My Debt to Henry P. McKean Jr. 2

David Williams[1]

I was very privileged to have had as research supervisors, David Kendall and Harry Reuter. I learnt a great deal from them and from Eugene Dynkin, André Meyer, and of course, Paul Lévy. But it has been to Henry McKean that I have most often turned for inspiration.

There is a simple reason for this. If anyone else writes a book on Stochastic Integrals, Fourier Series and Integrals, or Elliptic Curves, they might produce a fine book on the topic. But Henry (either alone, or with Harry Dym, or with Victor H. Moll) writes *Mathematics*, not mere exposition of a topic. One is left awestruck by the rich interconnectedness of the subject as evidenced by a dazzling array of examples and (usually very challenging) exercises. What a great antidote to the too prevalent 'elegant abstraction' culture in which (for example) Number Theory is OK provided one keeps away from those common-or-garden numbers, and in which even the Generalized Riemann Hypothesis, astoundingly deep though it is, is perhaps rather closer to the ground than one should be flying.

A few years ago, I had to have a brain tumour removed in something of an emergency. It was explained to me that the operation might seriously impede my ability to understand Mathematics. (I had great surgeons, and I don't think it has!) How glad I was that I had McKean and Moll [14] with me for what might have been my last few hours of Mathematics! Yes, there are a few slips in the book, but these are fussed over only by those who could never write anything a tenth as inspirational.

I started my research career on Markov-chain theory, and soon became haunted by the then recent paper by William Feller and McKean [4]. This showed that there exists a chain with all states instantaneous, counter to what Lévy had thought, though it was he who then gave the beautiful probabilistic construction of the F-M chain. From the viewpoint of the time, the F-M chain was even more amazing because all its off-diagonal jump rates are zero. I became rather obsessed by the Q-matrix problem of characterizing what could be the off-diagonal jump rates of a totally instantaneous chain.

When I realized that I could then make no progress with this problem, I decided to switch fields and to read the great Itô-McKean book on diffusion processes. Again I had to work very hard to do the exercises. I felt that Itô and McKean had calculated everything there is to calculate about Brownian motion. (This was in the days before Marc Yor and coworkers had found, and solved, lots more explicit problems.)

When it came to the famous Section 2.8 of Itô-McKean on local time, I despaired of

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having a full understanding. I therefore tried hard to decompose the problem into simpler ones. Henry included a nice exposition of my efforts as part of his paper [13].

What I had never expected was that thinking hard about Brownian local time would lead me to solve that Q-matrix problem. (I also made heavy use of ideas of Kendall, Reuter, Jacques Neveu and Lèvy.) Things really are interconnected!

For many years, I had been intrigued by McKean's paper [9] on a winding problem driven by white noise. In this, he looks at windings around the origin of the two-dimensional process with Brownian motion as one component and its integral up to current time as the other. The winding process is not at all easy to analyze. However, in this case, the joint process is Gaussian, and this allows one to describe a key distribution by an integral equation. In a typical *tour de force* of transform theory, McKean obtained the explicit solution as an unfamiliar distribution, and derived striking probabilistic consequences. This was the first paper on what came to be known as Markovian Wiener-Hopf theory.

I became interested in more general Wiener-Hopf winding problems in which one component of the two-dimensional process is a Markov process, and the other a fluctuating additive functional of that process. One of the simplest problems required calculation of a jump distribution from 0 of an induced Brownian motion on a half-line of the type studied by Feller and, more fully, by Ito and McKean [5]. When Henry visited Swansea, I told him that I conjectured that the jump distribution in the W-H context must be totally monotone. A day later (I recall Michael Atiyah's saying to me that Henry thinks at a million miles per hour!), Henry told me that he had proved both this conjecture and, by using Krein theory as in Dym and McKean, that all totally monotone jump laws arise this way. See London, McKean, Rogers and Williams [6].

My most recent paper (`http://arxiv.org/abs/1011.6513`), which I hope to be an amusing *divertissement*, is on the simplest non-linear version of Markovian W-H problems. Its W-H aspects can all be traced back to McKean on windings.

The paper's non-linear aspects can be traced back to McKean's paper on travelling waves and the KPP equation [12] which prompted massive developments on branching diffusions, measure-valued processes, and the like. How often he has sparked off new fields of investigation! He was way ahead of the field even in financial mathematics [10].

McKean has also been interested in winding problems of non-W-H type: in particular, topological problems on windings of the Brownian path. See, in particular, the papers [7, 15] by McKean with Lyons and with Sullivan, a formidable trio indeed, and doing Mathematics, not mere Probability.

His expertise in Gaussian processes was also used to great effect in his paper [8] on Lèvy's Brownian motions in multi-dimensional (and even Hilbert-space) time. There is profound work here on splitting fields, etc., and there are really surprising results. See also papers [2, 3] with Dym.

In 1980, I organized a conference at Durham in which the then-brand-new Malliavin calculus played a large part. I decided to write an introduction to the proceedings, and found McKean's paper [11] on the geometry of differential space invaluable for this. The fundamental Malliavin process is Henry's Brownian motion on an infinite-dimensional sphere of radius the square-root of infinity. (Surreal?!)

Though perhaps primarily interested in seeing principles put to good use in the concrete, he is a master of the abstract too. His paper [1] with Blumenthal and Getoor, proves that two Markov processes with identical hitting distributions are time-changes of each other. No result in Markov-process theory is deeper than this.

The above is just a hint as to how Henry's work has enriched the life of one probabilist. I have concentrated on books and papers which, as it were, have become part of me:

ones on which I do not need to refresh my memory. Other, better, probabilists could say much more.

I know that Henry's work is regarded with the same admiration and gratitude by people working in differential equations and in other fields.

If there is an explicit solution to be found, then Henry is the man to find it. But his almost unique skill at calculation is always combined with deep new insights into the underlying principles.

My sincere thanks, and my very best wishes, Henry!

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