Chapter 2
Introduction and Theoretical Background

2.1 Preamble

The nature of light and its main properties, either seen classically as an electromagnetic wave or quantum mechanically as a photon, is not only the focus of much ongoing research, but has been covered by many excellent textbooks. Thus, only a short introduction to the theory will be conducted in the following chapter mainly to establish the employed nomenclature.

2.2 Light as Electromagnetic Waves

I will follow essentially the discussions and calculations of the standard text books in optics of Born and Wolf, Saleh and Teich and Siegman [1–3], the textbook of Andrews and Babiker [4] and the PhD-thesis of A. Mair [5], adapting them to the nomenclature used in this thesis.

2.2.1 Paraxial Wave Equation

The most basic and general way to describe classically the phenomena of light is the theory of electromagnetic fields. Here, the basic equations are the Maxwell equations, which fully describe all phenomena known to classical light fields as electromagnetic waves. However, in this thesis, a simpler but still powerful approximation can be used as the basic theoretical background, the so-called paraxial approximation. The time-independent complex amplitude of the electromagnetic wave can be written as

\[ U(r) = u(r) \exp(ikz), \]  

(2.1)
where the vectorial character of light is assumed to be uniform (and will be disregarded for the moment), \( z \) is the direction of propagation and \( r \) describes the three-dimensional coordinate in space. The simplifying assumption is now that the field varies slowly with respect to the propagation direction \( z \) (within a wavelength \( \lambda: \Delta u \ll u \)). Hence, the complex amplitude can be considered as a scalar wave with a “modulated” envelope \( u \) (herein called spatial mode), which travels in \( z \)-direction. From this approximation the paraxial wave equation (PWE) can be derived\(^\text{1}\)

\[
\left( \Delta_\perp + 2ik \frac{\partial}{\partial z} \right) u(s, z) = 0,
\]  

(2.2)

where \( \Delta_\perp \) represents the transverse Laplacian operator and \( s \) labels the position in the transverse plane in the corresponding coordinate system (e.g. \((x, y)\) for Cartesian, \((r, \theta)\) for cylindrical and \((u, v)\) for elliptical coordinates).

It can be shown that the paraxial approximation is justified if the focusing and diverging angles of the light field are smaller than approximately 30° before notable corrections have to be taken into account \([3]\). Although the PWE (2.2) accurately describes all experimental investigations presented in this thesis, one should always bear in mind that it is only an approximation. Moreover, it was pointed out in detail by Lax et al. \([6]\) that the paraxial approximation, applied to the Helmholtz equation for electric vectorial fields leads to a contradiction. For transversely-polarized fields, i.e. the electric field oscillates transverse to the optical axis, the transverse directional derivative of the electric field is equal to zero. Thus, the transverse Laplacian \( \Delta_\perp \) is also equal to zero, which contradicts paraxial assumption. It was shown by Lax et al. \([6]\) (and later by Davis \([7]\) in a simpler way for vector potentials) that the paraxial approximation can be naturally obtained from lowest-order terms by scaling the Maxwell equations and expanding the field appropriately.

Fields outside of this simplified description might be a future research direction to investigate unknown properties of light fields. Although outside of the scope of this thesis, the interesting approach from Bialynicki-Birula introducing the Riemann-Silberstein vector \([8]\) might be an avenue to investigate the full properties of light described by the Maxwell equations. In the following sections, a detailed discussion of the transverse spatial modes, the polarization of the field and their connection to the angular momentum of light will be conducted. After that, it will be shown that non-trivial combinations of both lead to interesting, complex modes of light.

### 2.2.2 Transverse Spatial Modes

The PWE (2.2), which was derived in the last section, can be solved in different coordinate systems, each resulting in a different spatial mode family. Since the solution

\(^{1}\)The derivation from the Maxwell equations can be found for example in the textbook \([4]\) or the thesis of A. Mair \([5]\).
2.2 Light as Electromagnetic Waves

is restricted to the paraxial regime, where the z-axis is the Cartesian axis of propagation, only solutions for different, orthogonal coordinate systems in the 2-dimensional transverse plane are considered. The coordinate systems can be classified depending on their polynomial degree [9] and the separability of their coordinates. This leads to different spatial modes (a more detailed characterization can be found in the Master’s thesis of M. Krenn [10]). In the following, only coordinate systems with polynomial degree one or two will be taken into account, i.e. solved by the separation of the variable ansatz. Furthermore, only the elliptical coordinate system rather than the parabolic coordinates will be investigated (although solutions have been found [11] and experimentally demonstrated [12]). In the next section two special cases of the solutions will be described first: Cartesian coordinates [13], from which the known Hermite-Gauss beams can be derived, and circular coordinates [14], which result in the mode family of Laguerre-Gauss beams. After that, the general solutions of elliptical coordinates systems [15] and their mode family of Ince-Gauss beams [16] will be discussed.

Hermite-Gauss Modes

If the PWE (2.2) is solved in Cartesian coordinates \( u(x, y, z) \), the complete orthogonal set of Hermite-Gauss (HG) modes is obtained:

\[
HG_{m,n} := u_{m,n}(x, y, z) = \frac{\sqrt{2}}{\sqrt{\pi m!n!}2^{m+n}} \frac{1}{w(z)} H_m \left( \sqrt{2} \frac{x}{w(z)} \right) H_n \left( \sqrt{2} \frac{y}{w(z)} \right) \exp \left[ ik(x^2 + y^2) \right] \frac{1}{2R(z)} - \left( \frac{x^2 + y^2}{w^2(z)} \right) \exp[j\phi_{m,n}],
\]

(2.3)

where \( w(z) = w_0(1 + (z/z_r)^2)^{1/2} \) is the beam diameter at the z-position, \( R(z) = z(1 + z_r^2/z^2) \) is the radius of curvature of the mode, \( z_r = \frac{\pi w_0^2}{\lambda} \) is the Rayleigh length, and \( w_0 \) is the beam waist.

The Hermite polynomials, which give the mode family its name, are defined as

\[
H_m(x) = (1)^m \exp(x^2) \frac{d^m}{dx^m} \exp(-x^2),
\]

(2.4)

where \( m \in \mathbb{N} \) (analogous for \( H_n \)) and the Gouy phase of Hermite-Gauss modes is

\[
\phi_{m,n} = \left( 1 + \frac{m + n}{2} \right) \arctan(z/z_r).
\]

(2.5)

The Gouy phase describes the change of the phase when passing through the focus (point of minimal beam waist \( w_0 \)) and might be used to classify different mode “generations” [18]. Superpositions built by modes of the same generation, i.e. modes with the same Gouy phase, are stable under z-propagation (the transverse spatial structure does not change besides the natural spreading of the beam). In Fig. 2.1, the

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2 A detailed derivation can be found e.g. in [3, 17].
Hermite-Gauss modes

\[
\text{LG}_{l,n} := u_{l,n}(r, \theta, z) = \frac{\sqrt{2n!}}{\sqrt{\pi(|l| + n)!}} \frac{1}{w(z)} \left( \frac{\sqrt{2}r}{w(z)} \right)^{|l|} L_n^{(|l|)} \left( \frac{2r^2}{w^2(z)} \right) e^{i\phi_{l,n}} e^{il\theta},
\]
(2.6)

where the beam diameter \( w(z) \), the radius of curvature \( R(z) \), and the Rayleigh length are defined analogously as for HG modes. The Laguerre polynomials are

\[
L_n^l(r) = \frac{e^r r^{-l}}{n!} \frac{d^n}{dr^n} \left( e^{-r} r^{l+n} \right),
\]
(2.7)

with \( l \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Similarly to HG modes, LG modes have a mode dependent Gouy phase \( \phi_{l,n} \), which is

\[
\phi_{l,n} = (|l| + 2n + 1) \arctan(z/z_r).
\]
(2.8)
Fig. 2.2 Laguerre-Gauss mode gallery. The upper row shows calculated intensity structures (normalized), the lower row the corresponding helical phase distribution. The azimuthal phase dependence corresponds to the first characteristic number \( l \) and is directly related to the orbital angular momentum of the light (see Sect. 2.2.4). The phase singularity in the center leads to a vanishing intensity along the optical axis. The second number \( n \) is related to the number of radial phase steps, which can be seen as nodal rings in the intensity structure.

Again, the Gouy phase can be used to formulate different mode generations, where superpositions of the generation have a stable transverse beam profile when propagating. LG mode superpositions that are built by modes with different Gouy phase rotate during propagation. The intensity structure and phase distribution of various LG modes can be seen in Fig. 2.2. The two characteristic numbers \( l \) and \( n \) describe the azimuthal phase structure and the number of radial \( \pi \)-phase steps. If \( l \) is unequal to zero, the phase forms a helical structure. The steepness of helical phase front is defined by the number \( l \), which is related to the orbital angular momentum of light and is called the topological charge (see Sect. 2.2.4). As a result of the screw-like phase structure, these beams exhibit a phase singularity in the center. Thus, they have an intensity null along the optical axis. For higher orders of \( n \), nodal rings can be found in the intensity structure.

Due to the angular momentum associated with LG modes, they have attracted a lot of attention over the last 20 years. For a more detailed discussion of these modes, i.e. their properties as well as recent applications, the interested reader is directed to the recent book by Andrews and Babiker [4] or the review article by Franke et al. [19].

**Ince-Gauss Modes**

A more general mode family is constituted by the Ince-Gauss (IG) modes [16]. They are solutions to the PWE in elliptical coordinates \( u(u, v, z) \), which are related to Cartesian coordinates as follows:

\[
\begin{pmatrix}
\frac{x}{y}
\end{pmatrix} = f(z) \begin{pmatrix}
\cosh(u) \cos(v) \\
\sinh(u) \sin(v)
\end{pmatrix}.
\]  

(2.9)

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\(^3\)A more detailed description can be found here [18].
The function $f(z)$ is called the semi-focal separation (eccentricity) at position $z$ and given by

$$f(z) = f_0 \frac{w(z)}{w_0}.$$  \hfill (2.10)

The set of Ince-Gauss modes can be written as (for the sake of clarity $z = 0$)

$$IG^e_{p,m} := u_{p,m}(u, v, \epsilon) = N_e C_{p,m}(i u, \epsilon) C_{p,m}(v, \epsilon) \exp \left[ \frac{-r^2}{w_0^2} \right]$$  \hfill (2.11)

$$IG^o_{p,m} := u_{p,m}(u, v, \epsilon) = N_o S_{p,m}(i u, \epsilon) S_{p,m}(v, \epsilon) \exp \left[ \frac{-r^2}{w_0^2} \right],$$  \hfill (2.12)

where $C_{p,m}, S_{p,m}$ are the Ince polynomials, $p, m \in \mathbb{N}$, the ellipticity parameter is $\epsilon = 2 f_0^2/w_0^2$, $N_o, N_e$ are normalization constants, and $e, o$ label the even and odd Ince-Gauss modes. The radial-cylindrical coordinate $r$ is defined as

$$r = f_0 \sqrt{\cosh^2(u) \cos^2(v) + \sinh^2(u) \sin^2(v)}. \hfill (2.13)$$

A detailed discussion about solving the PWE in elliptical coordinates can be found in the seminal paper of Bandres and Vega [16] or the Masters thesis of Mario Krenn [10].

The superposition of the even and odd Ince-Gauss modes (Eqs. (2.11) and (2.12)) leads to the helical Ince-Gauss modes (herein called “Ince-Gauss” modes)

$$IG^\pm_{p,m} := u_{p,m}(u, v, \epsilon) = \frac{1}{\sqrt{2}} \left( u_{p,m}^{IG^e}(u, v, \epsilon) \pm i u_{p,m}^{IG^o}(u, v, \epsilon) \right).$$  \hfill (2.14)

The two characteristic numbers of Ince-Gauss modes $p$ and $m$ are restricted to have the same parity, i.e. both are even or both are odd, and to fulfill $m \leq p$. Various types of these modes can be found in Figs. 2.3 and 2.4.

In Fig. 2.3, where modes for a fixed ellipticity are shown, it can be seen that $m$ corresponds to the number of hyperbolic nodal lines in the intensity structures. The number of elliptic nodal lines is given by the number $(p - m)/2 + \gamma$, where $\gamma$ is 1 for odd modes and 0 for even modes. In Fig. 2.4, modes with fixed $m$ and $p$ are presented while the ellipticity is continuously increased. If the ellipticity $\epsilon$ is zero, the IG modes become LG modes with the topological charge $l = m$ and $n = (p - m)/2$ radial nodes. In the limit of infinite ellipticity $\epsilon \rightarrow \infty$, the IG modes become helical Hermite-Gauss modes [20] (which can be decomposed into two HG modes). Another interesting feature is the continuous splitting of the vortices with increasing ellipticity. Depending on the topological charge of the vortex if $\epsilon = 0$ (the quantum number $m$), the modes show splitting into $m$ vortices along the transverse horizontal axis if the ellipticity is increased.
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Fig. 2.3 Ince-Gauss mode gallery for ellipticity $\epsilon = 4$. Analogous to the previous figures, the calculated intensity structures and the corresponding phase structure is shown in the upper and lower row. The second number $m$ is related to hyperbolic nodal lines intensity structure, whereas from a combination of $p$ and $m$ the elliptical nodal lines can be deduced (formula given in the main text).

Fig. 2.4 Ince-Gauss mode gallery for one mode with increasing ellipticity. The upper and lower row demonstrate the change in the intensity and phase structure. If the ellipticity equals zero, the IG mode becomes a LG mode. If the ellipticity goes to infinity, the IG mode turns into a helical HG mode.

Similarly to LG modes, an orbital angular momentum can also be assigned to IG modes. A detailed discussion of the OAM value per photon can be found in [10, 21] and reveals a continuously varying OAM (incl. increasing and decreasing fractional OAM values) depending on the ellipticity.

Before the angular momentum connected to electromagnetic fields is discussed in Sect. 2.2.4, a brief investigation of another important property of light is conducted in the following section, namely the vectorial nature of light fields i.e. the polarization.

2.2.3 Polarization

So far, the classical light field was considered to be only a transverse, scalar field $u(s, z)$. However, as stated earlier this is just an approximation. The electromagnetic
field fully described by Maxwell’s equations has a vectorial characteristic. The polarization of the light is such a vectorial property. It is determined by the temporal change of the direction of the electromagnetic field and might vary for different positions $\mathbf{r}$. In the paraxial approximation the field vector lies in the transverse plane (spanned by the vectors $\mathbf{s}$). The curve that is described by the movement of the endpoint of the field vector is called polarization ellipse. Usually, this polarization ellipse is uniformly distributed in the paraxial regime. However, in following Sect. 2.2.5 the coherent superpositions of two different transverse modes, each having a different polarization, will be discussed. These combinations lead to modes where the polarization varies across the transverse beam extend.

To account for the vectorial nature of the field, the complex amplitude $u(\mathbf{s})$ is now described as a vector field and written in the Cartesian coordinate system as

$$u = u_x x + u_y y.$$  \hspace{1cm} (2.15) 

Note that it is most convenient to use the Cartesian coordinate system although all considerations also hold for the other systems (cylindrical or elliptical).

The electric field $E(z, t)$ that travels in $z$-direction can now be separated into its transverse components (for the sake of simplicity a plane wave is assumed)

$$E_x = u_x \cos \left[2\pi \nu \left(t - \frac{z}{c}\right) + \delta_x\right] \hspace{1cm} (2.16a)$$

$$E_y = u_y \cos \left[2\pi \nu \left(t - \frac{z}{c}\right) + \delta_y\right]. \hspace{1cm} (2.16b)$$

These formulas can be rewritten into the parametric equation of the ellipse

$$\frac{E_x^2}{u_x^2} + \frac{E_y^2}{u_y^2} - 2 \frac{E_x E_y}{u_x u_y} \cos \delta = \sin^2 \delta,$$  \hspace{1cm} (2.17) 

where the phase difference between the two components is $\delta = \delta_x - \delta_y$.

**Different Types of Polarization**

The state of the polarization is described by the ellipse of Eq.(2.17). Hence, \textit{elliptically-polarized} light is defined by three parameters: the amplitudes $u_x$ and $u_y$ and the phase difference $\delta$ (see Fig. 2.5). However, there are two special cases, which are of particular interest (especially because they are often used as a basis to describe the polarization state in the quantum regime, see Sect. 2.3.2).

The first important polarization state arises when the phase difference is $\delta = \pm \pi/2$ and both amplitudes are equal $u_x = u_y$. As seen in Eq. (2.17), the ellipse turns into a circle. Thus, the light is \textit{circularly polarized}. The common convention is that if $\delta = +\pi/2$ the light is said to be right handed circularly polarized (at a fixed position, the electric field rotates clockwise around the optical axis $z$ if seen from the plane that the light approaches). For $\delta = -\pi/2$, the light is called left handed circularly polarized.
The second special case is the linear polarization. If the phase difference $\delta$ is either 0 or $\pi$, the ellipse collapses to a straight line. The slope of the linear polarization depends on the ratio of $\pm u_y/u_x$, where the sign corresponds to the phase difference $\delta$ of 0 or $\pi$. Following the standard convention for linearly-polarized light, the polarization will be called horizontal ($H$), vertical ($V$), diagonal ($D$) or anti-diagonal ($A$) if $u_y = 0$, $u_x = 0$, $u_y/u_x = +1$ or $u_y/u_x = -1$, respectively.

Another convenient description of the polarization state of the light are the so-called Stokes parameters. They can easily be measured (see Sect. 5.1.2) and are related to the amplitudes and phases in the following way:

$$
S_0 = u_x^2 + u_y^2 \\
S_1 = u_x^2 - u_y^2 \\
S_2 = 2u_x u_y \cos \delta \\
S_3 = 2u_x u_y \sin \delta.
$$

Here, $S_0$ depicts the intensity of the light and the parameters $S_1$, $S_2$, and $S_3$ correspond to the ratio of the amplitudes, the ellipticity, and orientation of the electromagnetic light field. For a fully polarized beam, the Stokes parameters can be considered as a real three-dimensional vector in Cartesian coordinates, which can be assigned to spherical coordinates

$$
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix} = S_0 \begin{pmatrix}
\cos(2\epsilon) \cos(2\theta) \\
\cos(2\epsilon) \sin(2\theta) \\
\sin(2\epsilon)
\end{pmatrix},
$$

where the ellipticity $\epsilon$ ranges from $-\pi/4$ to $\pi/4$ and the orientation $\theta$ can take any value between 0 and $\pi$. The sphere, which is spanned by all possible configuration ($S_0$ constant) of $\epsilon$ and $\theta$, is called the Poincaré sphere. Every point on its surface corresponds to a specific polarization. Linear polarizations can be found on the equator, circular polarizations on the poles, and elliptical polarizations everywhere between (see Fig. 2.11 for more information).
2.2.4 Optical Angular Momentum

The two properties—transverse spatial phase structures and polarization—can be linked to an interesting and important feature of light: the angular momentum. Already in 1909, Poynting showed that it is possible to assign an angular momentum to circularly-polarized light in analogy to the one known from mechanical motions [22]. Later, Beth experimentally demonstrated the angular momentum of light. He showed that it can be understood not only classically but quantum mechanically by assigning $\pm \hbar$ of angular momentum to the circularly-polarized photons as a spin angular momentum (SAM) [23]. In 1992, Allen, Woerdmann, and coworkers discovered that an orbital angular momentum (OAM) can be identified, if the light field has a helical phase structure, e.g. the phase term $\exp(il\theta)$ in LG modes [24]. Depending on the inclination of the screw-like phase structure, thus depending on the $l$ value, $l$ quanta of OAM can be assigned to each photon. This type of angular momentum was later demonstrated for the first time in experiments by the group of Rubinstein-Dunlop [25, 26], where particles were set in rotation. Intuitively (and seen in experiments), the difference between the two angular momenta can be understood as follows: if circularly-polarized light is absorbed from trapped particles, they start to spin around their own axis, whereas OAM containing light fields, e.g. LG modes, lead to a rotation of the particles around the optical axis [27].

Classically, a straightforward separation between the two angular momenta arises, if the total angular momentum of light is considered

$$\mathbf{J} = \int dV \varepsilon_0 \mu_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{H}).$$

(2.20)

This can be transformed with the vector potential $\mu_0 \mathbf{H} = \nabla \times \mathbf{A}$ and Gauss’ theorem (quickly declining electromagnetic fields assumed) to

$$\mathbf{J} = \int dV \varepsilon_0 E_i (\mathbf{r} \times \nabla) A_i + \int dV \varepsilon_0 \mathbf{E} \times \mathbf{A}$$

(2.21)

$$= \mathbf{L} + \mathbf{S},$$

(2.22)

where the analogy to the quantum mechanical angular momentum operator suggests that $\mathbf{L}$ is related to the orbital part of the angular momentum. The dependence of $\mathbf{S}$ on the vectorial nature of the light field, i.e. the polarization, assigns the second term to the spin part of the angular momentum.

Despite this seemingly obvious separation into SAM and OAM, there has been a lot of discussions and confusions about its meaning. The confusion mainly arose due to the fact that neither of the two momenta itself, regarded as quantum mechanical operators $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$, fulfills the commutation relation for angular momenta (for more details see [4, 28] and citations therein). However, for paraxial beams it was shown that the separation into SAM and OAM can be done in a meaningful way [29], hence paraxial beams have a well defined spin and orbital angular momentum in the
direction of propagation. Moreover, in paraxial beams the defining operators $\hat{S}_z$ and $\hat{L}_z$ fulfill the commutation relation for angular momentum.

This is somewhat remarkable because it implies that single photons are not limited to one quantum of angular momentum. Theoretically, single photons can have arbitrary large angular momentum values, e.g. $l \cdot h$ of OAM for LG modes depending on the phase factor $\exp(il \theta)$. This result is of great importance for experiments presented later, where the entanglement of high angular momenta is investigated.

For reasons of completeness, it is worthwhile mentioning that modes can carry fractional OAM as well, e.g. IG modes [21] or beams with a helical phase structure that is not a multiple of $2\pi$ (non-integer values of $l$) [30, 31].

### 2.2.5 Vector Beam Families

In the previous sections, either the transverse spatial mode structure was investigated assuming the same polarization across the whole beam or polarization was discussed disregarding the modal structure of the beam. In the following section, the discussions will be generalized by investigating a superposition of two different transverse spatial modes, that are orthogonally polarized to each other. The common feature of such superpositions is a transverse varying polarization, which led to their name “vector beams” [4, 32].

A general vector beam can be described by

$$\text{Vec} = \frac{1}{\sqrt{2}} (u_{spM}(s) \cdot e_{pol} + e^{i\varphi} u_{spM}(s) \cdot e_{pol}),$$

(2.23)

where $u_{spM}$ labels different spatial modes, $e_{pol}$ is the unit vector related to a specific polarization, and $\varphi$ can take any value between 0 and $2\pi$. For the sake of simplicity, the considerations will be restricted to vector modes at the position $z = 0$. Specific features connected to propagation will be mentioned when appropriate (e.g. Poincaré beams).

Similar to the spatial mode families from Sect. 2.2.2, they can be classified into different families depending on their type polarization pattern. The pattern is naturally linked to the spatial modes and polarizations used to construct the vector modes. In the following sections, only vector beams constructed by zeroth or first-order spatial modes are considered, although vector beams can also be built by higher-order modes [33, 34]. However, they do not show other properties than first-order vector beams, apart from a higher frequency of the polarization change across the beam extent (higher-order symmetries).

**Cylindrical Vector Beams**

An frequently studied family of vector beams is that of so-called “cylindrical vector beams”, which have a cylindrically symmetric polarization pattern [32]. The large interest in these beams can be attributed to different applications of which a few
are mentioned in the following. It was shown, for example that such beams, more specifically radially-polarized beams, can increase the excitation rate of surface-plasmons [35, 36]. Moreover, they can be used to focus light beams to smaller spots [37, 38], which increases the resolution in microscopy applications. In addition, they allow photon-atom coupling efficiencies up to 100% [39, 40], which could be very beneficial if quantum networking tasks between photons and single atoms are realized.

Cylindrical vector beams can be constructed by superposing either linearly-polarized HG modes or circularly-polarized LG modes. For an easier distinction, the first types will be called “HG vector beams”, while latter ones will be named “LG vector beams”.

**HG Vector Beams**

One way to construct cylindrical vector beams is to superpose two HG modes with linear polarization. Hence, Eq. (2.23) can be rewritten as

\[ \text{Vec}^{HG} = \frac{1}{\sqrt{2}} (HG_{1,0} \cdot e^{iH/V} + e^{i\varphi} HG_{0,1} \cdot e^{iV/H}) , \] (2.24)

following the nomenclature introduced in the previous sections. The arising polarization pattern of first-order HG vector beams for different phases \( \varphi \) can be seen in Fig. 2.6 (columns three to six). For specific superpositions, the popular cylindrical vector beams with azimuthal (1st row, \( \varphi = 0^\circ \)) or radial polarization (2nd row, \( \varphi = 0^\circ \)) is found. The so-called anti-vortex configuration (1st and 2nd row, \( \varphi = 180^\circ \)) shows a rotation of the linear polarization vector contrary to its angular position.

**Fig. 2.6** Polarization pattern of HG vector beams. The calculated intensity is shown in gray scale in the background and is overlaid by the vector beam polarization pattern. The shapes and colors of the polarization vectors depict the type of polarization (color coding as introduced in Fig. 2.5: blue = linear, red = right elliptic, green = left elliptic). The components out of which the vector beams are constructed are shown in the first two columns (small insets depict the phase of the beam).
LG vector beams

A second way to construct cylindrical vector beams is the superposition of circularly-polarized LG modes

\[
\text{Vec}^{LG} = \frac{1}{\sqrt{2}} \left( \text{LG}_{1,0} \cdot \mathbf{e}_{R/L} + e^{i\varphi} \text{LG}_{-1,0} \cdot \mathbf{e}_{L/R} \right).
\] (2.25)

The polarization pattern of LG vector beams composed of circularly-polarized \( \text{LG}_{1,0} \) and \( \text{LG}_{-1,0} \) is visualized in Fig. 2.7. All polarization pattern in the upper row have only linear polarization vectors with the cylindrical symmetric azimuthal (\( \varphi = 180^\circ \)), radial (\( \varphi = 0^\circ \)), and spiral (\( \varphi = 90^\circ, 270^\circ \)) configuration. The combination depicted in the lower row leads to the anti-vortex structure, where different phases \( \varphi \) result in a rotation of the polarization pattern.

With both types, LG and HG vector beams, radially and azimuthally-polarized vector beams can be generated for different configurations (described above). However, the difference between the two types is clearly visible (see Figs. 2.6 and 2.7). The pattern of LG vector beams exhibit only linear polarizations in all superpositions. Another crucial difference becomes apparent, if one compares specific transverse positions throughout one type of superposition (one row in Fig. 2.6 or 2.7). For HG vector beams, the horizontal and vertical central regions always show the same polarization, whereas for LG vector beams any transverse position has a different polarization if the phase changes. This feature can be explained intuitively by the modal structure of the spatial components (HG or LG modes). At regions where no modal overlap occurs (e.g. for HG vector beams at horizontal and vertical central regions), the polarization is independent of the phase \( \varphi \). This special characteristic is important and will be discussed later in more detail, when entanglement of such modes is investigated.

![Fig. 2.7](image_url) Polarization pattern of LG vector beams. Arrangement and color coding as in the previous Fig. 2.6
Poincaré Beams

Only recently, a novel class of vector beams containing every polarization on the Poincaré sphere was presented [41] (a short discussion including their specific polarization patterns can already be found in the PhD thesis of Mair [5]). These so-called “Poincaré beams” have attracted much attention, because they exhibit interesting features like different types of polarization singularities and a change of the polarization pattern while freely propagating. Polarization singularities are points or lines in the transverse polarization pattern, where a property of the polarization vector is undefined. In Poincaré beams, so-called C-point singularities, i.e. points where the orientation of the polarization ellipse is undefined (circular polarization) and L-line singularities, where the handedness of the polarization is undefined (linear polarization), can be found [34, 42–46]. Another interesting effect, the rotation of the pattern during free propagation, can be explained by the different Gouy phases of the two superposed modes [47].

The simplest way to construct Poincaré beams is to superpose a circularly-polarized Gauss mode with an orthogonally-polarized LG mode of the first order

\[
\text{Vec}^P = \frac{1}{\sqrt{2}} \left( L G_{0,0} \cdot e_{R/L} + e^{i\phi} L G_{1,0} \cdot e_{L/R} \right).
\]

(2.26)

The polarization pattern of Poincaré beams (Fig. 2.8) exhibits every polarization, starting with circular polarization in the center and continuously changing to the orthogonal circular polarization depending on the radial position. Depending on the azimuthal position, the slope of the elliptical (or linear) polarization varies through all possible angles. Hence, the whole Poincaré sphere is stereographically projected onto the real, physical plane of the transverse light field. Furthermore, the pattern of Poincaré beams contains different polarization singularities. In the center of the beam a C-point singularity is located. Around the center, where the light is linearly polarized, lies the L-line singularity. Both singularities can be seen in Fig. 2.8 (the L-line singularity is depicted by a dashed ring). The center of the beam (and to some extent the outermost regions) has an additional feature that was already discussed for HG vector photons. Here, the polarization does not change throughout all modes, i.e. it is independent of the phase \( \phi \). Analogously to the HG vector beams, this can be easily explained by the overlap of the two constructing modes and will be more important for quantum mechanical effects.

Custom-Tailored Poincaré Beams

To understand the polarization pattern of vector beams more intuitively, “artificially” created vector beams will be investigated. Here, the transverse spatial mode that constructs the vector beam is not a simple, “natural” solution to the PWE, but a

\footnote{This effect is similar to the rotation of the intensity mentioned earlier, if two LG modes with different Gouy phase are superposed.}
2.2 Light as Electromagnetic Waves

Fig. 2.8  Polarization patterns of Poincaré beams. Again, the background shows the calculated intensity in gray scale and is overlaid by the polarization pattern (shapes and colors of the polarization vectors depict the type of polarization; color coding as in Fig. 2.6). On the left side (first two columns), the components to build the Poincaré beams are shown including the corresponding phase distribution (small insets). The upper row shows the so-called lemon configuration, the lower row demonstrates a star configuration (following the characterization of surface singularities—umbilic points—by Berry and Hannay [48]). Both patterns exhibit a C-point polarization singularity in the center and L-line singularities around the optical axis (dashed circle)

square shaped beam. Within the square shaped mode, the intensity varies linearly from left to right. For one mode it increases from zero to maximum intensity, for the other mode it decreases from maximum to zero intensity. Hereby, the modal overlap between the two modes varies from left to right with a maximal overlap in the center of the beam. In addition, the phase of each mode is changed from the top to the bottom of the square shaped beam. For one component the phase increases from 0 to \( \pi \), for the other mode it decreases from \( \pi \) to 0. To finally create the custom-tailored Poincaré beam, one beam has to be vertically polarized and the other horizontally polarized (following the ideas of the former vector beams). In mathematical terms the obtained Poincaré beam can be written as

\[
\text{Vec}^{etP} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} u \exp(i \downarrow) \cdot e_V + e^{i\varphi} u \exp(i \uparrow) \cdot e_H \end{array} \right).
\]  

(2.27)

where \( u \) stands for the square shaped spatial mode and the arrows depict the direction of increasing intensity (horizontal arrows) or phase (vertical arrows). The resulting Poincaré beam pattern can be seen in Fig. 2.9. Similar to Poincaré modes every polarization can be found in the arising polarization patterns. Hence, the term “custom-tailored Poincaré beams” is justified. The generated pattern changes continuously as

---

5 Although the square shaped modes might not be a simple solution to the PWE, they should be describable with a complex superposition of many “simple” modes (maybe an infinite sum of modes). This stems from the fact that all earlier discussed modes form a complete set of spatial modes. Thus, they are able to describe any paraxial light field. A more detailed discussion however is outside of the scope of this thesis.
2.3 Photons as Quantum Systems

Photonic quantum information is one of the most popular fields in quantum information science. It has already matured to a stage where complex informational tasks can be implemented with advanced technologies and complex experiments [49–51]. Therefore, the following introduction will be kept as short as possible but as elaborate as necessary to understand the discussed topics in the subsequent chapters. The interested reader is directed to the excellent standard textbooks by Nielsen and Chuang [52], Bouwmeester et al. [53] or Audretsch [54] for further information.

2.3.1 Quantum States

So far, light has been described classically by electromagnetic fields. However, in the limit of very weak intensities this wave-like picture in not tenable anymore and light has to be described by single photons and their quantum states (for a detailed discussion about the theoretical justification, I refer to the book [55]). Generally, quantum states are represented by state vectors in a complex Hilbert space. For the sake of simplicity, the following considerations will be restricted to states of the two-dimensional Hilbert space, called “qubits”. In this thesis all investigated states are mere qubit states. However, most photonic degrees of freedom can be used to encode quantum states of higher dimensionality of which a few will be named in Sect. 2.3.2.

A general qubit state in the Hilbert space $\mathcal{H}$ is described in Dirac notation by

$$|\psi\rangle = \cos(\theta/2) |0\rangle + e^{i\phi} \sin(\theta/2) |1\rangle,$$  

(2.28)
2.3 Photons as Quantum Systems

![Bloch sphere for qubit systems](image)

**Fig. 2.10** Bloch sphere for qubit systems. Any two-dimensional pure state is related to a specific point on the surface of the Bloch sphere. The state vectors of the computational basis $|0\rangle$ and $|1\rangle$ are found to be on the poles, equally weighted superpositions along the equator. As an example, the state $\psi$ from Eq. (2.28) with the according angles is shown in red

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$.

The states $|0\rangle$ and $|1\rangle$ are usually called computational basis and label two mutually exclusive states of the system, i.e. properties of a two-level system described by quantum mechanical states. This means that the physical property under investigation is in a superposition of two different states, each will be detected with a probability of $|\cos(\theta/2)|^2$ for “0” and $|\sin(\theta/2)|^2$ for “1”. Besides the computational basis $|0\rangle$ and $|1\rangle$, one can define more, fully equivalent bases to describe the qubit state. Three equivalent bases can be constructed in total such that the measured result in one basis does not give any information about possible results in another basis. Hence, they are mutually unbiased. All three bases together form a set of mutually unbiased bases (MUB). For qubits they can be written as

$$
|0\rangle, |1\rangle; \quad \frac{1}{\sqrt{2}} |0\rangle + |1\rangle, \frac{1}{\sqrt{2}} |0\rangle - |1\rangle; \quad \frac{1}{\sqrt{2}} |0\rangle + i|1\rangle, \frac{1}{\sqrt{2}} |0\rangle - i|1\rangle. \quad (2.29)
$$

MUBs play an important role in various quantum informational tasks like the verification of quantum entanglement (as it will shown in Sect. 2.4), optimal state tomography [56], quantum cryptography schemes, and quantum computation [52, 53].

All states governed by Eq. (2.28) can be represented by the surface of a unit sphere, called the Bloch sphere (see Fig. 2.10). Note, that even mixed states are represented as they would lie within the Bloch sphere with the completely mixed state located in the centre.\(^6\) Physical implementations of the states might be (apart from photonic degrees of freedom discussed in the following section) 2-level atoms, ions in two energy levels, the spin of electrons or nucleus, quantum dots, and superconducting currents, to name a few.

---

\(^6\)I abstain from discussing in detail the more realistic description of physical states in terms of density matrices for mixed states and only refer the reader to the corresponding literature [52, 54]. Nevertheless, the simplifying assumption of pure states is a very good approximation to all real physical systems in the presented results. Thus, all drawn conclusions hold.
The aforementioned link between the unobservable state $|\psi\rangle$ and the real observations, namely the detection probabilities $|\cos(\theta/2)|^2$ and $|\sin(\theta/2)|^2$, already shows one of the most important distinctions of quantum mechanics from all classical theories in physics. It is not possible to assign a real physical property to the abstract concept of a quantum state before and independent of any measurement. In the following sections it will be shown, why this conclusion has to be drawn by the example of a specific bi-photon qubit state, namely an entangled pair of qubits.\footnote{Note that even single higher-dimensional systems—so-called qudit states—could also be used to justify the statement (for example it was shown for single qutrits in \cite{57}). However, they are not in the scope of this thesis, where qubit systems are the focus of all investigations.}

**Quantum Entanglement**

To justify the latter statement, qubit states with at least two systems will be investigated. Hence, the simplest system to discuss is a bipartite system.\footnote{Although multi-partite systems of more than two parties can form very interesting in quantum states, e.g. GHZ states \cite{58}, they are not in the scope of this thesis.} A general two photon qubit-state in the product Hilbert space $H^{AB} = H^A \otimes H^B$ can be written as

$$|\Psi\rangle_{AB} = \sum_{r,s=0}^{1} \lambda_{rs} |r\rangle_A |s\rangle_B \quad (2.30)$$

$$= \sum_{r,s=0}^{1} \lambda_{rs} |rs\rangle_{AB}, \quad (2.31)$$

where the indices label the two photonic systems $|r\rangle_A \in H^A$ and $|s\rangle_B \in H^B$ and the amplitudes $\lambda_{rs}$ are complex numbers that satisfy $\sum \lambda_{rs}^2 = 1$.

The state $|\Psi\rangle$ is called separable if it can be written as a product state

$$|\Psi^{sep}\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B. \quad (2.32)$$

Here, each subsystem behaves as a fully separable, individual system without any relation to the other subsystem. However, the more interesting case appears when the amplitudes are $\lambda_{rs} = 0$ either for $r = s$ or $r \neq s$. The two possible states

$$|\Phi^{ent}\rangle_{AB} = \lambda_{00} |0\rangle_A |0\rangle_B + \lambda_{11} |1\rangle_A |1\rangle_B \quad (2.33)$$

and

$$|\Psi^{ent}\rangle_{AB} = \lambda_{01} |0\rangle_A |1\rangle_B + \lambda_{10} |1\rangle_A |0\rangle_B \quad (2.34)$$

are called entangled. They cannot be rewritten as product states. Entangled states describe the bipartite system as a superposition of a single, combined system [59], which leads immediately to surprising consequences. The two entangled subsystems can be physically separated but the observation of one is strongly correlated to the
measurement result of the other. Although a more detailed discussion will be pursued later in this chapter (Sect. 2.4), it becomes apparent that quantum entanglement is the mind boggling feature of quantum physics, which led to a drastic change in the understanding of nature.

Before discussing different ways to encode quantum information and various tests to verify entanglement, a frequently used subset of entangled bipartite qubit systems, the so-called Bell-states, should be introduced:

\[ |\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \]  \hspace{1cm} (2.35)

\[ |\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B) \]  \hspace{1cm} (2.36)

\[ |\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B) \]  \hspace{1cm} (2.37)

\[ |\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B) \]  \hspace{1cm} (2.38)

Bell-states are maximally entangled states and form a basis for the bipartite qubit states. They play a special role in many experimental and theoretical investigations. Here, the Bell-state \( \Psi^- \) plays a special role because it is the only state that is anti-correlated in all three MUBs. It is often called the qubit singlet state.

### 2.3.2 Ways to Encode Photonic Quantum Information

In the previous section quantum states have been considered mainly as abstract vectors in a two-dimensional Hilbert space. This section will focus on physical realizations of photonic quantum states.

There are many different ways to encode quantum information on single photons. Theoretically, every photonic degree of freedom (DOF) could be utilized to generate superpositions or entanglement. However, technical difficulties complicate the implementations in quantum optical laboratories. For the presented findings, two DOF are of special importance and will be discussed separately: polarization and transverse spatial modes. For the sake of completeness, other DOF for which photonic entanglement has been demonstrated should also be mentioned: phase and momentum [60], energy and time [61], position and momentum [62] or discrete colors [63]. Most of these systems work with continuous properties, which can be discretized to create not only qubit but higher-dimensional quantum states. Two popular examples are time-bin [64, 65] or path entangled photons [66, 67].
Polarization

Polarization is the most widely used DOF in quantum information science realized with single photons [68]. Ways to create, manipulate and measure the polarization states of single photons like down conversion crystals, wave plates, polarization filters, and polarizing beam splitters are matured technologies. As described in the previous sections for classical fields, polarization is limited to two mutually exclusive states. Therefore, polarization naturally forms a two-dimensional quantum state, which can be represented by the introduced qubit formalism.

The MUBs are represented by the following polarizations

\[
\begin{align*}
\text{horizontal} : |H\rangle &= |0\rangle \\
\text{vertical} : |V\rangle &= |1\rangle \\
\text{diagonal} : |D\rangle &= \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\
\text{anti-diagonal} : |A\rangle &= \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle) = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
\text{right-handed circular} : |R\rangle &= \frac{1}{\sqrt{2}} (|H\rangle + i |V\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + i |1\rangle) \\
\text{left-handed circular} : |L\rangle &= \frac{1}{\sqrt{2}} (|H\rangle - i |V\rangle) = \frac{1}{\sqrt{2}} (|0\rangle - i |1\rangle)
\end{align*}
\]

Again, the MUBs define the axes of a three-dimensional sphere, called Poincaré sphere (see Fig. 2.11 on the left). The linear polarization states, which include the computational basis (Eqs. (2.39a) and (2.39b)), are distributed around the equator, while the poles are constituted by right and left-handed circular polarization. Since all MUBs can be used to describe any state equivalently, the difference to the Bloch sphere (Fig. 2.10) is only a matter of convention.

Transverse Spatial Modes

Although spatial modes of light have been known for a long time, their use in quantum physics started quite recently. In the seminal paper from Mair et al. [70] in 2001 the entanglement of LG modes was shown for the first time. Shortly after that paper, an interesting feature of spatial mode entanglement, namely the possibility to create higher-dimensional entanglement, was shown in another experiment [71]. Since then, LG modes have been used in many different quantum experiments for various tasks like quantum bit commitment [72] quantum coin tossing [73], high-dimensional cryptography [74], higher-dimensional entanglement [75] even up to 100 dimensions [76], and novel uncertainty relations in angle and angular momentum [77, 78].

Although the main focus will lie on LG modes, it is worthwhile mentioning, that the entanglement of other modes, like OAM containing Bessel-Gauss modes [79] or Ince-Gauss modes [80], have also been demonstrated recently. Latter ones are especially interesting since they have an additional parameter, the ellipticity. Here,
two modes of the same order (same quantum numbers $p$ and $m$) can differ or even be close to orthogonal (mutually exclusive) if the ellipticity for both modes is different. This IG gauss specific feature might be advantageous as an extension to already known cryptography protocols (for more information see [80]).

The general notation for two-dimensional quantum states encoded in spatial modes will be analogous to the one presented for polarization. The analog to the Poincaré sphere, the two-dimensional OAM sphere for $l = \pm 1$ can be seen in Fig. 2.11. The poles are defined by the LG mode with right-handed or left-handed OAM, respectively, whereas along the equator equally weighted superpositions can be found. These superpositions show an interesting feature, namely they all have the same intensity structure, which rotates with the varying phase of the superposition. Because measurements in two of three MUBs, e.g. measurements around the equator, are sufficient to verify entanglement (which will be discussed in the following section in detail), it is interesting to investigate this feature in more detail. A superposition of LG modes of the same order $l$ can be written as

$$|\psi_{LG}\rangle = \cos(\theta/2) |LG_{l,p}\rangle + e^{i\varphi} \sin(\theta/2) |LG_{-l,p}\rangle,$$  

(2.40)

where the phase is $\varphi$ and the amplitudes are weighted by $\theta$. With equal weighting ($\theta = \pi/2$) and only looking at the angle $\vartheta$ dependence of the intensity $I(\vartheta)$, it follows

$$I(\vartheta) \propto \cos^2 \left[ \left( \frac{\varphi}{2l} - \vartheta \right) \cdot l \right].$$  

(2.41)
Fig. 2.12 Intensity structures of superpositions for different-order LG beams. The intensity shows $2l$ maxima arranged in a ring. The rotation angle $\gamma$ is depended on the phase $\varphi$.

There are $2l$ intensity maxima if the angle $\theta$ is varied from 0 to $2\pi$, which corresponds to a closed loop around the optical axis. Moreover, changing the phase $\varphi$ of the superposition leads to a simple rotation of the structure, where its angular position $\gamma$ (in degree) is connected to the phase as follows

$$\gamma = \frac{\varphi}{2l} \cdot \frac{360^\circ}{2\pi}. \quad (2.42)$$

The intensities of different superpositions and various phases can be seen in Fig. 2.12. Importantly, it can be seen that the angular position of the intensity for superpositions of $\pm l$ are directly related to the phase of the superposition. Hence, every equally weighted superposition (two of three MUBs) can be accessed by only measuring the intensity structure and its angular position. Before introducing a novel way of measuring LG mode superpositions, which takes advantage of this feature (Chap. 3), it will be shown in the following section that measuring two of three MUBs is enough to prove entanglement.

### 2.4 Tests of Quantum Entanglement

There are many different methods to show entanglement of the created quantum state. A very powerful way is to identify strict bounds for classical (separable) states, which have to be tested under specific, properly defined assumptions. For the experimental detection of entanglement, this generally leads to an inequality, which can only be violated if the measured data is based on entangled quantum systems.

Three different popular types of such entanglement tests will be briefly discussed in the following sections (for more details see [81]), starting with the one that makes the strongest assumptions, a so-called entanglement witness.
2.4 Tests of Quantum Entanglement

2.4.1 Entanglement Witness

Entanglement witnesses fully rely on quantum mechanical assumptions. The maximum expectation value of the witness for general bipartite separable states, like

\[
|\Psi^{sep}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \left( \cos(\theta_1/2) |0\rangle + e^{i\varphi_1} \sin(\theta_1/2) |1\rangle \right) \otimes \left( \cos(\theta_2/2) |0\rangle + e^{i\varphi_2} \sin(\theta_2/2) |1\rangle \right),
\]

is utilized as an upper bound. If the bound is exceeded, the system is entangled. In the following, two slightly different types of entanglement witnesses that measure the visibilities in the MUBs will be derived.

**Witness in 3 MUBs**

The first witness operator can be constructed as follows:

\[
\hat{W}_3 = \hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z,
\]

where \(\hat{\sigma}_x, \hat{\sigma}_y\) and \(\hat{\sigma}_z\) stand for the single-qubit Pauli matrices for the bipartite systems (positions denote the systems).

Measurements of the single-qubit Pauli matrices for a bipartite system correspond to the visibility measurements in the MUB denoted by the index. Thus, the different terms of the witness \(\hat{W}_3\) can be rewritten in terms of projection operators

\[
\hat{P}_{i,i} = |i, i\rangle \langle i, i|,
\]

where \(i\) can stand for any vector of all MUBs (see Eq. (2.29)). This leads e.g. for the \(x\)-basis to

\[
\langle \hat{\sigma}_x \otimes \hat{\sigma}_x \rangle = vis_x = \begin{vmatrix}
< \hat{P}_{+,+} > + < \hat{P}_{-,+} > - < \hat{P}_{+,-} > - < \hat{P}_{-,-} > \\
< \hat{P}_{+,+} > + < \hat{P}_{-,+} > + < \hat{P}_{+,-} > + < \hat{P}_{-,-} >
\end{vmatrix},
\]

where the indices + and − denote “\(|0\rangle + |1\rangle\)” and “\(|0\rangle − |1\rangle\)” respectively. In order to be independent from correlated or anti-correlated measurements the absolute value is used. The same holds for the other MUBs \(y\) and \(z\) too.

The index of \(\hat{W}_3\) labels the number of MUBs taken into account. The expectation value for general separable states can be calculated as

\[
W_3 := \langle \Psi^{sep} | \hat{W}_3 | \Psi^{sep}\rangle = \sin(\theta_1) \cdot \sin(\theta_2) \cdot \cos(\varphi_1 − \varphi_2) + \cos(\theta_1) \cdot \cos(\theta_2),
\]
which has a maximal value of 1. Hence, the entanglement criterion written in more “experimental” terms of visibilities $vis_i$ is

$$W_3 = vis_x + vis_y + vis_z \begin{cases} \leq 1 & \text{separable} \\ > 1 & \text{entangled.} \end{cases}$$ (2.50)

Maximally entangled states have perfect correlations (or anti-correlations) in all three MUBs, i.e. visibilities of 1 for each MUB, which leads to a maximum achievable value of 3 within the quantum mechanical framework. Therefore, the classical bound of 1 can be violated with entangled bipartite systems where an average visibility of $>30\%$ can be measured.

**Witness in 2 MUBs**

So far, the calculation of the classical bound for measurements in all three possible MUBs was considered. However, sometimes it might be simpler not to measure all of them due to experimental constraints. If only two of the three MUBs are taken into account e.g. $\sigma_x$ and $\sigma_y$ (as is it actually the case for some of the presented measurements in this thesis), the witness can be reformulated to

$$\hat{W}_2 = \hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y.$$ (2.51)

A similar straightforward calculation of the expectation value for separable bi-photon states yields

$$W_2 := \langle \Psi^{sep} | \hat{W}_2 | \Psi^{sep} \rangle = \sin(\theta_1) \cdot \sin(\theta_2) \cdot \cos(\varphi_1 - \varphi_2),$$ (2.52)

which is also upper bounded by 1. Again, the entanglement criterion for measurements in 2 MUBs can be written in a similar manner as before

$$W_2 = vis_x + vis_y \begin{cases} \leq 1 & \text{separable} \\ > 1 & \text{entangled} \end{cases}$$ (2.53)

Because the bound has the same value as before but less measurements are taken into account, the verification of entanglement gets more challenging in the experiment. In other words, the highest obtainable value for maximally entangled states is only 2. Hence, it is harder (the average visibility in both MUBs needs to be $>50\%$) to prove entanglement by only measuring the visibilities in two MUBs.

### 2.4.2 Steering-Inequality

A second way to determine if the bipartite system is entangled or not is the test of a steering-inequality. The concept of “steering” as a feature of quantum mechanics was originally discussed by Schrödinger [82]. Although it can be used to test
entanglement, i.e. it is a subclass of more general tests for non-separability like entanglement witnesses, it actually makes weaker assumptions in its derivation, thus excludes a broader class of states. In the derivation of the steering-inequality, which was introduced by Wiseman and co-workers [83], only one party requires measurable probabilities in agreement with quantum mechanics (like in the witness derivation), while the other accepts probabilities obeying local realism. Therefore, by demonstrating non-classicality with a steering-inequality not only non-separability is proven but also the non-local steering of the investigated property (the interested reader is directed to [82–85] for more information and the derivation of the inequality).

The steering-inequality is

\[
S_{\text{St}} = |\hat{\sigma}_x \otimes \hat{\sigma}_x|^2 + |\hat{\sigma}_y \otimes \hat{\sigma}_y|^2 + |\hat{\sigma}_z \otimes \hat{\sigma}_z|^2 \leq 1.
\] (2.54)

Compared to the witness criterion (2.50), the steering-inequality is violated by a smaller class of states. Mathematically, this is expressed by squaring of the three terms. Again, the quantum mechanical prediction for maximally entangled states is three. The steering-inequality was already successfully tested in various experiments [86, 87] even in a loophole free manner [88].

### 2.4.3 Bell-CHSH-Inequality

The oldest and by far most popular way to show entanglement in an experiment, is the test of a Bell-inequality. Nearly twenty years after the first discussion of Einstein et al. [89], Bell discovered a way to experimentally distinguish between the local-realistic description of the world and the one, which quantum mechanics describes [90]. Besides all philosophical implications, this enabled experimentalists to test if entanglement can be found in the investigated system. Bell found an experimentally testable inequality, which is upper bounded by any local realistic theory. A separable two qubit quantum state can be regarded as such a local-realistic model. If the experiment shows a violation of this so-called Bell-inequality, there is no separable description of the state. Thus, the state must be entangled. As seen in the previous sections, nowadays there are different ways to verify entanglement. However testing a Bell-inequality is still the most important proof, since in its derivation nothing but a local-realistic description of the world is assumed (for both parties). Therefore, only the experimental violation of a Bell-inequality strictly proves the non-classicality of nature (assuming all loopholes are closed, which has not yet been achieved in a single experimental test).

---

9Without explaining the concept of local realism in more detail, I use the modern, standard way of defining locality, i.e. two space-like separated systems cannot influence each other faster than the speed of light, and realism, i.e. measurable properties do not exist before and independent of the measurements.
In this thesis the well-known Bell-CHSH-inequality will be used (for its derivation I refer to the original work by Clauser et al. [91]):

\[ S = |E(\alpha, \beta) - E(\alpha', \beta') + E(\alpha, \beta')| \leq 2 \] (2.55)

where \( \alpha, \alpha', \beta, \) and \( \beta' \) label different measurement settings and \( E \) is the normalized correlation function for photon pairs to be found with these settings. \( E \) can be calculated from the measurements as follows:

\[ E(\alpha, \beta) = \frac{C(\alpha, \beta) - C(\alpha^\perp, \beta) - C(\alpha, \beta^\perp) + C(\alpha^\perp, \beta^\perp)}{C(\alpha, \beta) + C(\alpha^\perp, \beta) + C(\alpha, \beta^\perp) + C(\alpha^\perp, \beta^\perp)} \] (2.56)

where \( C \) denotes the measured coincidence count rates at the settings \( \alpha, \beta \). The superscript \( \perp \) indicates the orthogonal setting.

If the measured result surpasses the upper limit of 2, the measured state is entangled, to be more precise, it is not explainable by local realism. Maximally entangled states can reach the quantum mechanical limit of \( 2\sqrt{2} \).

All three presented types of entanglement tests can be used to reveal entanglement and show a certain hierarchy: Demonstrating entanglement by a Bell-test is harder than by a steering-inequality. Similarly, testing entanglement by a steering-inequality is more challenging than proving it by an entanglement witness. The hierarchy directly follows from the strength of the assumptions that have to be made during the derivation and was proven using Werner states and projective measurements in [83, 84].

With this, the main theoretical considerations, which will be required later, have been completed. In the following, three different experiments that have been conducted during the time of the thesis will be presented and the results will be discussed. At first, a novel flexible way to efficiently create arbitrary spatial mode entanglement will be introduced. It will be used in a first experiment to entangle two photons with very high orbital angular momentum quanta.

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