}^2} M_{i,j}$$

with the property that $R_{i,j}M_{k,l} \subseteq M_{i+k,j+l}$ for all $(i,j),(k,l) \in \mathbb{N}^2$. If I is a bihomogeneous ideal of R, then I and R/I are both examples of bigraded R-modules. Another example is the polynomial ring R but with a *shifted grading*. Specifically, let $(a,b) \in \mathbb{N}^2$. Then $R(-a,-b)$ is the polynomial ring, but the (i,j)-th graded piece of $R(-a,-b)$ is defined by

$$R(-a,-b)_{i,j} := R_{i-a,j-b}.$$

Note that $R_{i,j} = 0$ if $(0,0) \not\preceq (i,j)$.

Remark 2.2. Any bigraded R-module $M = \bigoplus_{(i,j)\in\mathbb{N}^2} M_{i,j}$ can also be viewed as a graded R-module if we view R as a standard graded ring, i.e., $\deg x_i = \deg y_i = 1$ for $i = 0,1$. Indeed, for each $t \in \mathbb{N}$, set

$$M_t := \bigoplus_{i+j=t} M_{i,j}.$$

Then $M = \bigoplus_{t\in\mathbb{N}} M_t$ is a graded R-module. So, for example, if I is a bihomogeneous ideal of $R = k[x_0,x_1,y_0,y_1]$, then we can also view R/I as a graded ring using the standard grading. We will find it expedient from time-to-time to "forget" the bigraded structures of our modules and consider only the standard graded structure.

Because of our interest in the interpolation problem, we shall be interested in the bigraded Hilbert function, and its associated first difference function, both of which are defined below.

Definition 2.3. Let I be a bihomogeneous ideal of $R = k[x_0,x_1,y_0,y_1]$. The *Hilbert function of* R/I is the numerical function $H_{R/I} : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$H_{R/I}(i,j) := \dim_k (R/I)_{i,j} = \dim_k R_{i,j} - \dim_k I_{i,j}.$$

Example 2.4. As a simple example, we compute the Hilbert function of R/I when $I = (0)$. In this case, $H_{R/I}(i,j) = \dim_k R_{i,j}$. But we now have

$$\dim_k R_{i,j} = (i+1)(j+1) \text{ for all } (i,j) \in \mathbb{N}^2$$

because there are precisely $(i+1)(j+1)$ monomials of the form $x_0^a x_1^b y_0^c y_1^d$ with $a+b=i$ and $c+d=j$, and furthermore, these monomials form a basis for the k-vector space $R_{i,j}$.

Notation 2.5. When R/I is a bigraded ring, we write the output of the Hilbert function of R/I as an infinite matrix, where the initial row and column are indexed with 0 as opposed to 1. By Example 2.4, $\dim_k R_{i,j} = (i+1)(j+1)$ for all $(i,j) \in \mathbb{N}^2$. Consequently, the Hilbert function $H_{R/I}$ when $I = (0)$ is given by

$$H_{R/I} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \\ 2 & 4 & 6 & 8 & 10 & \cdots \\ 3 & 6 & 9 & 12 & 15 & \cdots \\ 4 & 8 & 12 & 16 & 20 & \cdots \\ 5 & 10 & 15 & 20 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example 2.6. We consider a slightly more interesting example. Let

$$I = (x_0,y_0) \cap (x_1,y_1) = (x_0x_1, x_0y_1, x_1y_0, y_0y_1).$$

The ideal I is a monomial ideal and clearly bihomogeneous. Now $\dim_k I_{0,0} = \dim_k I_{1,0} = \dim_k I_{0,1} = 0$, which can be seen by the fact that the generators of I have degree $(2,0)$, $(1,1)$, or $(0,2)$. Now consider any $(i,j) \in \mathbb{N}^2 \setminus \{(0,0),(1,0),(0,1)\}$. Every monomial of degree (i,j) is an element of $I_{i,j}$ except $x_0^i y_0^j$ and $x_1^i y_1^j$, whence $\dim_k I_{i,j} = (i+1)(j+1) - 2$. So, the Hilbert function of R/I is given by

$$H_{R/I} = \begin{bmatrix} 1 & 2 & 2 & \cdots \\ 2 & 2 & 2 & \cdots \\ 2 & 2 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In the standard graded case, information about R/I can be found in the first difference of its Hilbert function. An analog of the first difference Hilbert function can also be defined for bigraded Hilbert functions.

Definition 2.7. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. The *first difference function of H*, denoted ΔH, is the function $\Delta H : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\Delta H(i,j) := H(i,j) - H(i-1,j) - H(i,j-1) + H(i-1,j-1)$$

where $H(i,j) = 0$ if $(i,j) \not\geq (0,0)$.

Example 2.8. We computed the Hilbert function of R/I when $I = (x_0,y_0) \cap (x_1,y_1)$ in Example 2.6. The first difference function of $H_{R/I}$ is then given by

$$\Delta H_{R/I} = \begin{bmatrix} 1 & 1 & 0 & \cdots \\ 1 & -1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

An R-module homomorphism between two bigraded R-modules M and N, say $\varphi : M \to N$, has *degree* $(0,0)$ if $\varphi(M_{i,j}) \subseteq N_{i,j}$ for all $(i,j) \in \mathbb{N}^2$. Just as in the graded case, if

$$0 \to M_p \to M_{p-1} \to \cdots \to M_1 \to M_0 \to 0$$

is an exact sequence of bigraded R-modules with all morphisms with degree $(0,0)$, then we have the following relation among the Hilbert functions of $M_0,\ldots,M_p$:

$$H_{M_0}(i,j) - H_{M_1}(i,j) + H_{M_2}(i,j) - \cdots + (-1)^p H_{M_p}(i,j) = 0$$

for all $(i,j) \in \mathbb{N}^2$.

Associated with any bigraded R-module M is a *bigraded minimal free resolution*, that is, an exact sequence with degree $(0,0)$ homomorphisms of the form

$$0 \longrightarrow \bigoplus_{(j_1,j_2)\in\mathbb{N}^2} R(-j_1,-j_2)^{\beta_{l,(j_1,j_2)}(M)} \longrightarrow \cdots \longrightarrow$$

$$\bigoplus_{(j_1,j_2)\in\mathbb{N}^2} R(-j_1,-j_2)^{\beta_{1,(j_1,j_2)}(M)} \longrightarrow \bigoplus_{(j_1,j_2)\in\mathbb{N}^2} R(-j_1,-j_2)^{\beta_{0,(j_1,j_2)}(M)} \longrightarrow M \longrightarrow 0.$$

For details on how to construct this resolution, see [20, Chapter 6]; note that while this reference describes how to construct the graded minimal free resolution of a graded ideal, by taking into account the bigrading, we can adapt this procedure to construct a bigraded minimal free resolution. By Hilbert's Syzygy Theorem [20, Theorem 2.1], $l \leq 4$ because we are considering ideals in a polynomial ring with four variables. The numbers $\beta_{i,(j_1,j_2)}(M)$ are the *bigraded Betti numbers* of M.

2.2 Biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$

We now generalize the definition of a projective space $\mathbb{P}^n$ and its subvarieties to a multiprojective setting. While multiprojective spaces can be defined using the modern scheme language, it will suffice for our purposes to only consider the classical construction, i.e., the algebra-geometry dictionary between radical ideals and closed subsets. We only sketch out the relevant details to describe the algebra-geometry dictionary for bihomogeneous ideals of $R = k[x_0,x_1,y_0,y_1]$ and closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$. The proofs are similar to the graded case (see, for example, the book of Cox, Little, and O'Shea [19]) so they are omitted.

The *biprojective space* $\mathbb{P}^1 \times \mathbb{P}^1$ is the set of equivalence classes

$$\mathbb{P}^1 \times \mathbb{P}^1 := \left\{ ((a_0,a_1),(b_0,b_1)) \in k^2 \times k^2 \;\middle|\; \begin{array}{c} \text{neither } (a_0,a_1) \neq (0,0) \\ \text{nor } (b_0,b_1) \neq (0,0) \end{array} \right\}_{/\sim}$$

where $\sim$ is the equivalence relation $((a_0,a_1),(b_0,b_1)) \sim ((c_0,c_1),(d_0,d_1))$ if there exist nonzero $\lambda,\mu \in k$ such that $(a_0,a_1) = (\lambda c_0, \lambda c_1)$ and $(b_0,b_1) = (\mu d_0, \mu d_1)$. An element of $\mathbb{P}^1 \times \mathbb{P}^1$ is called a *point*. We denote the equivalence class of $((a_0,a_1),(b_0,b_1))$ by $[a_0 : a_1] \times [b_0 : b_1]$. It follows that $[a_0 : a_1]$, respectively $[b_0 : b_1]$, is a point of $\mathbb{P}^1$.

If $F \in R$ is a bihomogeneous element of degree (i,j) and $P = [a_0 : a_1] \times [b_0 : b_1]$ is a point of $\mathbb{P}^1 \times \mathbb{P}^1$, then

$$F(\lambda a_0, \lambda a_1, \mu b_0, \mu b_1) = \lambda^i \mu^j F(a_0,a_1,b_0,b_1) \text{ for all nonzero } \lambda,\mu \in k.$$

To say that F vanishes at a point of $\mathbb{P}^1 \times \mathbb{P}^1$ is, therefore, a well-defined notion.

If T is any set of bihomogeneous elements of R, then we define

$$V(T) := \{P \in \mathbb{P}^1 \times \mathbb{P}^1 \;|\; F(P) = 0 \text{ for all } F \in T\}.$$

If I is a bihomogeneous ideal of R, then $V(I) := V(T)$ where T is the set of all bihomogeneous elements of I. If $I = (F_1, \ldots, F_r)$, then $V(I) = V(\{F_1, \ldots, F_r\})$.

The biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ can be endowed with a topology by defining the *closed sets* to be all the subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form $V(T)$ where T is a collection of bihomogeneous elements of R. If Y is a subset of $\mathbb{P}^1 \times \mathbb{P}^1$ that is closed and irreducible with respect to this topology, then we say Y is a *biprojective variety*, or simply, a *variety*.

If Y is any subset of $\mathbb{P}^1 \times \mathbb{P}^1$, then we set

$$I(Y) := \{F \in R \mid F(P) = 0 \text{ for all } P \in Y\}.$$

The set $I(Y)$ is a bihomogeneous ideal of R and is called the *bihomogeneous ideal associated with* Y, or simply, the *ideal associated with* Y. If $P \in \mathbb{P}^1 \times \mathbb{P}^1$ is a point, then we abuse notation and write $I(P)$ instead of $I(\{P\})$. We call $R/I(Y)$ the *bihomogeneous coordinate ring of* Y, or simply, the *coordinate ring of* Y. If $H_{R/I(Y)}$ is the Hilbert function of $R/I(Y)$, then we sometimes write H_Y for $H_{R/I(Y)}$, and we say H_Y is the *Hilbert function of* Y.

By adapting the proofs from the well-known homogeneous case, one can prove the following facts.

Theorem 2.9. *(i) If $I_1 \subseteq I_2$ are bihomogeneous ideals, then $V(I_1) \supseteq V(I_2)$.*
(ii) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^1 \times \mathbb{P}^1$, then $I(Y_1) \supseteq I(Y_2)$.
(iii) For any two subsets Y_1, Y_2 of $\mathbb{P}^1 \times \mathbb{P}^1$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

Example 2.10. Consider the ideal of Example 2.6, that is, $I = (x_0, y_0) \cap (x_1, y_1)$. Since $V((x_0, y_0)) = \{[0:1] \times [0:1]\}$ and $V((x_1, y_1)) = \{[1:0] \times [1:0]\}$, we have

$$V(I) = \{[1:0] \times [1:0], [0:1] \times [0:1]\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1.$$

One can build an algebra-geometry dictionary between bihomogeneous ideals of R and subvarieties of $\mathbb{P}^1 \times \mathbb{P}^1$ similar to the standard algebra-geometry dictionary between graded ideals and varieties in $\mathbb{P}^n$. The correspondence between these objects requires a bigraded version of the Nullstellensatz. Again, the proof follows as in the graded case, so it is omitted.

Theorem 2.11 (Bigraded Nullstellensatz). *If $I \subseteq R = k[x_0, x_1, y_0, y_1]$ is a bihomogeneous ideal and if $F \in R$ is a bihomogeneous polynomial with $\deg F \neq (0,0)$ such that $F(P) = 0$ for all $P \in V(I) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, then $F^t \in I$ for some $t > 0$.*

One difference between the standard graded case and the bigraded case is the notion of irrelevant ideals.

Definition 2.12. A bihomogeneous ideal I of $R = k[x_0, x_1, y_0, y_1]$ is called *projectively irrelevant* if either $(x_0, x_1)^t \subseteq I$ or $(y_0, y_1)^t \subseteq I$ for some integer t. An ideal $I \subseteq R$ is *projectively relevant* if it is not projectively irrelevant.

Example 2.13. To understand the nomenclature, suppose that I is an ideal with $(x_0,x_1)^t \subseteq I$. In particular, x_0^t and $x_1^t \in I$. If we now consider $V(I)$, we must have $V(I) = \emptyset$ since there is no point $P = [a:b] \times [c:d]$ that vanishes at both x_0^t and x_1^t. If P did vanish, we must have $a = b = 0$, which is not allowed.

The following result can be proved by adapting the proof of the graded case and using the Bigraded Nullstellensatz (Theorem 2.11).

Theorem 2.14. *There is a bijective correspondence between the non-empty closed subsets of* $\mathbb{P}^1 \times \mathbb{P}^1$ *and the bihomogeneous ideals of* R *that are radical, i.e.,* $I = \sqrt{I}$*, and projectively relevant. The correspondence is given by* $Y \mapsto I(Y)$ *and* $I \mapsto V(I)$*.*

We conclude this section with a special case of Bezout's Theorem for curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.15 (Bigraded Bezout (Special Case)). *Let* $F \in R = k[x_0,x_1,y_0,y_1]$ *be a bihomogeneous form with* $\deg F = (a,b)$*, and let* $H \in R_{1,0}$*. If the curves* $V(F)$ *and* $V(H)$ *meet at more than* b *points (counting multiplicities), then* $F = HF'$*.*

Proof. After a change of coordinates, we can assume that $H = x_0$. After applying the Division Algorithm for polynomials (see [19, Theorem 2.3.3]), we have $F = F'x_0 + F''$ where $F'' = x_1^a G(y_0,y_1)$ and $G(y_0,y_1)$ is a homogeneous polynomial of degree b in the y_is.

Any point $P \in V(F) \cap V(H)$ must have the form $P = [0:1] \times [b_1:b_2]$ because P lies on $V(H)$. But then $[b_1:b_2] \in \mathbb{P}^1$ is a point that must vanish on $G(y_0,y_1)$. Because $G(y_0,y_1)$ is homogeneous polynomial of degree b, it has exactly b roots (counting multiplicity). So, if there are more than b points (counting multiplicities) that lie in the intersection, we must have $G(y_0,y_1) = 0$, as desired.

Although we will not require it in this monograph, here is the more general statement of Bezout's Theorem for the bigraded ring $R = k[x_0,x_1,y_0,y_1]$.

Theorem 2.16 (Bigraded Bezout). *Let* $F,G \in R = k[x_0,x_1,y_0,y_1]$ *be a bihomogeneous form with* $\deg F = (a,b)$ *and* $\deg G = (c,d)$*, and furthermore, suppose that* G *is irreducible. If the curves* $V(F)$ *and* $V(G)$ *meet at more than* $ad+bc$ *points (counting multiplicities), then* $F = GF'$*.*

Theorem 2.15 is just Theorem 2.16 when G is the irreducible form $H \in R_{1,0}$. Note that a result similar to Theorem 2.15 holds if we take an irreducible form $V \in R_{0,1}$. We leave it to the reader to verify the details.

2.3 Cohen-Macaulay rings

In this section, we review the relevant background on Cohen-Macaulay rings. We continue to work within the context of the bigraded ring $R = k[x_0,x_1,y_0,y_1]$, although these definitions and results hold more generally.

Definition 2.17. If $I \subseteq R$ is a bihomogeneous ideal, then a sequence $F_1, \dots, F_r$ of elements is a *regular sequence modulo I* if and only if

(i) $(I, F_1, \dots, F_r) \subseteq \mathbf{m}$,
(ii) $\overline{F}_1$ is not a zero-divisor in R/I, and
(iii) $\overline{F}_i$ is not a zero-divisor in $R/(I, F_1, \dots, F_{i-1})$ for $1 < i \leq r$.

The *depth of* R/I, denoted depth(R/I), is the length of the maximum regular sequence modulo I.

Definition 2.18. If $I \subseteq R$ is a bihomogeneous ideal, then the *height* of a prime ideal $\wp$ in R/I, denoted $\mathrm{ht}_{R/I}(\wp)$, is the largest integer t such that there exist prime ideals $\wp_i$ of R/I such that $\wp_0 \subsetneq \wp_1 \subsetneq \cdots \subsetneq \wp_{t-1} \subsetneq \wp_t = \wp$. For any ideal I of R, the *Krull dimension of* R/I, denoted K-dim(R/I), is

$$\text{K-dim}(R/I) := \sup\{\mathrm{ht}_{R/I}(\wp) \mid \wp \text{ a prime ideal of } R/I\}.$$

One always has the inequality depth$(R/I) \leq$ K-dim(R/I) (see, [88, Corollary 16.30]). The equality of these two invariants leads to an important class of rings.

Definition 2.19. If $I \subseteq R$ is a bihomogeneous ideal, then the ring R/I is *Cohen-Macaulay* if depth$(R/I) =$ K-dim(R/I).

In this monograph, we are primary interested in arithmetically Cohen-Macaulay varieties of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.20. If $I = I(Y)$ is the bihomogeneous ideal of the variety $Y \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, then we say Y is *arithmetically Cohen-Macaulay* (ACM) if $R/I(Y)$ is a Cohen-Macaulay.

The following results from homological algebra allow us to link the depth of a ring to its projective dimension.

Definition 2.21. The *projective dimension* of an R-module M, denoted proj-dim(M), is the length of a minimal free resolution of M.

Given a homogeneous ideal I of the ring R (not necessarily a bihomogeneous ideal), a beautiful result of Auslander and Buchsbaum links the projective dimension of R/I to the depth of R/I. We continue to assume that $R = k[x_0, x_1, y_0, y_1]$, although this result holds much, much more generally.

Theorem 2.22 (Auslander-Buchsbaum Formula). *Let I be a homogeneous ideal in the ring* $R = k[x_0, x_1, y_0, y_1]$. *Then*

$$\text{proj-dim}(R/I) + \text{depth}(R/I) = \text{K-dim}(R) = 4.$$

Proof. See [79, 15.3] for both the general statement and its proof.

We can use short exact sequences to place bounds on the projective dimensions of R-modules. The following results can deduced from the Depth Lemma [98, Lemma 1.3.9] and the Auslander-Buchsbaum Formula.

Lemma 2.23. *Let M_1, M_2, and M_3 be R-modules, and suppose that we have a short exact sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then

(i) proj-dim$(M_3) \leq \max\{\text{proj-dim}(M_2), \text{proj-dim}(M_1)+1\}$.
(ii) proj-dim$(M_2) \leq \max\{\text{proj-dim}(M_1), \text{proj-dim}(M_3)\}$.

Two special classes of Cohen-Macaulay rings that we will encounter are complete intersections and artinian rings. We recall their definitions and properties.

Definition 2.24. An ideal $I \subseteq R$ is a *complete intersection* if it is generated by a regular sequence.

A complete intersection is also Cohen-Macaulay because of the following result.

Lemma 2.25. *Suppose that $\{F_1, \ldots, F_s\}$ is a regular sequence of homogeneous elements in the polynomial ring $R = k[x_0, \ldots, x_n]$. If I is the complete intersection generated by $\{F_1, \ldots, F_s\}$, then R/I is Cohen-Macaulay and* $\dim R/I = (n+1) - s$.

Proof. See [98, Lemma 1.3.10] and [98, Proposition 1.3.22]. Note that for both of these references, the ring is assumed to be a local ring. However, the results also hold for graded ideals in a polynomial ring.

The following lemma provides an explicit description of the bigraded minimal free resolution of a complete intersection generated by two bigraded forms in R. We shall find this a very useful tool.

Lemma 2.26. *Suppose that $I = (F, G) \subseteq R = k[x_0, x_1, y_0, y_1]$ is a complete intersection. Furthermore, suppose that* $\deg F = (a_1, a_2)$ *and* $\deg G = (b_1, b_2)$. *Then the bigraded minimal free resolution of I is given by*

$$0 \longrightarrow R(-a_1-b_1, -a_2-b_2) \xrightarrow{\phi_2} R(-a_1, -a_2) \oplus R(-b_1, -b_2) \xrightarrow{\phi_1} I \longrightarrow 0$$

where $\phi_1 = [G \;\; F]$ *and* $\phi_2 = \begin{bmatrix} F \\ -G \end{bmatrix}$.

Proof. This result is simply a special case of the fact that the Koszul complex is a minimal free resolution of a complete intersection (see [79, Theorem 14.7]). We also take into account the fact that the generators of I are bigraded.

Definition 2.27. Let $S = k[x_1, y_1]$ be a bigraded ring with $\deg x_1 = (1, 0)$ and $\deg y_1 = (0, 1)$. A bihomogeneous ideal $J \subseteq S$ is an *artinian ideal* if $\sqrt{J} = (x_1, y_1)$. The ring S/J is a *bigraded artinian quotient* if J is a bihomogeneous artinian ideal.

Remark 2.28. Because the dimension of an artinian ring is zero, and because we always have $0 \le \operatorname{depth}(S) \le \text{K-dim}(S)$ when S is a quotient of a polynomial ring, we deduce that an artinian ring is Cohen-Macaulay.

Although we do not have a complete classification of bigraded Hilbert functions, we can classify the bigraded Hilbert functions of artinian quotients of $k[x_1, y_1]$.

Theorem 2.29. *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of a bigraded artinian quotient of $k[x_1, y_1]$ if and only if*

(i) $H(0,0) = 1$,
(ii) $H(i,j) = 0$ *or* 1 *for all* $(i,j) \in \mathbb{N}^2$,
(iii) $H(i,j) = 1$ *for only finitely many* $(i,j) \in \mathbb{N}^2$, *and*
(iv) $H(i,j) = 0$ *implies that* $H(k,l) = 0$ *for all* $(i,j) \preceq (k,l) \in \mathbb{N}^2$.

Proof. $(\Rightarrow)$ Suppose that there is a bihomogeneous artinian ideal $J \subseteq S = k[x_1, y_1]$ such that $H = H_{S/J}$. Because J is artinian, $J \neq (1)$, so $\dim_k (S/J)_{0,0} = 1$, that is, $H_{S/J}(0,0) = 1$, thus proving (i).

Because $\dim_k S_{i,j} = 1$ for all $(i,j) \in \mathbb{N}^2$, we have

$$H(i,j) = H_{S/J}(i,j) = \dim_k S_{i,j} - \dim_k J_{i,j} = 1 - \dim_k J_{i,j} \text{ for all } (i,j) \in \mathbb{N}^2.$$

Thus $H_{S/J}(i,j) = 0$ or 1 for all $(i,j) \in \mathbb{N}^2$, proving (ii).

Because J is artinian, there exist positive integers a and b such that $x_1^a \in J$ and $y_1^b \in J$. Thus, for all $(i,j) \in \mathbb{N}^2$, if $i \ge a$ or if $j \ge b$, then $\dim_k J_{i,j} = 1$, whence $H_{S/J}(i,j) = 0$. So, if $H_{S/J}(i,j) = 1$, then $(i,j) \preceq (a-1, b-1)$. Because there are only a finite number of such (i,j), this proves (iii).

Finally, if $H_{S/J}(i,j) = 0$, this means that $\dim_k J_{i,j} = 1$, and thus $\dim_k J_{k,l} = 1$ for all $(i,j) \preceq (k,l)$. This, in turn, implies that $H_{S/J}(k,l) = 0$, thus proving (iv).

$(\Leftarrow)$ Let H be a numerical function that satisfies (i) through (iv). In $S = k[x_1, y_1]$, let J be the ideal generated by $\{x_1^i y_1^j \mid H(i,j) = 0\}$. We claim that $H = H_{S/J}$ and J is artinian.

First, note that $\dim_k J_{i,j} = 1$ if and only if $H(i,j) = 0$. One direction of this statement follows directly from the definition of J. On the other hand, if $\dim_k J_{i,j} = 1$, then $x_1^i y_1^j \in J_{i,j}$. Thus, there is some generator $x_1^k y_1^l \in J_{k,l}$ with $(k,l) \preceq (i,j)$ that divides $x_1^i y_1^j$. Because $x_1^k y_1^l$ is a generator of J, $H(k,l) = 0$. But then by (iv), $H(i,j) = 0$ since $(k,l) \preceq (i,j)$. Note that $J \neq (1)$ since $H(0,0) = 1$ by (i). Finally, since $H(i,j) = 1$ for only a finite number of (i,j), there must exist an (i,j) of the form $(a,0)$ such that $H(a,0) = 0$. But this means $x_1^a \in J$. Similarly, we can find some $(0,b)$ such that $H(0,b) = 0$, which means $y_1^b \in J$. So, J is artinian.

2.4 Additional notes

The material in this chapter is quite standard. The first two sections are a natural generalization of the algebra-geometry dictionary from the graded case to the multigraded case. While we have focused on the case of $\mathbb{P}^1 \times \mathbb{P}^1$, everything we have presented in this chapter extends naturally to the case of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. To the best of our knowledge, Van der Waerden [90] was the first to consider the multigraded version of the algebra-geometry dictionary (although he focused on the bigraded case, and left the general situation to the reader). Readers may also be interested in a later paper of Van der Waerden [91]. The construction of the multigraded dictionary can also be found in a paper of the second author [93]. A different approach using the language of schemes and the notion of Proj can be found in the PhD thesis of Vidal [97]. The results on Cohen-Macaulay rings can be found, in much greater generality, in most graduate level books on commutative algebra, e.g., [12, 24, 88].

As mentioned in Chapter 1, classifying bigraded (or multigraded) Hilbert functions remains an open problem. Ideally, one would like to find a multigraded analog of Macaulay's classification [68] of Hilbert functions of graded quotients of $k[x_1,\ldots,x_n]$. Van der Waerden's work [90] contains some early results on this problem. More recently, Aramova-Crona-De Negri [2] and Crona [22] presented some necessary conditions on the growth of bigraded Hilbert functions. The case of the bigraded ring $k[x_1,y_1,\ldots,y_n]$ with $\deg x_1 = (1,0)$ and $\deg y_i = (0,1)$ with $i = 1,\ldots,n$ was studied by the second author [92]. In this chapter, we have only presented this result for the case of $k[x_1,y_1]$. Any improvements on these results would be a welcome addition to the literature.

http://www.springer.com/978-3-319-24164-7

Arithmetically Cohen-Macaulay Sets of Points in P^1 x P^1

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2015, VIII, 134 p. 25 illus. in color., Softcover
ISBN: 978-3-319-24164-7