The Theory of Parallel Lines in the Age of Leibniz

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2.1 From Antiquity to the Renaissance.

It may be useful to give a picture of the discussions on the Parallel Postulate in the age of Leibniz, as well as a list of his mathematical sources on the topic.[1]

The definition of parallel lines, in Euclid's wording, is that of "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction". The Parallel Postulate appears as Postulate Five in the First Book of the *Elements*, and states that "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than the two right angles".[2] In Early Modern editions of Euclid, however, the Postulate was normally arranged among the *axioms*. Axioms were normally distinguished from postulates, meaning that the latter were principles employed in licensing geometrical constructions (e.g. Postulate One: "to draw a straight line from any point to any point") while the former expressed states of affairs (for instance: "two straight lines do not enclose a space").[3] In this systematization, the Parallel Postulate

[1] The classical reference for the history of non-Euclidean geometry is Bonola 1906, to which at least Pont 1986 should be added.

[2] Here and elsewhere, I take the English wording from Heath 1925.

[3] The classification of the Parallel Postulate among the axioms (or common notions) dates back at least to Geminus (first century BC), and was discussed by Proclus, *In Euclidis* 182–83. In modern times, Simon Grynaeus, the editor of the *editio princeps* of the *Elements* (1533), accepted Proclus' observation and classed (following a Greek manuscript he found) the Parallel Postulate as Axiom 11 (after the original Euclidean Common Notions and a few newly interpolated axioms). As most of the modern editions of Euclid were based on the Greek text established by Grynaeus, his classification of the principles was widely accepted, and the naming of the Parallel Postulate as Axiom 11 is very recurrent. Clavius adopts Grynaeus' numbering in the first edition of his commentary, but after having introduced one more axiom in his attempt to prove the Parallel Postulate (in the second edition of his commentary), he rearranges the order of the principles and the Parallel

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seems to fit better among the axioms, and Leibniz follows the Early Modern use in this respect. In any case, Leibniz' program to prove all the principles of geometry applies uniformly to both axioms and postulates.

Innumerable difficulties have been raised since Antiquity with regard to both the definition of parallels and the Parallel Postulate; with regard to the definition, because it refers to infinity, or at least to indefinite extension of straight lines; with regard to the postulate, as it seems too complex to be an authentic principle, and because (moreover) the inverse implication (*Elements* I, 28: "if a straight line falling on two straight lines make the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another") is in fact provable without the need of any special principle.

In Late Antiquity, Proclus reported that several attempts had been made in the past to prove the Parallel Postulate, and he discussed some of them, eventually accepting one demonstration as sound. Roughly in the same era, the Aristotelian commentator Simplicius also transmitted a different proof of the same statement.[4] While Proclus' commentary was basically lost during the Middle Ages, Simplicius' work circulated in the Islamic world, engendering a long and beautiful tradition of attempts to prove the Parallel Postulate. The level of sophistication of these discussions was very high, and the works by Thābit ibn Qurra, Ibn al-Haytham, Omar Khayyām, and Nasīr ad-Dīn at-Tūsī on the theory of parallels introduced concepts and techniques which are among the most important developments in the history of the foundations of geometry.[5]

In the West, Simplicius was already known as early as the Middle Ages, and Proclus was rediscovered in the early sixteenth century; none of the more advanced Arabic sources, however, was available for a long time. The Parallel Postulate began to be widely discussed when the Jesuit mathematician Christoph Clavius (1538–1612) published his all-important edition of Euclid's *Elements* in 1574, including in it Proclus' proof of the Postulate and claiming it be a theorem.[6] Some years later, Clavius

Postulate becomes Axiom 13. In Leibniz' papers, it is sometimes called "Axiom 11", but more often "Axiom 13" as he was working mainly on Clavius' (later) edition. See De Risi [c].

[4] Proclus, *In Euclidis* 364–73. Simplicius *apud* an-Nayrizi, in Lo Bello 2003a, pp. 157–66 (original Arabic text in Besthorn&Heiberg 1932). On the latter text see Sabra 1969, who maintains that the proof appearing in an-Nayrizi is in fact by one Agapius ("Aganis" in the text; cf. Sabra 1968), while the original proof by Simplicius is to be found in another Arabic manuscript that Sabra edited (as well as in a couple of other Arabic texts).

[5] For a French translation of the most relevant Arabic texts, see Jaouiche 1986. Among the histories of non-Euclidean geometry, Rosenfeld 1988 is especially concerned with the developments in the Islamic World. As for the knowledge of medieval sources that Leibniz may have had, we are informed that several medieval Latin manuscripts were held in the library of Wolfenbüttel, that Leibniz directed for 25 years: see Folkerts 1981. A few of these Latin treatises were translated from Arabic, and were drawing on independent Arabic sources. These works, however, were limited to the most elementary developments of the rich season of Islamic medieval mathematics. Moreover, at the times of Leibniz all the information concerning these texts might be found in Renaissance editions of Euclid; we are thus not compelled to claim that Leibniz was actually reading Wolfenbüttel's manuscripts.

[6] Truth be told, the proof of Proclus was already published in the important Latin edition of the *Elements* by Federico Commandino, which appeared in 1572 (and in Italian translation in 1575).

came across an Arabic manuscript containing several versions of Nasīr ad-Dīn's commentary on the Parallel Postulate, and was able to extract some information from it. The text disclosed to him the rich and complex tradition of the geometrical researches of the Islamic world, and Clavius was forced to change his mind, disregard Proclus' demonstration of the Parallel Postulate as weak and flawed, and deeply revise Euclid's system of principles adding one more axiom and several demonstrations. He devised, then, two proofs of the Parallel Postulate, largely dependent on the Arabic sources, and published them in the second edition of his Euclidean commentary (1589).[7] Clavius' book was later re-issued several times and for two centuries it was considered the best and most complete edition of the *Elements* available to advanced scholarship, and the one text to be referred to on foundational issues in geometry. Thus, the entire discussion on the Parallel Postulate in the Early Modern Age started from, and continuously referred to, Clavius' work.

Clavius' first demonstration of the Postulate is especially relevant for us, as it relies on the concept of equidistance. Since Antiquity, in fact, various geometers had proposed to change Euclid's definition of parallels as non-incident straight lines into that of "equidistant straight lines". Starting with the latter definition, and implicitly assuming that the defined object (the equidistant straight lines) exists, it was possible to (correctly) prove the Parallel Postulate: and Proclus' demonstration was in fact relying on this definition and hidden existential assumption.[8] The mistake is

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Clavius' 1574 edition is not very different on this issue, but it would improve after the second edition.

[7] In his commentary, Clavius opened the discussion of the Parallel Postulate acknowledging his Arabic sources: "Id quod in Euclide quodam Arabico factum etiam esse accepi, sed nunquam facta mihi est copia demonstrationem illam legendi, etsi obnixe illud iterum atque iterum ab eo, qui eum Euclidem Arabicum poßidet, flagitavi" (*Euclidis*, p. 50; in his copy of Clavius' book, Leibniz has underlined the passage). Nasīr ad-Dīn had written two different treatises on the Parallel Postulate (the second being a variation of the first), to which one should add a spurious essay, the so-called *Longer Version* edited by a pupil of his in 1298, a few years after Nasīr ad-Dīn's death. A copy of the three works was owned (along with many other Arabic manuscripts, such as that of Books 5–7 of Apollonius' *Conics*) by the Medici family; their librarian and orientalist Raimondi was appointed to publish most of these manuscripts, and to this end he founded and directed the Typographia Medicea in Rome. Raimondi published Nasīr ad-Dīn's *Longer Version* (in Arabic) in 1594. Nasīr ad-Dīn's works, however, had already been known in Rome for some years (the edition had started in 1588), and Clavius was therefore able to make some use of them in 1589, even though his relations with Raimondi were not excellent (see Baldini&Napolitani 1992). In fact, Clavius' proof in the *Euclidis* shows that he could also rely on the two original essays by Nasīr ad-Dīn, which are more complex and deeper than the published *Longer Version*. The three Arabic works are now published in French in the above-mentioned Jaouiche 1986. On the edition in Rome, see Knobloch 2002a, and (most of all) the volume edited by Cassinet 1986.

[8] This definition should date back to Posidonius (second century BC) and is reported by Heron, *Definitiones* 70, and Proclus, *In Euclidis* 176, who employed it in a proof of the Parallel Postulate in *In Euclidis* 371–73. It was discussed by Simplicius (or Agapius) and an-Nayrizi, too (see Tummers 1994, pp. 23–25, for the Latin text; cf. Lo Bello 2003b, pp. 39–41), who not only advanced a proof of the Parallel Postulate relying on equidistance, but also simply called "equidistant" the non-intersecting lines (as they assumed the truth of the Postulate). Since an-Nayrizi was well-known in the West from the Middle Ages, it happens that Renaissance and Early Modern editions of Euclid (like the one by Commandino) still called "equidistant" two straight lines that are defined as non-

that the definition assumes that the line equidistant to a straight line is itself straight. The assumption of the existence of equidistant *straight* lines is in fact equivalent to the Parallel Postulate, and in hyperbolic geometry (where the Postulate is false) the line equidistant to a straight line is not straight (it is a curve called *hypercycle*).[9] Clavius, who had accepted Proclus' proof in the first edition of his commentary, later realized that he had to prove that the equidistant line is straight, and tried to establish this result, first, by simply relying on a "good" definition of a straight line (which was itself one of the biggest foundational problems in the Early Modern Age, as Euclid's definition seems to be vague and obscure); and, second, with a kind of kinematic argument. This is as follows: let a straight segment make a right angle with a given straight line, and let it move along the line preserving the right angle (that is, it moves parallel to itself); then the flow of the other end of the segment describes a line which is equidistant to the given straight line. This equidistant line, moreover, has to be straight (this is the argument) as the motion is *uniform* and it always keeps the same direction; it seems impossible that such a flow could draw a curved line. This is false, of course, but the conclusion is almost irresistible for the imagination, and the mistake was hard to detect before a complete mathematical explanation of Gaussian curvature.[10] Clavius' argument from motion (which had been already employed by the Arabs, and by Thābit in particular) was repeated and modified innumerable times after him, and was still alluring geometers in the nineteenth century. We will see that some of the most important contributions of Leibniz to the foundations of geometry deal with the definition of a straight line, and with the exact, mathematical definition of uniformity, homogeneity, and other similar properties of space (and motion) that classical geometry had left vague. It will be clear, then, that any proof of the straightness of the equidistant line would be far from being elementary.

Clavius' second demonstration of the Parallel Postulate is less important for us (and for Leibniz). It is a simple variation on Nasīr ad-Dīn's original proof, and stumbles upon a wrong limiting process extending a local property to a global result.

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intersecting. These editions then simply inadvertently skipped from the *name* of equidistance (that should characterize non-incident lines) to the *fact* of their actual equidistance, and thus were able to prove the Parallel Postulate through Proclus' demonstration (see Commandino, *Euclidis*, p. 20). These texts contributed to blurring the fact that one has to prove the equidistance of non-intersecting straight lines.

[9] We may remark that the above-mentioned Heron (*Definitiones* 70) only defines equidistant *lines* (γραμμαί), and not equidistant *straight lines* (εὐθεῖαι); properly speaking, he is not making any mistake or surreptitious assumption. The *title* of his definition, however, explicitly mentions equidistant straight lines (but this title may be spurious) and his following definition of converging straight lines (*Definitiones* 71) also explicitly refers to εὐθεῖαι. We don't know whether Heron may have been aware of any foundational difficulty on this point. To my knowledge, the first mathematician to explicitly recognize the *petitio* is Alfonso de Valladolid (1270–1350), even though the repeated attempts of the Arab and Persian geometers to produce a proof of the Fifth Postulate not grounded on equidistance may reveal some awareness of the problem. It is clear, in any case, that the difficulty was not easily detected in the Renaissance and the Early Modern Age. On Alfonso's criticisms of the proof, see Lévy 1992.

[10] For Clavius' proof, see the long *Scholium* before *Elements* I, 29, in his *Euclidis*, pp. 48–53.

Basically, it doesn't take into proper consideration the possibility of asymptotic lines, and thus fails to recognize the central issue in hyperbolic geometry. Men like Wallis and Leibniz, more used than Clavius (or Nasīr ad-Dīn) to dealing with the infinite in mathematics, would not fall into this trap, even though a complete and perspicuous explanation of Nasīr ad-Dīn's misstep would only be given by Saccheri.[11]

It is hard, in any case, to overemphasize the importance of Clavius' commentary for Leibniz. He studied Euclid's *Elements* from this work, and filled it with marginal notes. Almost all the references to Euclid in Leibniz' works are in fact references to Clavius, and it seems that he always read Euclid in Clavius' translation and accompanied by Clavius commentary. Leibniz' own foundational studies on elementary geometry often took the form of a literal commentary on Clavius' edition. Given the large time span in which these essays were written, we can imagine that Leibniz kept Clavius' book on his working desk for most of his life, and it surely represents the major source for his geometrical researches.[12]

After Clavius, the seventeenth-century discussion on the Parallel Postulate took several directions.

2.2 Jesuit and French Attempts in the Seventeenth Century.

The Jesuits went on to produce important commentaries on Euclid, but none of them reached the level of their common master. The greater part of these works were aimed at students of Jesuit colleges, and tried to simplify Clavius' presentation. It was common, for them, to present parallel lines as equidistant lines without any foundational concern; the Parallel Postulate was thus generally considered provable from this (flawed) definition. Among Jesuit textbooks, we have to mention the *Cursus mathematicus* (1661) by Kaspar Schott (1608–1666), which Leibniz read and annotated. Schott, a Jesuit disciple of Athanasius Kircher, defined parallel lines as non-incident straight lines (even though he immediately remarked that they are in fact equidistant) and thought to prove the Parallel Postulate just by noting that the inverse proposition (*Elements* I, 28) is provable – a line of reasoning that had been

[11] We know that Wallis was not satisfied with Nasīr ad-Dīn's proof, but he doesn't offer a refutation of it. In fact he could have been simply unsatisfied by the explicit assumptions made by the Persian mathematician, which may be not more evident than the Postulate itself. The spurious *Longer Version* of Nasīr ad-Dīn's commentary, as a matter of fact, assumed several unproved Lemmata, which Nasīr ad-Dīn had demonstrated in his original essays on the subject. It is true, however, that Wallis had access to the original essay as well (see below, *note* 26); in any case, he doesn't explicitly criticize any demonstrative step in Nasīr ad-Dīn's proof. See Wallis, *De postulato quinto*, p. 673. Saccheri's complete refutation of Nasīr ad-Dīn's proof is in the Third Scholium after Proposition 21 of *Euclides vindicatus* (pp. 38–41; see my notes *ad loc.* in De Risi 2014 [a]).

[12] Leibniz was using the 1607 edition of Clavius' commentary, which does not differ sensibly from the final 1612 edition (from which we quote). Leibniz' copy of Clavius' commentary is currently kept in the *Landesbibliothek* in Hannover, and has several handwritten marginal notes. Among them, we find a heavily annotated index, in which Leibniz spelled out the propositions needed to prove every theorem of Euclid, so to expose the deductive structure of the *Elements*.

followed, in the Renaissance, by Pierre de la Ramée, other Ramist logicians, and an otherwise fine geometer like Peletier.[13]

Among Jesuit texts that Leibniz knew, we should also include the important *Synopsis geometrica* (1669) by Honoré Fabri (1607–1688). This book, which raised a few interesting foundational issues (Fabri was a logician of some value), simply defined parallel lines as equidistant, and proved in passing the Parallel Postulate (as an uninteresting corollary). Similar stances were taken by André Tacquet (1612–1660) and Milliet Dechales (1621–1678) in their editions of the *Elements*.[14] The only exception to this trend in Jesuit schoolbooks is represented, of course, by Gerolamo Saccheri (1667–1733), who strongly criticized the many simplifications of his Jesuit fellows in the foundational domain, and aspiring to produce a thorough analysis of Euclid's axioms ended up with the first ever treatise on non-Euclidean geometry. Even though Leibniz knew about Saccheri and read (and criticized) his work on statics, however, the outstanding accomplishments on the theory of parallels of *Euclides vindicatus* (1733) arrived too late for him.[15]

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[13] Peletier's edition of Euclid's *Elements* seems to be the source of Schott's "demonstration". As for Peletier, see his *Demonstrationum in Euclidis elementa*, p. 10. As for Schott, he defined parallel lines as non-incident and then as equidistant straights in these two statements: "34. Parallelae rectae lineae sunt, quae cum in eodem sint plano, et ex utraque parte in infinitum producantur, in neutram sibi mutuo incidunt. 35. Parallelogrammum est figura quadrilatera, cujus bina opposita latera sunt parallela, seu aequidistantia" (Schott, *Cursus*, Book 3, p. 64; cf. also Book 1, Chap. 3, art. 4, p. 6). He had already explained how to trace, in several ways, a parallel line to a given straight (*Cursus*, Book 1, Chap. 4, art. 2, pp. 9–10). And before proving *Elements* I, 29, he gave his proof of the Parallel Postulate based on the conversion of *Elements* I, 28. It goes as follows. *Elements* I, 28 has shown that if two straights are cut at right angles by a third line, they don't meet. It should follow, then, that if the sum of the angles formed by two straights with a transversal is less than π, then the two straight lines are "less divaricated" than earlier and thus they meet; should they not meet, in fact, they would be as divaricated as earlier, and thus would form two right angles with the transversal, which is absurd. Here the original of this very clumsy passage: "si enim duae rectae, AB, CD, tunc sunt parallelae, quando recta linea EF incidens facit angulos internos duobus rectis aequales, eo quod tunc a se invicem ita sunt divaricatae, ut concurrere non possint versus illas partes, ad quassunt recti anguli: ergo si facit eosdem duobus rectis minores, necessario concurrere debent ad partes, ubi minores sunt anguli, eo quod tunc minus a se invicem divaricentur quam antea: alioquin si aeque divaricarentur ut antea, recta linea EF incidens faceret duos angulos internos duobus rectis aequales ut antea. Adhiberi ergo in posterum potest Axioma illud 13. Euclidis, si non ut Axioma, saltem ut Theorema evidenter deductum ex demonstratione: tametsi id aliter et multis demonstrare conetur Proclus, Clavius, Tacquet, et alii" (*Cursus*, Book 3, p. 73). We may remark that the philosopher Salomon Maimon (1753–1800) was to advance a proof of the Parallel Postulate based on the conversion of *Elements* I, 28, as late as 1797; he grounded it on the Kantian claim that geometrical statements are synthetic *a priori* judgements, and on a further (non-Kantian) argument stating that in this class of propositions the subject and the predicate have to be coextesive: see the *Schlußanmerkung* to his *Kritische Untersuchungen über den menschlichen Geist* (pp. 361–70), as well as the commentary in Freudenthal 2006.

[14] Fabri's definition of parallels is in *Synopsis*, p. 30. His proof of the Parallel Postulate appears later as Corollary 3 to Proposition 1 (p. 83). Apart from reading Fabri's and Tacquet's works, Leibniz met Dechales in person when he was in Paris.

[15] Leibniz read Saccheri's *Neo-statica* (1708) and criticized it in a letter to Des Bosses dated August 23rd, 1713 (in GP ii, p. 482). During his lifetime, Saccheri enjoyed some celebrity in the Jesuit milieu, mostly because of his reputation as an extremely quick-minded thinker (he was a chess

In open rivalry with Jesuit textbooks, Antoine Arnauld (1612–1694) wrote a book on Euclidean geometry, the *Nouveaux elémens de géometrie* (first edition 1667; second revised edition 1683), which was meant to be the Jansenist treatise on mathematics, as well as (and more importantly) the Cartesian handbook on elementary geometry. Arnauld's attitude toward the Parallel Postulate is pretty equivocal, though. He defines parallel lines as non-incident lines, claiming that this is a "negative" definition, while the "positive" definition of parallelism is given through equidistance; the two definitions are then (rightly) regarded as equivalent by means of the Parallel Postulate.[16] As for proving the Postulate itself, Arnauld thinks that it is useless to try, since it is self-evident enough. In the end, however, he also gives Nasīr ad-Dīn's proof which he takes from Clavius (but without quoting him, as he was a Jesuit). Leibniz, who had an important correspondence with Arnauld on metaphysical topics, read and commented on the *Nouveaux elémens* in 1693.

Among the developments of the foundations of geometry in France, we should mention the geometrical work of Gilles Personne de Roberval (1602–1675), who already in 1642 was planning a new and improved version of the *Elements* (along the same lines as Arnauld). Roberval returned to the project in his old age, and was working on certain *Elemens de geometrie* in 1673–1675, the same years in which Leibniz joined the French community of mathematicians in Paris. On January 2nd, 1675, in a joint session of the *Académie des Sciences*, Leibniz presented to the scientific world his new calculating machine and Roberval his work on the foundations of geometry, which was intended to appear around Easter of the same year. In October, however, Roberval died without completing this book; and already in December Leibniz wrote to Oldenburg that he had been able to consult the manuscript. Roberval's *Elemens* were only published in the twentieth century, but Leibniz was greatly impressed by them and referred to Roberval's project throughout his life.[17] Roberval's aim was a complete reform of elementary geometry, as he was not satisfied by the carelessness of some Greek geometrical proofs, as well as by the system of principles governing Euclid's work. Roberval's theory of parallels did not offer any truly new solution to the problem, as he defined parallel lines as equidistant lines and then assumed an axiom stating that if two straight lines approach one another, they will continue to approach further on; a principle that had been already stated by Nasīr ad-Dīn and is

champion, for instance). Although he worked on the theory of parallels for more than thirty years, he didn't publish or anticipate anything on the subject and the *Euclides vindicatus* only went to print in the last weeks of his life.

[16] See ARNAULD, *Nouveaux elémens*, pp. 391–92 (ed. Descotes). The proof follows immediately thereafter.

[17] The edition of Roberval's *Elemens* is JULLIEN 1996, which also gives some details about the history of the manuscript. Leibniz' letter to Oldenburg dates from the 28th of December 1675 is in A III, 1, n. 70, p. 328, and is dismissive about the actual results of Roberval's work (he does not suggest the publication of the work). Either because Leibniz changed his opinion about Roberval's accomplishments, or because he nevertheless endorsed his epistemological program, in the following years he often mentioned Roberval's *Elemens* with great approval. For some quotes, see below *note* 7 in § 3.

in fact equivalent to the Parallel Postulate.[18] This notwithstanding, the *Elemens de geometrie* are of interest for our problem for several reasons. First of all, they expressed an epistemological and logical program for a reform of elementary geometry that was rarely stated with similar clarity before Leibniz. Second, their rigor in proving the results of the theory of parallels is truly remarkable, and many elementary results on quadrilaterals (proven without assuming the Parallel Postulate) are quite advanced and very relevant for the general theory (they would later be rediscovered by Saccheri). Third, Roberval had a fondness for purely logical reasoning that pushed him to prove simple results, such as the symmetry of the relation of parallelism, that was later to be shared by Leibniz as well.[19]

2.3 Italian and British Attempts in the Seventeenth Century.

Another line of development of the theory of parallels in the seventeenth century is to be found in the Italian school (consisting mostly in Galileo's legacy). Giovanni Alfonso Borelli (1608–1679) wrote an important *Euclides restitutus* (first edition 1658; third emended edition 1679), which is one of the most daring attempts in the foundations of geometry of the Early Modern Age. Borelli dealt mainly with the theory of proportions, but also with some minor issues in the theory of parallels. He believed that Euclid's definition of parallel lines is defective since it contains a reference to infinity, but recognized that the definition of parallels as equidistant

[18] Roberval's axiom, to be found in Jullien 1996, p. 201, rules out *ultraparallel* lines in hyperbolic geometry (straight lines that approach one another until they reach a minimal distance between them, and then diverge again to infinity) and is thus strong enough to force a Euclidean structure on the model. Roberval's general strategy in the Book IV of the *Elemens* is to prove a large number of results on quadrilaterals, assuming Euclidean hypotheses about them, and then prove through the above-mentioned axiom the truth of these hypotheses. The previous theorems, then, allow him to establish several Euclidean results (the first of which, in Proposition 19, is that two perpendiculars to the same straight line are equidistant – which is false in hyperbolic geometry) and finally prove the original Parallel Postulate in Proposition 27. (Note that Roberval proves *Elements* I, 29 as Proposition 17 without the need of his axiom, because he expresses the statement employing the concept of parallels as equidistant lines, and under this definition the proposition is in fact a theorem of absolute geometry).

[19] Roberval defines the distance between a point and a line through the length of the perpendicular from the point to the line. Then he says that two straight lines are *equidistant* if the distances from all the points of one of them to the other line are equal. Finally, he calls two lines *parallels* if one is equidistant from the other and the second is equidistant from the first. In Proposition 16, Roberval demonstrates the symmetry of the relation of equidistance, thus proving (in his terminology) that equidistant lines are also necessarily parallels. The proof, which does not require further principles (nor the Parallel Postulate), is relevant because it opens the way to the consideration of parallelism as an equivalence relation, that was to interest Leibniz in his formal essays on *characteristica geometrica* and was to represent, later on, an important development in the attempts to prove the Parallel Postulate. Kant himself, in his attempt to prove the Postulate, was to insist much on the symmetry of the relation of equidistance (see his *Reflexionen* 8–10, in KgS xiv, pp. 33–51). The antecedent of Roberval's proof may be found in Thābit ibn Qurra, even though the two treatises on parallel lines by the great Arab mathematician were probably unknown to both Roberval and Leibniz (see Rashed&Houzel 2005).

lines is faulty. Thus, he defined parallel lines as straight lines which have a common perpendicular, and then stated the Parallel Postulate (which he did not try to prove) saying that parallel lines (so defined) are equidistant.[20] Borelli's fine argumentative turn, which dispensed with the attempts to prove the Postulate, was followed by Angelo Marchetti (1674–1753), whose *Euclides reformatus* (1709) accepted both Borelli's definition of parallel lines and the new formulation of the Postulate.[21] It was not followed by others, though. Vitale Giordano da Bitonto (1633–1711) wrote in 1680 (second edition 1686) an *Euclide restituto*, in Italian, which was intended as a posthumous homage to Borelli. Giordano saw with great clarity all the difficulties involved in proving the Parallel Postulate (the paralogisms deriving from the notion of equidistance, or the uniform motion, etc.) and accepted a definition of parallels as equidistant lines only to say that their actual existence should be proved through the Euclidean Postulate, and thus attempted a new and original demonstration of it (which takes up many pages of his volume). In the end, he also stumbled upon a limiting process, and claimed to have proven the famous statement; however, his attempt was complex and elegant, and is considered one of the best constructions before Saccheri.[22] Leibniz was well acquainted with these developments. He had an interesting correspondence with Giordano on the foundations of geometry (even though not on the theory of parallels properly speaking) in 1689, and was able to

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[20] Borelli introduced his new definition of parallel lines because it refers to a most clear and evident property ("cum exponatur per passionem possibilem & evidentissimam", as we read in the *Preface* of his *Euclides*). Moreover, since Euclid is able to construct the perpendicular to a given straight line without resorting to the Parallel Postulate (by *Elements* I, 10 and 11), one is able to show that Borelli's parallels exist without appealing to the Postulate itself (as Euclid does, apparently, in his *Elements* I, 31, at least to guarantee the uniqueness of the construction), or to another axiom about equidistant lines. Borelli's epistemological concerns about definitions, thus, are very similar to those by Leibniz about the "real definition" of a geometrical object (see below § 3). Borelli's formulation of the Parallel Postulate, in fact, directly assumes what Clavius had claimed to prove with his clumsy reference to the uniformity of motion, i. e. that the movement of a segment perpendicular to a straight line traces another straight line: "Axioma XIV. Si recta linea, in suo extremo semper perpendiculariter constituta super aliam rectam lineam, moveatur in transversum in eodem plano: alterum punctum extremum translatae rectae lineae in eius fluxu rectam lineam describet" (*Euclides restitutus*, 1658, p. 32; as Axiom 13 in the 1679 edition, p. 23). To this axiom, Leibniz remarked (in his copy of Borelli's book) that there is no necessity to assume that the flowing line is perpendicular to the given straight.

[21] See Marchetti, *Euclides reformatus*, pp. 10–11.

[22] Giordano criticisms of the Euclidean definition of parallel lines follow Borelli's own qualms about the use of infinity in a definition (see *Euclide restituto*, pp. 12–13). His demonstration of the Parallel Postulate occurs just before *Elements* I, 29, and can be read in *Euclide restituto*, pp. 60–66. Giordano's attempt in this direction is especially relevant because he discusses at some length the properties of the so-called "Saccheri quadrilaterals". Even though these figures (quadrilaterals with two equal sides at right angles with the base) were already treated in Medieval proofs (especially in Khayyām) and in Clavius, no one before Saccheri saw their foundational importance as Giordano did. On this point see Bonola 1905. It may be remarked that the last demonstration of the Parallel Postulate attempted by Leibniz (in the *In Euclidis* πρῶτα) also discusses these quadrilaterals (the Parallel Postulate being proved if one can show that the opposite angles in a Saccheri quadrilateral are also right angles; see *text 36* below).

read and annotate Giordano's *Euclide*.[23] By 1678, moreover, Leibniz had acquired Borelli's book (from the library of Martin Fogel), and Giordano recommended that he read it attentively; we have, in fact, Leibniz' marginalia to Borelli's *Euclides* as well.[24] Later in his life, Leibniz also had an exchange with Marchetti, but apparently they did not correspond on the theory of parallels.[25]

A further important line of development of the theory of parallels in the seventeenth century was engendered by the above-mentioned opinion of Henry Savile (1549–1622) that the Parallel Postulate is an ugly *naevus* in Euclid. Decades later, John Wallis (1613–1703), who was appointed Savillian professor of geometry in Oxford, decided to honor the will of his benefactor and directed his genius to the Euclidean blemish. In 1651 Wallis had Nasīr ad-Dīn's book translated from Arabic into Latin by his colleague Edward Pocock (thus reading for the first time the original Medieval treatise instead of Clavius' interpretation of it), and in 1663 he himself

[23] Leibniz read Giordano's *Euclide* (in the 1686 edition) while he was traveling in Rome, and wrote to him shortly thereafter. The whole correspondence amounts to only three letters (two by Leibniz and one by Giordano), which are however very rich in epistemological and geometrical remarks. They were first published in GM I, pp. 195–200, but the critical edition in A III, 4, nn. 216–18, pp. 420–28, is also useful as Leibniz' draft of the second letter to Giordano is much clearer than the letter he actually sent. The correspondence is mostly concerned with providing a good definition of a straight line, and touches in passing the definition of parallels. As for Leibniz' marginalia to Giordano's book, they only concern Proclus' proof (reported by Giordano on p. 22 of his *Euclide*) that two straight lines cannot have any common segment. To this proof, Leibniz objects that it should be extended taking into consideration that two straight lines might have even more than one segment in common (while Proclus' proof is based on the fact that, *per absurdum*, they might have one; and it only works in this restricted hypothesis). Leibniz expressed his dissatisfaction about this proof in his first letter to Giordano. As Leibniz' marginal note in Giordano's *Euclide* is still unpublished, I transcribe it here (it is in Italian, like Giordano's book): "Si suppone che G et F sono punti differenti, il che non è necessario, perché non è dimostrato ancora che le due rette AD, CD, non possono concorrere nel punto G overo F dove il circolo le sega. Questo difetto è perdonabile, perché al meno non tutti [i] puncti possono essere li mede[si]mi, dove un altro circolo descritto dal centro B sega queste rette altrimenti non si darebbero …". Leibniz adds at the beginning of p. 24 (Proclus' proof that two straight lines do not enclose a space) a similar remark: "Si suppone ancora che siano diversi". There is nothing about Giordano's proof of the Parallel Postulate.

[24] We may also mention that Michelangelo Fardella had been a pupil of Borelli, and Leibniz met him during his trip to Italy in 1690 (at the same time in which he met Giordano), and later had an important correspondence with him (see GARBER 2004). Leibniz' reading of Borelli's *Euclides* (or better a re-reading of it) may have been very late (around 1712): see DE RISI 2007, pp. 119–20. Among the many marginalia in this book, we may mention a few about the concept of dimension (DE RISI 2007, pp. 207–208), a couple concerning the definition of a straight line, and two complaining that Borelli (correctly, we would say) assumed so many unproved axioms in his book ("hoc demonstrandum"!). Leibniz' final judgement on Borelli was however quite dismissive: "In Borello multum diligentiae agnosco, ingenio mediocri fuisse videtur" (Leibniz to Johann Bernoulli, October 23rd, 1716; in GM III, p. 971).

[25] Leibniz had a brief exchange with Marchetti in the 1690s (A III, 6, n. 244, pp. 806–807; A III, 7, n. 196, pp. 779–81) about the latter's new theory of proportion published in 1695. Here, Leibniz approves the foundational stance of Marchetti and Borelli before him, only regretting that the analysis of the principles was not complete (A III, 7, p. 780). Marchetti's book on Euclid (dealing with the theory of parallels) was only published some fifteen years later, when apparently Leibniz was no longer in contact with him (cf. BODEMANN 1889, p. 167).

attempted a demonstration of the Postulate. Both essays (Wallis' and Nasīr ad-Dīn's) were later (1693) published in an appendix to Wallis' treatise on algebra.[26] Wallis' proof is of the utmost importance, as it shows that the Parallel Postulate is in fact equivalent to the possibility of transformations through similarity. This means that in non-Euclidean spaces there are no (non-trivial) similar figures; for instance, any two triangles with the same angles are in fact isometric (congruent) with one another. Wallis considered this result to be not so much a characterization of the Parallel Postulate (an alternative formulation of it), as an actual proof, as he could not conceive that the possibility of similarities could be denied. The reasoning behind this last claim seems to be (besides an appeal to the principle of continuity) that similarities preserve the quality (i.e. the shape) of the figures, while they do not preserve their quantity (i.e. length, area or volume); but since quality and quantity are (in Aristotle) different categories of being, it must be possible to change a feature in a figure pertaining to one of them while leaving the other untouched (i.e. change the quantity while preserving the quality).[27] Wallis' proof convinced few mathematicians, and already Saccheri complained that his demonstration of the Parallel Postulate was more metaphysical than mathematical, but it was widely discussed by philosophers up to the nineteenth century.[28] In 1696, Leibniz wrote a review of Wallis' *Algebra*, mentioning the appendix, but he didn't enter into the philosophical discussion about similarity.[29] In his writings on *analysis situs*, however, Leibniz developed a vast theory of similarity which he considered to be the cornerstone of his new geometry; a

[26] Wallis' treatise of algebra appeared in English in 1685, and in Latin (with some appendices) in 1693. In the latter form, it was printed as the second volume of Wallis' *Opera mathematica*, and included the short essay from 1663 entitled *De Postulato Quinto et Definitione Quinta Libri 6 Euclidis Disceptatio geometrica* at pp. 665–78 (pp. 669–73 being Pocock's Latin translation of Nasīr ad-Dīn). We know that Pocock translated into Latin both the original work of Nasīr ad-Dīn and the spurious essay that was published in Rome. Wallis' *Algebra* only contains the latter, but Wallis himself had access to Pocock's translation of the former, which he added in manuscript to his own copy of the *Opera mathematica*. Pocock's unpublished translation is still available in manuscript at the Bodleian Library, and was reprinted in Cassinet 1986.

[27] *De postulato quinto*, Lemma 8, p. 676. But cf. also Wallis' *Institutio logicae*, in *Opera*, vol. 3 ii, p. 105 (on the concept of figure as quality).

[28] We know Saccheri's opinion on Wallis' proof from a letter sent on the 12th of July 1713 by Tommaso Ceva to Guido Grandi, in which Ceva reports Saccheri saying: "… quantunque il suo lemma ottavo resti da lui provato più tosto con la metafisica che con rigore geometrico, onde pare che fosse necessaria la costruzione problematica per torre ogni sospetto di petizione di principio" (in Tenca 1952, p. 35). The opinion was probably shared by many other mathematicians, as no one was prevented by Wallis' alleged proof from attempting to demonstrate the Parallel Postulate. A similar position was later held by Lambert, *Theorie der Parallellinien*, §§ 79–81 (pp. 350–2), who however was willing to accept Wallis' principle on similarity as an unprovable axiom.

[29] Leibniz' review of Wallis appeared in the 1696 issue of the *Acta Eruditorum* (pp. 249–59). We have also several unpublished notes on Wallis' volume and drafts of the review: see LH XXXV, vii, 22; and cf. LH XXXV, xiv, 2, Bl. 66–75 and 90. Among these papers, however, there is not much about Wallis' proof of the Postulate, as Leibniz preferred to concentrate on the algebraic results proper. In the published review, the whole appendix on the Parallel Postulate is disposed of with a dismissive note: "Peculiari etiam dissertatione defenduntur definitio quinta & postulatum quintum libri sexti Euclidis, quae Savilius ipse inter propositiones demonstrabiles rectius referri in suis Lectionibus judicaverat" (p. 259). As even the quotation is wrong (the Parallel Postulate

theory, moreover, that had important consequences in epistemology and metaphysics. Thus, Wallis may have at least convinced him that a proof of the Parallel Postulate was needed not only to fill a gap in the system of principles of elementary geometry, but also to give a foundation to his own philosophical projects.[30]

Lastly, we should mention a clumsy move in the development of the theory of parallels, by Thomas Hobbes (1588–1679). In his influential work on natural philosophy from 1655, the treatise *De corpore*, Hobbes advanced several mathematical theses that were in turn attacked by Wallis himself; Hobbes answered with *Six lessons to the Savillian professors of mathematics* that appeared as an appendix to the 1656 English edition of the same treatise. In these essays, which predated Wallis' attempts to prove the Postulate and in fact even Borelli's criticisms of the *Elements*, Hobbes claimed that the Euclidean definition of parallel lines is obscure and that a sound mathematical definition should show the way to generate its object (which the Euclidean definition does not). Later on, however, he gave a non-constructive definition of parallels, saying that they are equidistant lines, that is, lines "between which every line drawn, in any angle, is equal to any other line drawn in the same angle". After claiming that this definition is quite general and good for both straight and curved lines (and surfaces), Hobbes simply assumed that one can find straight lines behaving in this way, and thus (making the old mistake of Proclus) proved the Parallel Postulate from a definition that would need itself need a proof. Hobbes concluded that he had "done that business for which Dr. Wallis receives the wages"

being the Fifth Postulate of Book One, not Book Six), one suspects that Leibniz didn't grasp the full importance of Wallis' proof at the time.

[30] Among Borelli's reflections on the role of principles in geometry (in the first edition of his *Euclides restitutus*, pp. 14–17), we find a discussion on the necessity of proving the existence of a defined mathematical object, as for instance the parallel lines in his own definition (see above at *note* 20 his reference to *Elements* I, 10 and 11 to this effect). It is very remarkable that Borelli's examples of definitions, whose objects are doubtful before an existential or constructive proof, are: parallel lines (in Euclid's definition), squares, similar figures. There is no further textual hint that he connected these various examples under the same label. Since however he must have tried hard to prove the Parallel Postulate and surely read the extant literature on the subject, he could not have failed to notice that the theory of parallels was implied in the possibility of constructing a square (as it is clear in Clavius and in the Arabs through the use of Saccheri quadrilaterals). It is not unlikely, then, that Borelli also realized that the Parallel Postulate is needed to ground the possibility of similar figures. If this is true, he may have conceived Wallis' "proof" of the Postulate and disregarded it as too philosophical (one should also bear in mind that a sketch of Wallis' proof may be recognized in Clavius: cf. Maierù 1978; and possibly even in Euclid, *Data* 34–38). Be that as it may, Leibniz (who seemed unreceptive to Wallis' own demonstration) carefully underlined Borelli's sentence on his copy of *Euclides restitutus*. Here is the original quote: "Deinde si constructio aut passio nominata, sit quidem possibilis, & vera, sed nobis ignota, aut dubia; tunc bona definitio non erit: Nam conclusiones ab ignoto, & dubio principio ortae, incertae quoque, & dubiae erunt; & ideo suspicionem, aut opinionem, non autem scientiam certam afferent. Ut cùm dicitur in vulgata definitione parallelarum: Duae rectae lineae in eodem plano ex utraque parte non concurrentes, vocetur parallelae. Ignoratur an dari possint duae rectae lineae, habentes hanc conditionem. Similiter ignoratur, an reperiri possit in natura figura quadrilatera, in qua omnes anguli sint recti, & quatuor latera sint inter se aequalia, quae vocatur quadratum. Pari modo incertum est an reperiri possint figurae planae aequiangulae, habentes circa angulos aequales latera proportionalia, quae similes vocantur" (*Euclides restitutus*, p. 17).

as the Savillian Professor of mathematics.[31] Even though Hobbes' proof is so badly conceived as not to have any place in the history of the attempts to prove the Postulate, Leibniz knew Hobbes' mathematical books very well, and was influenced by Hobbes' attempt to give a general definition of parallel lines that could be applied to both straight and curved lines. Most of all, Hobbes' theory of constructive definitions (forsaken by Hobbes himself) was quite important in Leibniz' epistemology of mathematics, and this, in turn, heavily affected his geometrical studies.

[31] On the need for constructive definitions, see for instance the *Six lessons*: "And where there is place for demonstration, if the first principles, that is to say, the definitions contain not the generation of the subject, there can be nothing demonstrated as it ought to be" (*The works*, vol. 7, p. 184). A similar passage is to be found in *De corpore* I, vi, § 13, but the reason Hobbes provides for the need for causal definitions is here to be found in the Aristotelian idea that scientific knowledge is causal knowledge (the διότι), rather than in the ontological doubt about the real possibility of an object. (see below § 3.2). Hobbes' original definition of parallels, in Latin, is in *De corpore*, II, xiv, § 12, and with it Hobbes proves *Elements* I, 29 (equivalent to the Parallel Postulate) as a Corollary, as well as the theorem that the interior angle sum of a triangle is π (*Opera*, vol. 1, pp. 163–65). The definition and the proofs were correctly criticized by Wallis in his *Elenchus geometriae hobbianae* (pp. 29–32), explaining several difficulties but not pointing out the main flaw (that Hobbes had to prove that a parallel to a straight line is straight as well), and Hobbes could re-state his original definition and demonstration (with only a small and insignificant correction) in the English edition of the *De corpore* (*The works*, vol. 1, pp. 189–91; where he also mentions similar criticisms raised by Mylon) and the *Six lessons*, in which he also better explains the alleged shortcomings of the Euclidean definition (*The works*, vol. 7, pp. 205–207, from which my quote is taken; and again pp. 254–55). Hobbes' claim to have settled Savile's problem better than Wallis is again in the *Six lessons* (*The works*, vol. 7, p. 185). On Hobbes' long and savage dispute with Wallis, see Jesseph 1999 and Beeley&Probst 2005.

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